

# Module **B4**

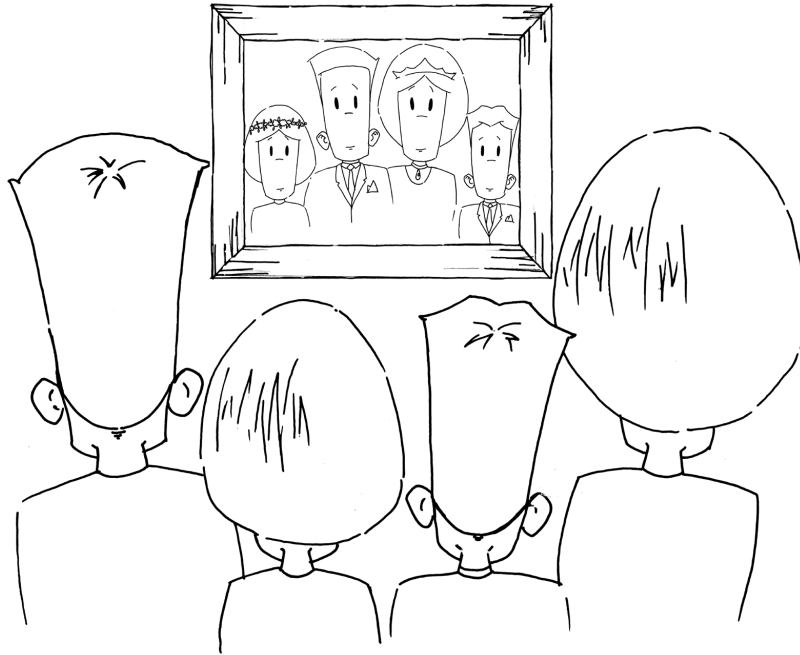
Relations and functions

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## Introduction

The title of this module might conjure up images of aunts and grandfathers, weddings and funerals. You might think that this does not have much to do with mathematics.



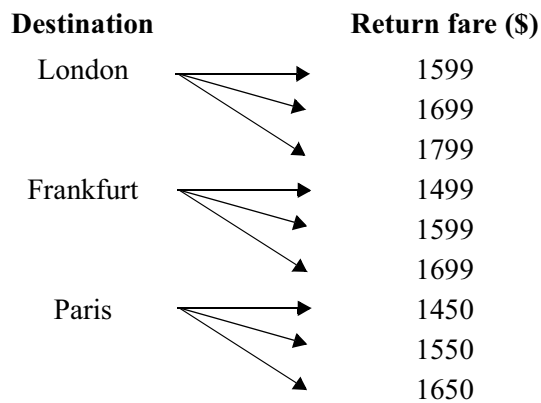
But mathematics and its applications in business, science and engineering etc. are primarily about relationships. Just as we might have a range of different types of relationships with our relatives, or friends for that matter, so different variables can be related in different ways. If we are lucky, or unlucky depending on the situation, we can usually visualize our relatives and friends (or get a photograph if necessary). This module is all about using visual images of relationships (graphs) to help us understand the nature of relationships between variables.

More formally when you have successfully completed this module you should be able to:

- demonstrate an understanding of the concept of a function
- use function notation
- demonstrate an understanding of the relationship between an algebraic expression (linear) involving two variables and its graph on the Cartesian plane
- recognize, sketch and use linear functions
- demonstrate an understanding of the nature of change (gradient) in a linear function
- demonstrate an understanding of and find the inverse of a linear function
- recognize, sketch and use quadratic functions
- demonstrate an understanding of the nature of change in a quadratic function
- recognize and describe other functions.

## 4.1 What are relations and functions?

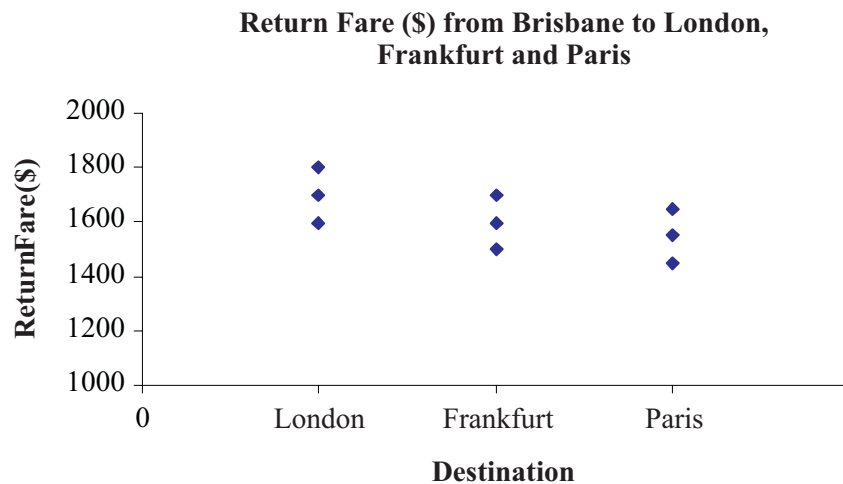
We know from module 3 that if a quantity is not fixed then we call it a variable and symbolize it with a letter or pronumeral. Often one variable is related to one or more other variables, so we say a relationship exists between the variables. The most simple case exists between two variables represented by a set of ordered pairs. For example, consider the situation of return airfares around the world depicted below. Each destination from Brisbane has three different return fares associated with it (they charge different amounts in different seasons).



If we recall some of our previous knowledge we could write this as a set of ordered pairs;

(London, 1599)  
 (London, 1699)  
 (London, 1799)  
 (Frankfurt, 1499)  
 (Frankfurt, 1599)  
 (Frankfurt, 1699)  
 (Paris, 1450)  
 (Paris, 1550)  
 (Paris, 1650)

and represent it as a graph.



In relationships of this type between two variables, we say one of the variables is dependent on the other. In the case above, we do not know what the return fare could be unless we first know the destination. The Return Fare is dependent on Destination. Because of this we call Return Fare the **dependent variable** and Destination the **independent variable**. If you are confused by these terms another way to think of them is as **input** and **output** variables. The independent is always the input variable, while the dependent is the output variable. Conventionally in most disciplines, the independent variable (input) is the first member of the ordered pair and is placed on the horizontal axis on a graph (often called the  $x$ -axis). The dependent variable (output) is the second member of the pair and is placed on the vertical axis (often called the  $y$ -axis).

### Example

In the following expression which variable is the independent variable and which is dependent?

In a small business the selling price of a commodity is related to the number ordered, so that if you buy one tee shirt it costs \$10 each while if you buy 20 they will cost \$5 each.

The selling price is determined by the number of shirts ordered so we say ‘Number ordered’ is the independent variable and ‘Selling Price’ is the dependent variable.

### Example

Pressure and depth underwater were both recorded regularly throughout an experiment on the effectiveness of a new wet suit. If you had to graph this relationship which variable is the independent variable and should be placed on the horizontal axis.

In this instance pressure is dependent on the depth of the diver, so we say that depth is the independent variable and should be placed on the horizontal axis. Pressure will be the dependent variable and should be placed on the vertical axis.

## Activity 4.1

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1. In the following expressions, which variable is the independent variable and which is dependent?
  - (a) In Australia, the amount of income tax paid is related to the amount of income earned. For example, for any income earned over \$50 000, 48 cents in the dollar is paid as tax.
  - (b) Reading the label on a medicine bottle, a parent finds that the number of millilitres to be given to a sick child is related to the age of the child. If a child is 3 years old, then a 5 mL dosage is sufficient, but a 12 year old child requires 15 mL.
2. If you had to graph the following relationships decide which variable should be placed on the horizontal axis (independent variable) and which should be placed on the vertical axis (dependent variable).
  - (a) The daily demand for the bulk purchasing of bottled water is related to the selling price of the water. When the selling price is \$6.00 per unit, then 20 units are sold, but when the selling price is \$6.50 per unit then only 14 units are sold.

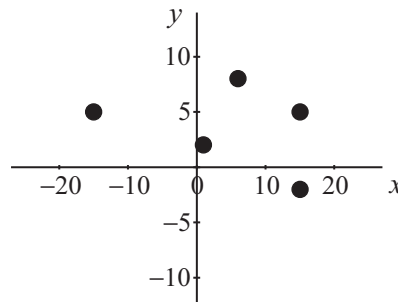
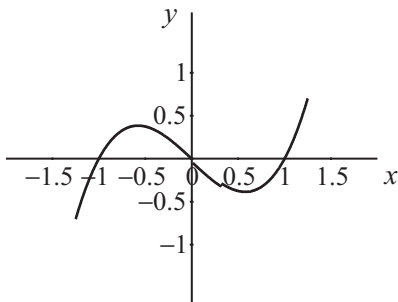
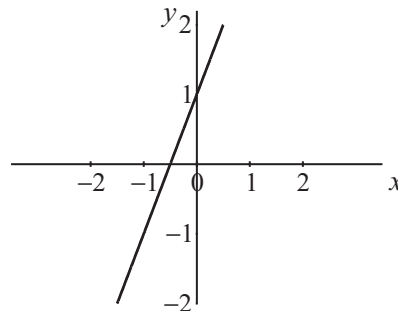
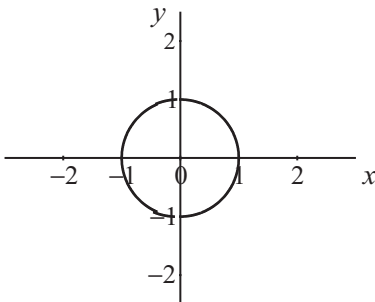
- (b) Scientists recorded data on the area covered by a plant species resistant to cold temperature extremes and the number of frost-free days throughout winter.
- (c) Safety tests performed on a new model car required data to be collected on stopping distances of the car for a range of velocities.

If we return to the airfares problem we saw that this relationship was a set of ordered pairs...we call this a relation.

**A relation is a set of ordered pairs.**

Relations or sets of ordered pairs can be represented in three ways, often depending on the type of relation. They can be represented as:

- groups of points as in the airfares example above or as in  $\{(1,0), (2,3), (4,1), (7,6)\}$ ;
- formulas or equations like  $y = 2x + 1$ , or  $z = \pm\sqrt{1 - p^2}$  (recall that given the value of either  $x$  or  $p$  in these equations we can calculate the value of  $y$  or  $z$  and so form sets of ordered pairs); or
- graphs of the relationships (see below).

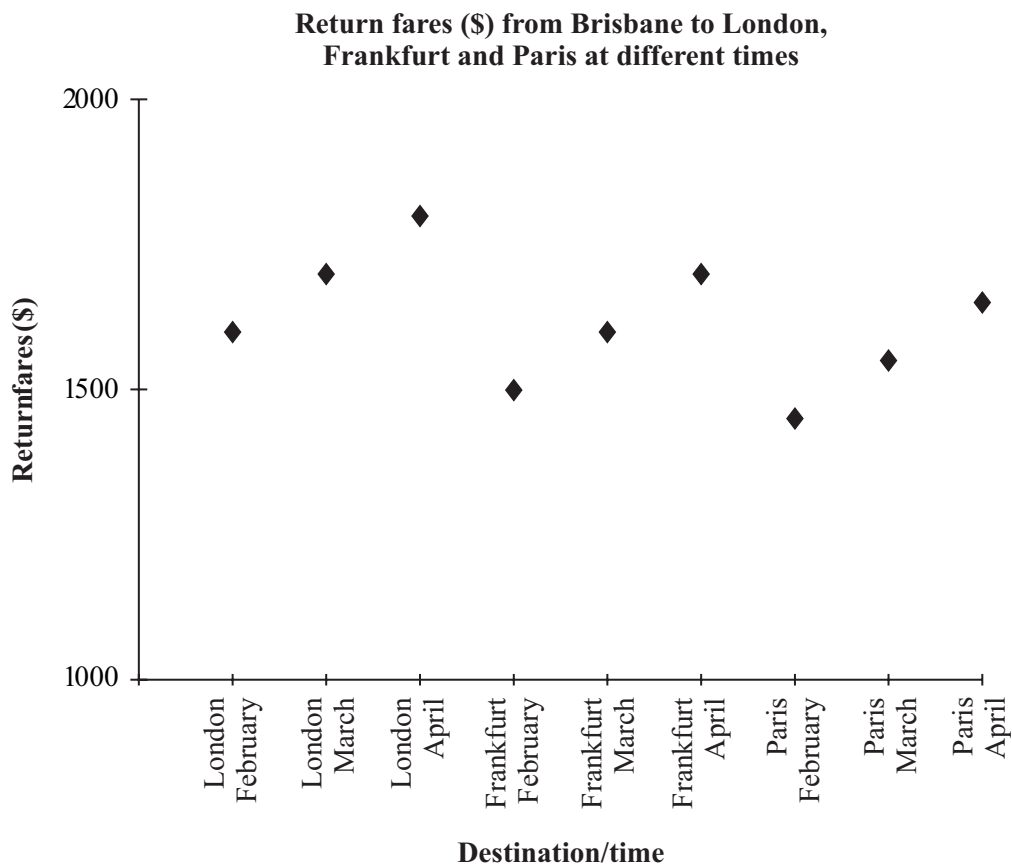


All of the above are called **relations**. But let's go back again to the airfares problem. This relation does tell us something about the cost of flying to specific places but unless we had some more information then we could not really use the relation to calculate one price...there are three different fares for each destination.

It might be more useful to have a relationship that looked like this, with destination and time together as one variable.

Destination/time	Return fare (\$)
London in February	1599
London in March	1699
London in April	1799
Frankfurt in February	1499
Frankfurt in March	1599
Frankfurt in April	1699
Paris in February	1450
Paris in March	1550
Paris in April	1650

It would be graphed like this:



In this relation we see that each destination/time has only one cost. It doesn't matter if the costs are the same for different destinations/time, as is the case for London in February and Frankfurt in March, because we are trying to predict cost from destination/time not vice versa.

In many mathematical situations it is often useful to have a relationship from which we can calculate just one value of the dependent variable from one value of the independent variable...this type of relation is called a **function**.

**A function is a relation in which there is only one value of the dependent variable for each value of the independent variable.**

**Something to talk about...**

You will see the definition of a function written in many different ways in different books. Write a definition of a function in your own words and share it with other students either through personal contact or through the Discussion Group. Think about which definition best captures the essence of the nature of a function and send some examples of this in practice to the group.

**Example**

The following are relations. Indicate which relations are also functions. Give reasons for your choices.

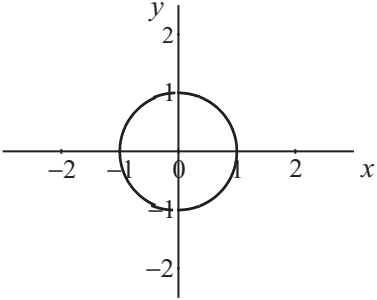
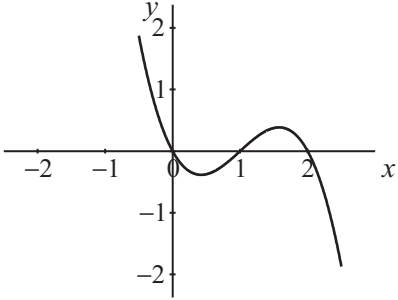
Relation	Function?	Reason
Rainfall in a small town is measured once per day in a mm rain gauge and is graphed against day of the month. The graph is shown at a town meeting.	Yes	In this relation, the day of the month is the independent variable and the amount of rain (mm) is the dependent. As the rainfall is measured only once per day there must only be one value of volume for each day. The relation must be a function.
$\{(1,0), (0,1), (1,1), (0,0)\}$	No	In ordered pairs the first coordinate is traditionally the independent variable and the second the dependent variable. For $x = 1$ there are two alternatives of $y = 0$ or $y = 1$ . The relation has two values of the dependent variable ( $y$ ) for the same value of the independent variable ( $x$ ). The relation is thus not a function.
$\{(7,10), (3,4), (5,6), (-1,10)\}$	Yes	In ordered pairs the first coordinate is traditionally the independent variable and the second the dependent variable. For each of the $x$ values shown in the set of ordered pairs there is only one $y$ value. It does not matter that there are repeated values of $y$ ( $y = 10$ ) for different $x$ values. The relation is a function.

	<p>No</p>	<p>In graphs the horizontal axis traditionally represents the independent variable and the vertical axis the dependent variable. At one point on the graph for one value of <math>x</math> (the independent variable) there are two values of <math>y</math> (the dependent variable). The relation is thus not a function.</p>
<p><math>k = \pm\sqrt{q}</math></p>	<p>No</p>	<p>For every value of <math>q</math> (the independent variable) we can calculate a positive and a negative value of <math>k</math> (the dependent variable). That is, we have two values of the dependent variable for one value of the independent variable e.g. if <math>q = 25</math> then <math>k = 5</math> or <math>-5</math>. The relation is not a function.</p>
	<p>Yes</p>	<p>As you move along the <math>x</math>-axis (the independent variable) there is only ever one value on the <math>y</math>-axis (the dependent variable) that corresponds to it. The relation must be a function.</p>

### Activity 4.2

1. Complete the following table indicating which of the following relations are also functions. Give reasons for your choices.

Relation	Function?	Reason
(a) $y = x + 5$		

<p>(b)</p> 		
<p>(c) <math>\{(-4,0), (3,2), (-1,2), (6,0)\}</math></p>		
<p>(d)</p> 		
<p>(e) The amount of potassium absorbed by the leaf tissue of a plant when placed in darkness is graphed against time.</p>		
<p>(f) <math>\{(3,27), (-9,2), (11,27), (3,5)\}</math></p>		

### 4.1.1 Domain and range of relations and functions

Let's return to the relation we used in the airfare problem, this time represented as a set of ordered pairs.

(London, 1599)  
(London, 1699)  
(London, 1799)  
(Frankfurt, 1499)  
(Frankfurt, 1599)  
(Frankfurt, 1699)  
(Paris, 1450)  
(Paris, 1550)  
(Paris, 1650)

You will notice that there are restrictions on the values of the independent variable. We could not use the relation to determine the cost of flying to San Francisco because it does not exist in the relation. The conditions under which a relation will work are called the domain a range.

**The domain is the set of values for which the independent variable is defined.**

All relations, including functions, will have domains, sometimes the domain will be very obvious as in the following:

**Example**

Function	Values of independent variable	Domain										
<table border="0"> <tr> <td><b>Fish Species</b></td> <td><b>Cost</b></td> </tr> <tr> <td>Guppy</td> <td>40c</td> </tr> <tr> <td>Goldfish</td> <td>50c</td> </tr> <tr> <td>Rainbow</td> <td>\$2.50</td> </tr> <tr> <td>Cichlid</td> <td>\$10.50</td> </tr> </table>	<b>Fish Species</b>	<b>Cost</b>	Guppy	40c	Goldfish	50c	Rainbow	\$2.50	Cichlid	\$10.50	Guppy, Goldfish, Rainbow, Cichlid	{Guppy, Goldfish, Rainbow, Cichlid}
<b>Fish Species</b>	<b>Cost</b>											
Guppy	40c											
Goldfish	50c											
Rainbow	\$2.50											
Cichlid	\$10.50											
(1,2), (2,3), (4,5)	$x = 1, x = 2, x = 4$	{1, 2, 4}...this would include only three values										
$p = 4a + 1$ , where $a$ is measured between 1 and 10.	$1 < a < 10$	$1 < a < 10$ ...this would include an infinite number of values between 1 and 10										
	Graph is only drawn between the $x$ values of $-1.3$ and $1.5$	$-1.3 < x < 1.5$										

Notice that the domain can be either a set of discrete values (e.g. 1, 10, 4, 22) or an infinite number of points as in a continuous expression (e.g.  $-2 < a < 4$ )

Sometimes, however, the domain of a function is not so obvious.

Consider these examples.

**Example**

Final speed of a car ( $v$ ) is equal to the starting speed of 0 m/s plus the product of acceleration (10 m/s/s) and time of travelling ( $t$ ). What is the domain of this function?

If we translated this from words to an algebraic formula we would get

$$v = 0 + 10t$$

$$v = 10t$$

In this instance the independent variable is time ( $t$ ). Are there any restrictions on the values of time?

If you said ‘yes’ you would be correct. In normal circumstances we would not be able to travel back in time and so time would always have to be a positive number or zero. We would say that the domain of this function would be  $t \geq 0$  and theoretically would extend to infinity.

**Example**

Consider the function

$$y = 2x + 1, \text{ what is its domain?}$$

By convention the independent variable in this function would be  $x$ . Are there are restrictions on the values of  $x$  for this function?

If you said ‘no’ you would be correct. The domain for this function contains an infinite number of values...in fact it will contain all real numbers. We would say that the domain would be all real values of  $x$ .

**Example**

Consider the function

$$p = \frac{1}{a}, \text{ what is the domain of this function?}$$

In this function, unless otherwise indicated, we would assume the independent variable to be  $a$ . Are there any restrictions on the values of  $a$  for this function?

If you said ‘yes’ you would be correct. Try and calculate the value of  $p$  for  $a = 0$ ...it is not possible.  $\frac{1}{0}$  is not a number so the function is not defined for  $a = 0$ .

We would say that the domain of the function is all real values of  $a$  except  $a = 0$

**Example**

Consider the function

$$y = \sqrt{x-1}, \text{ what is the domain of this function?}$$

In this function we would assume that the independent variable would be  $x$ . Are there any restrictions on the values of  $x$ ?

If you said ‘yes’ you would be correct.

Let’s think about the square root expression. Have you ever tried to find the square root of a negative number on your calculator? In the real number system it is not possible to calculate the square root of a negative number. So this means that any expression under the square root sign must not be negative.

In our example this means that  $x - 1 \geq 0$ .

If we solve this inequation for  $x$  we will get  $x \geq 1$ .

This means that the domain of the function is  $x \geq 1$ .

In the above examples the domain is sometimes called the natural domain.

### Activity 4.3

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1. State the natural domain of the following functions.

(a)  $y = \frac{1}{2x}$

(b)  $y = \frac{1}{x-1}$

(c)  $p = \sqrt{v-4}$

(d)  $y = \frac{1}{(x+3)(x-4)}$

(e)  $m = 8n$

Just as we have a word associated with the extent of the independent variable we also have a word associated with the extent of the dependent variable.

**The range is the set of values for which the dependent variable is defined.**

In most instances (unless we have a graph from which we can read off the values) we would first determine the domain and then use it to determine the range.

Look at our previous examples.

**Example**

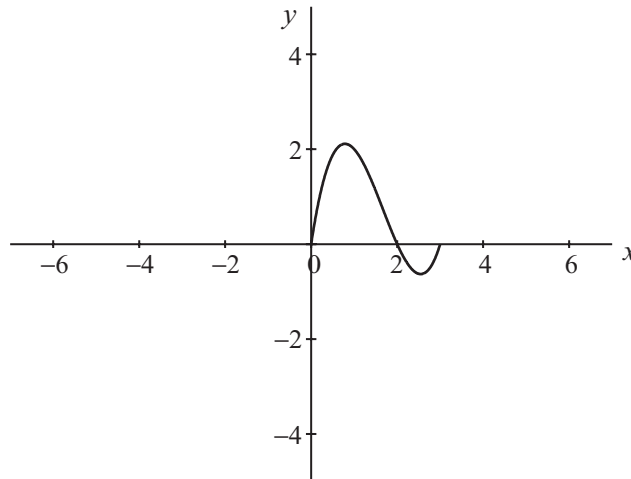
Function		Domain	Extent of the dependent variable	Range
<b>Fish species</b> Guppy Goldfish Rainbow Cichlid	<b>Cost</b> 40c 50c \$2.50 \$10.50	{Guppy, Goldfish, Rainbow, Cichlid}	All costs	{0.40, 0.50, 2.50, 10.50} (converted all cents to dollars)
(1,2), (2,3), (4,5)		{1, 2, 4}...	All $y$ coordinates	{2, 3, 5}
$p = 4a + 1$ , where $a$ is measured between 1 and 10		$1 < a < 10$	All values of $p$ when $1 < a < 10$ . Limiting value of $p$ is 5 when $a = 1$ , maximum value is 41 when $a = 10$ .	$5 < p < 41$
		$-1.3 < x < 1.5$	All values of $y$ when $-1.3 < x < 1.5$ . From the graph when $x = -1.3$ , $y = -0.9$ , when $x = 1.5$ , $y = 1.9$	$-0.9 < y < 1.9$

Determining the range for many functions is often not as easy as in the functions we have described above. It involves knowledge of the domain of the function and an understanding of the shape of the function. We will investigate domains and ranges of specific functions later in this module.

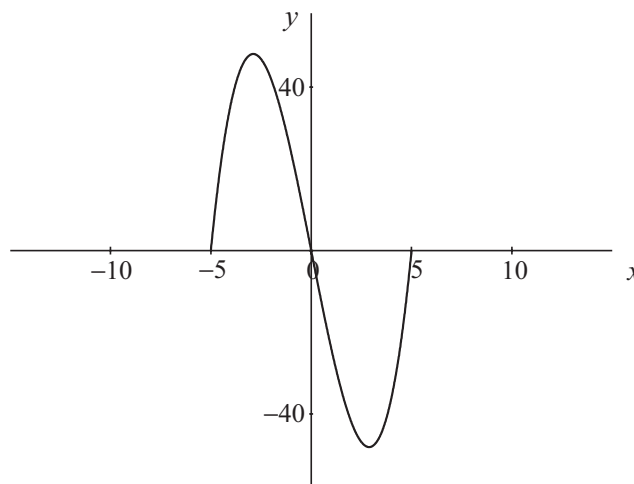
## Activity 4.4

1. Find the domains and ranges of the functions with these graphs.

(a)



(b)



2. State the range of the following functions.

(a)  $a = 2b + 3$ , where the domain is all real values of  $b$

(b) If  $r$  and  $q$  are continuous variables in the example detailed below.

$r$	2	5	-3	8	1	2
$q$	5	15	-10	23	2	5

(c)  $y = 2 + 4x$  where the domain is  $0 \leq x \leq 50$

## 4.1.2 Function notation

We know now what a function is and in some cases how to determine its domain and range, but because functions are so important and used so frequently a special notation has been developed to simplify their description.

Let's look at the following function from economics.

*In a monopoly, the selling price of goods can be set so that the total revenue is a function of the output quantity.*

If we call Total Revenue ( $R$ ) and quantity ( $q$ ) then more briefly we would say:

Total revenue is a function of quantity



$R$  is a function of  $q$



$$R = f(q)$$

In the monopoly example, a specific total revenue function is  $R = -0.6(q - 7)^2 + 40$ . If we wanted to calculate values of  $R$  for  $q$  we would do the following:

$$\text{If } q = 2, \text{ then } R = -0.6(2 - 7)^2 + 40 = 25$$

which we could now write more simply as:

$$f(2) = 25 \text{ in words we would say 'the value of the function } f \text{ at } 2 \text{ is } 25'.$$

The letter  $f$  was chosen to represent function for obvious reasons, but if we have different functions using the same independent variable we can choose any letter or symbol we like.

For example in the same monopoly, Price ( $P$ ) is also a function of output quantity ( $q$ ),  $P = q + 13$  so we could write this as  $g(q) = q + 13$ . If we wanted to find the Price when output quantity was 2 then we would calculate:

$$g(2) = 2 + 13 = 15 \text{ in words we would say 'the value of the function } g \text{ at } 2 \text{ is } 15'.$$

In these two examples above both  $f(q)$  and  $g(q)$  can still represent the dependent variable and when we graph the functions, these values would conventionally be graphed on the vertical axis. We will return to this later.

Let's look at some examples of the use of this notation.

**Example**

If  $y = f(x) = 2x - 1$  find the values of the function when  $x = -4$  and  $x = \frac{1}{2}$

$$f(x) = 2x - 1$$

$$f(-4) = 2(-4) - 1$$

$$f(-4) = -9$$

The value of the function at  $-4$  is  $-9$ .

$$f(x) = 2x - 1$$

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right) - 1$$

$$f\left(\frac{1}{2}\right) = 0$$

The value of the function at  $\frac{1}{2}$  is  $0$ .

**Example**

If  $q(t) = 3(t - 4)^2 + 2$ , evaluate  $q(a)$  and  $q(1 + a)$  if  $a$  is a constant.

$$q(t) = 3(t - 4)^2 + 2 \quad \text{everywhere } t \text{ occurs replace by } a$$

$$q(a) = 3(a - 4)^2 + 2$$

$$q(a) = 3(a^2 - 8a + 16) + 2 \quad \text{expand squared term}$$

$$q(a) = 3a^2 - 24a + 48 + 2$$

$$q(a) = 3a^2 - 24a + 50$$

$$q(t) = 3(t - 4)^2 + 2$$

$$q(1 + a) = 3((1 + a) - 4)^2 + 2 \quad \text{everywhere } t \text{ occurs replace by } 1 + a$$

$$q(1 + a) = 3(a - 3)^2 + 2$$

$$q(1 + a) = 3(a^2 - 6a + 9) + 2 \quad \text{expand squared term}$$

$$q(1 + a) = 3a^2 - 18a + 27 + 2$$

$$q(1 + a) = 3a^2 - 18a + 29$$

## Activity 4.5

- Write the following using function notation.
  - The maximum magnification,  $M$ , obtained when using a converging lens is a function of the focal length,  $f$ , of the converging lens, so that
 
$$M = 1 + \frac{25}{f}.$$
  - The daily cost,  $C$ , of producing a product is a linear function of the number,  $q$ , of the product made, so that  $C = 5q + 45$ .
  - The daily sales,  $S$ , of a product is a function of the time the product has been on the market, so that  $S = \frac{4000}{1 + 10e^{-0.03t}}$ .
  - The surface area of a sphere ( $S$ ) is a function of the radius ( $r$ ) of the sphere so that  $S = 4\pi r^2$ .
- Find the value of the following functions when  $x = 3$ ,  $x = -0.5$ ,  $x = 0$ ,  $x = (2+t)$ ,  $x = a$ 
  - $f(x) = 2x + 4$
  - $f(x) = \frac{4x}{2x - 3}$
  - $f(x) = x^2 - 2x + 9$
  - $f(x) = \sqrt{4x + 5}$
- Given the functional relationship,  $g(t) = 4 - 3t - 2t^2$  find
  - $g(-2)$
  - $g(a + b)$
- The weight,  $w$ , in kilograms of an object is a function of the distance,  $d$ , in kilometres from the surface of the earth. For a person weighing 65 kg, the function can be written,

$$w(d) = \frac{6400^2 \times 65}{(d + 6400)^2}$$

Find the weight of a person standing on a mountain 3200 m above sea level.

## 4.2 The linear function

Linear functions are functions whose graphs represent straight lines. You should have come across functions like this before. They are called linear because in each case the variables are of power one and no two variables are multiplied together (we will add to this definition later in the module). They are called functions because each value of the input variable has only one value for the output variable.

Examples we have seen before include:

$I = 400P$ , where  $P$  was the price of shirts and  $I$  was the income

$F = \frac{9}{5}C + 32$ , where  $F$  was the temperature in Fahrenheit and  $C$  the temperature in Centigrade.

Other examples would be:

$B = 1.2S + 2$ , where  $B$  is the temperature in Brisbane and  $S$  the temperature in Sydney in °C.

$H = -3V + 69$ , where  $H$  is an animal's heartbeat (per min) and  $V$  is the volume of a drug (mL)

A business consultant's fee is a retainer of \$700 plus \$450 per day or part of a day.

All of the above can be graphed on sets of coordinate axes. Remember we choose the axes so that the independent variable is on the horizontal axis and the dependent is on the vertical.

You can graph linear functions in a range of ways. In the past you might have drawn up a table of values and then plotted the points. For straight line graphs you only need to plot two points with possibly a third as a check.

Let's have a look at another way to graph straight lines.

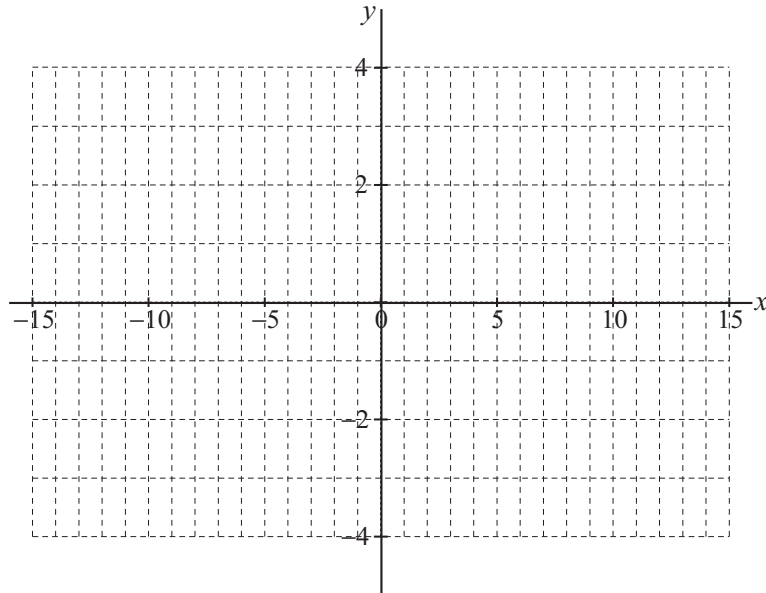
Consider the function

$B = 1.2S + 2$ , where  $B$  is the temperature in Brisbane and  $S$  the temperature in Sydney in °C.

We would first draw the axes with the independent variable ( $S$ ) on the horizontal axis and the dependent variable,  $B$ , on the vertical. Recall from our previous work on functions that we could have just as easily described this function as  $B = f(S) = 1.2S + 2$ .

The next step is to find and plot two points. We could choose any two points and many people would choose  $S = 0$  and  $S = 1$  because they are easy to calculate. But it is often useful to determine the intercepts on the vertical and horizontal axes. So we will follow that method. Note that in general where the vertical axis is the  $y$ -axis and the horizontal axis is the  $x$ -axis we would have the following.

Along the  $y$ -axis all  
values of  $x$  are  $x = 0$



Along the  $x$ -axis all values  
of  $y$  are  $y = 0$

So in our case where the vertical axis is the  $B$ -axis and the horizontal axis is the  $S$ -axis and  $B = 1.2S + 2$  to see where the line cuts the  $B$ -axis we would put  $S = 0$ .

If  $S = 0$ ,

then  $B = 1.2 \times 0 + 2 = 2$ .

So the vertical intercept is  $B = 2$ . This is the point  $(0, 2)$ .

To see where the line cuts the  $S$ -axis we would put  $B = 0$ . If  $B = 0$ , then

$$0 = 1.2S + 2$$

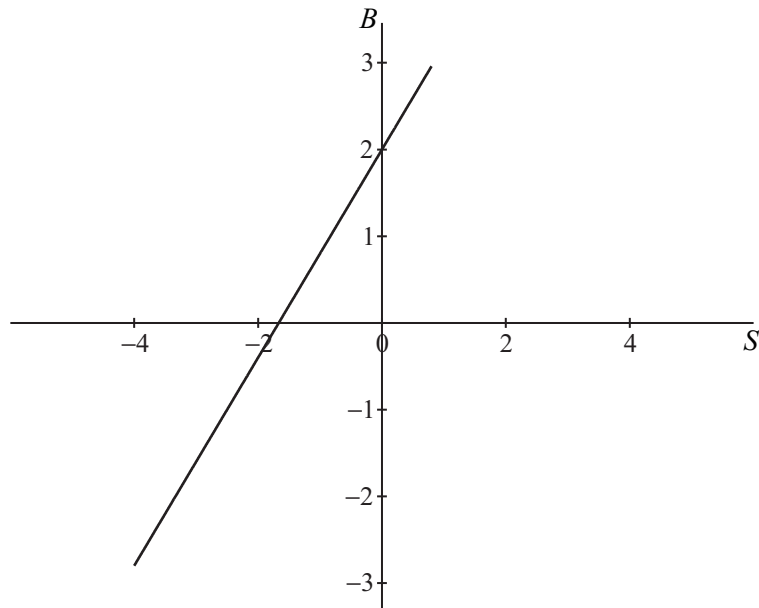
$$1.2S = -2$$

$$S = \frac{-2}{1.2}$$

$$S \approx -1.7$$

So the horizontal intercept is  $S \approx -1.7$ . This is the point  $(-1.7, 0)$ .

We can graph these two points on the axes and then draw a line through and beyond these two points.



If you have access to a graphing package then check the accuracy of this method by drawing the graph with the package and reading off the intercepts on the horizontal and vertical axes.

### Activity 4.6

---

- For each of the following linear functions find the vertical and horizontal intercepts.
  - $4x - 3y = 24$ ; where  $x$  is the independent variable.
  - $b = -2a + 5$ ; where  $a$  is the independent variable.
  - $\frac{4}{5}s - 8 = t$ ; where  $s$  is the independent variable.
  - $9 = 2m + 4n$ ; where  $m$  is the independent variable.
  - $3x = \frac{(x-y)}{2}$ ; where  $x$  is the independent variable.
  - $q - 3 = 5(p - 2)$ ; where  $p$  is the independent variable.
- Graph the linear functions in question 1 using the  $x$ - and  $y$ - intercepts.

3. Graph the following functions for the domains indicated. (Hint: you might want to rearrange the equation first.)

$$(a) \quad \frac{y-3}{2} = x \qquad 0 \leq x \leq 4$$

$$(b) \quad 2x - y = 6 \qquad 0 \leq x \leq 3$$

$$(c) \quad 5y = 15 \qquad -3 \leq x \leq 3$$

$$(d) \quad x + 2y = 8 \qquad 0 \leq x \leq 8$$

$$(e) \quad y = \frac{-x+2}{3} \qquad -4 \leq x \leq 0$$

This method is all very well when we are given the equation of the linear function but what is the procedure in a real world situation. In many instances we would just be presented with something like this.

*You are the finance manager for a small company and are asked to develop a 5 year plan for a business consultant's services over this time. The easy way to do this is to develop a function you can put into a spreadsheet from which you can predict consultant's fees from number of days employed. The only information you have is 'A business consultant's fee is a retainer of \$700 plus \$450 per day or part of a day'.*

(Note spreadsheets are tables of values. This terminology is often used when the table of values is generated by a specially designed computer package.)

Step 1: Name the variables and identify which would be the input variable and which would be the output variable. In this case it is clear that:

- number of days (or parts of a day) worked is the input (independent variable), call it  $n$
- total fee is the output variable (dependent variable), call it  $T$ .

Step 2: Generate the linear function. In this case it would be:

$$T = 700 + 450n \text{ in function notation this would be } f(n) = 700 + 450n$$

Step 3: Determine if there are any restrictions on the domain. In this case there are restrictions.  $n$ , the number of days (or parts of days) worked cannot be negative so there is a lower limit that  $n$  must be greater than zero. Also because we were asked to only develop a 5 year plan there is an upper limit of 5 years or 1825 days. The domain would thus be  $0 \leq n \leq 1825$ .

Once we have the above function and know its domain we can then happily construct a table of values to determine costs. If we had wanted to we could have also drawn a graph (as before) of the function from which we could have made approximate predictions.

**Example**

The temperature in a swimming pool cools down each night to  $20^{\circ}\text{C}$ . When the owner puts the heater on in the morning the temperature increases at  $0.1^{\circ}\text{C}$  per minute until it reaches  $27^{\circ}\text{C}$ , the temperature set on the thermostat of the pool heater. Draw a graph of the relationship between pool temperature and time in minutes, during this heating phase.

Step 1: The variables of interest are pool temperature in degrees Celsius (call it  $P$ ) and the time in minutes (call it  $t$ ).  $t$  is the independent variable and  $P$  is the dependent variable.

Step 2: The linear function in this case would be  
 $P = 0.1t + 20$  in function notation it would be  $f(t) = 0.1t + 20$

Step 3: There are restrictions on the domain of the function. There is a lower value because the time can never be negative. There will be an upper value because the temperature was only measured until it reached  $27^{\circ}\text{C}$  (i.e. during the heating phase). From the equation above we know that the temperature reaches  $27^{\circ}\text{C}$  after 70 minutes. So the domain will be  $0 \leq t \leq 70$ .

Step 4: To draw the graph it is conventional to place the independent variable on the horizontal axis and the dependent variable on the vertical axis. To calculate two points from which to sketch the straight line, we can choose the intercept on the vertical axis and the intercept on the horizontal axis.

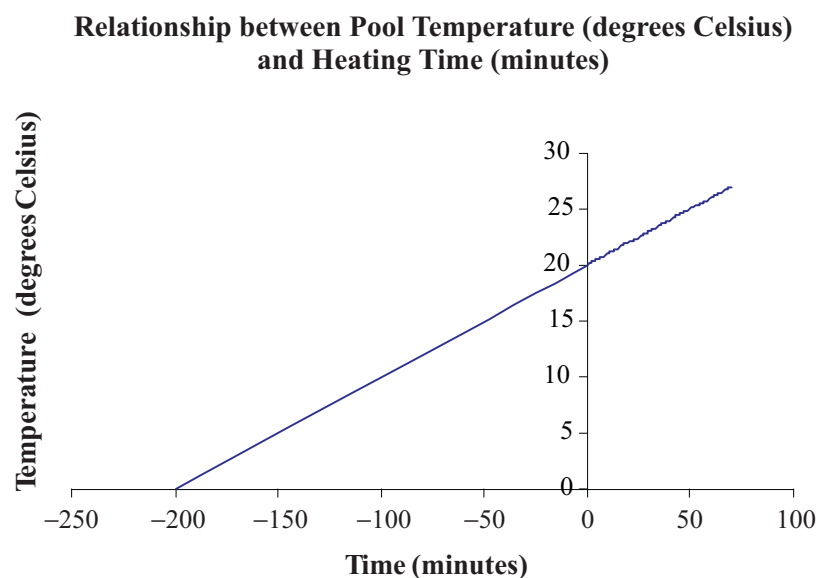
The intercept on the vertical axis occurs when  $t = 0$ , substituting this in the equation  $P = 0.1t + 20$ , we get  $P = 20$  giving us the point  $(0, 20)$ .

The intercept on the horizontal axis occurs when  $P = 0$ , substituting this in the equation  $P = 0.1t + 20$  we get

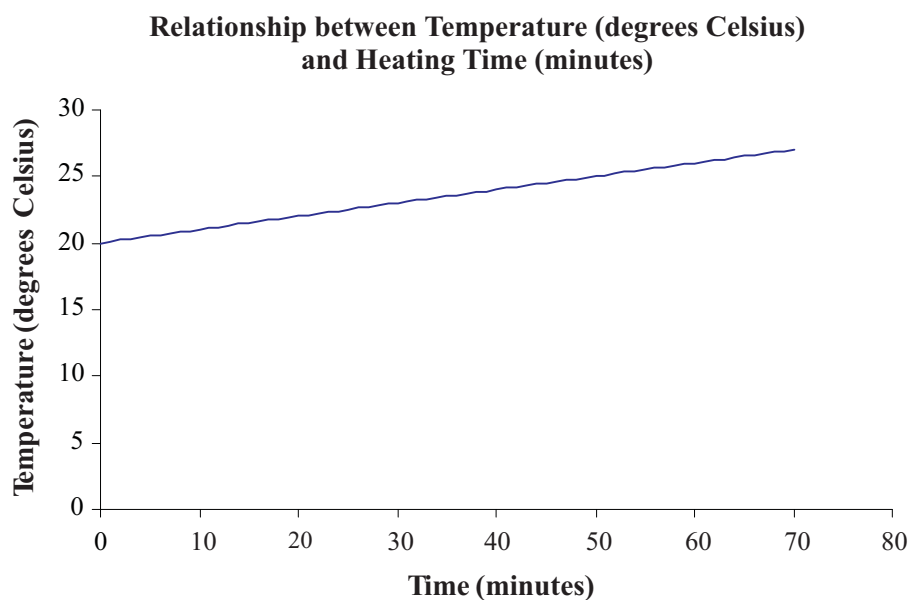
$$\begin{aligned}0 &= 0.1t + 20 \\-20 &= 0.1t \\t &= \frac{-20}{0.1} \\t &= -200\end{aligned}$$

giving us the point  $(-200, 0)$

Using these two points we can sketch the graph



Because of the restrictions to the domain the final graph would look like this (note we have modified the scale so that the graph is more usable).



Notice that the range of the function is from 20 to 27°C inclusive, which fits with the information given.

## Activity 4.7

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1. Establish a linear relationship for each of the following and specify the domain and range. Remember to define the variables.
  - (a) A candle 80 mm long burns so that every hour it is 10 mm shorter.
  - (b) A water tank has a capacity of 2 000 litres. Water leaks from two holes at the bottom of the tank so that after 1 hour it has lost 20 litres and after 2 hours has lost 40 litres.
  - (c) A screen printing firm is interested in its total costs for printing t-shirts for its customers. The fixed costs for the firm are \$8.50 for setting up the machine and the variable costs are \$140.00 to print every 20 t-shirts.
  - (d) A driver is interested in how far he is from his holiday house at a particular time on his journey. He starts his journey at his home, 540 km from his destination and then travels 90 km every hour.
2. Choose an appropriate scale to graph the linear functions in question 1.

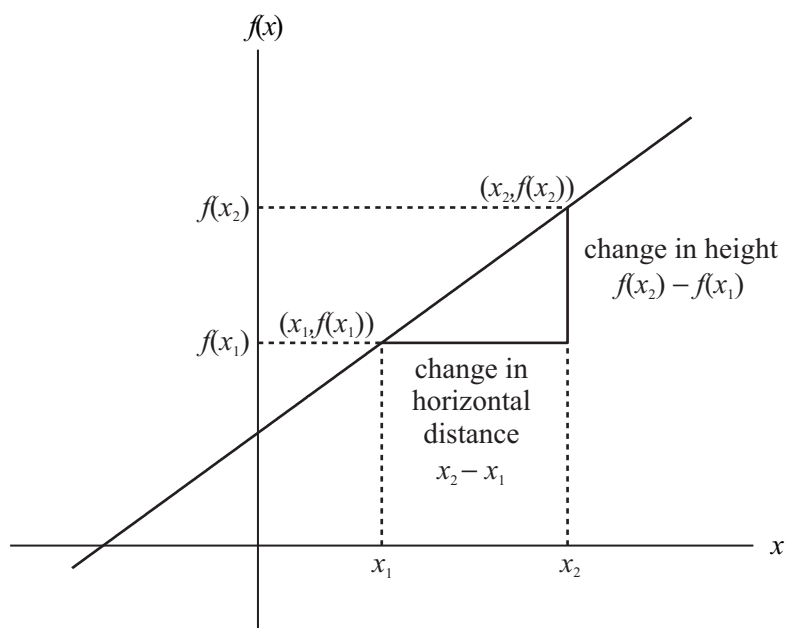
### 4.2.1 Rate of change of a linear function

Previously we have defined a linear function to be a graph of a straight line or a function in which the variables are only raised to a single power. But what does that mean? In both of these definitions what we are really saying is that no matter where we are in the graph, one variable is changing at the same rate as the other variable. The quantity that we use to measure how one variable changes with respect to another is the gradient or slope of the straight line. It is sometimes also referred to as the rate of change of the function.

Recall that the gradient puts a value on the steepness of a straight line by comparing the change in height with the change in horizontal distance.

$$\begin{aligned}\text{gradient} &= \frac{\text{change in height}}{\text{change in horizontal distance}} \\ &= \frac{\text{rise}}{\text{run}}\end{aligned}$$

Let's write this in function notation. If we have the linear function  $y = f(x)$ , which has two points on it called  $x_1$  and  $x_2$ .



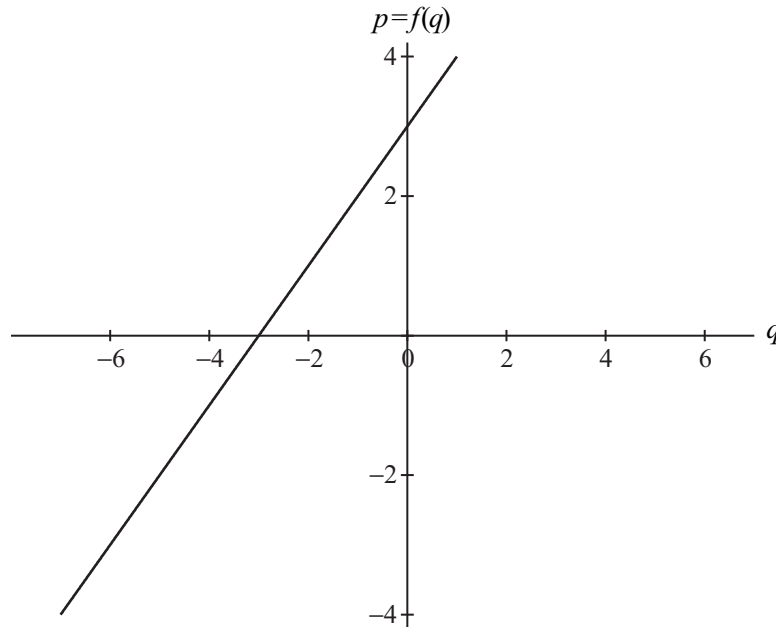
So the gradient (called  $m$ ) in function notation is written,

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

This means that if we know any two points on a straight line graph we can use them to calculate the gradient of the function.

**Example**

A function which relates two variables,  $p$  and  $q$ , where  $p$  is dependent on  $q$  is  $f(q) = p = q + 3$ . If  $f(-1) = 2$  and  $f(3) = 6$ , determine the gradient of the function.



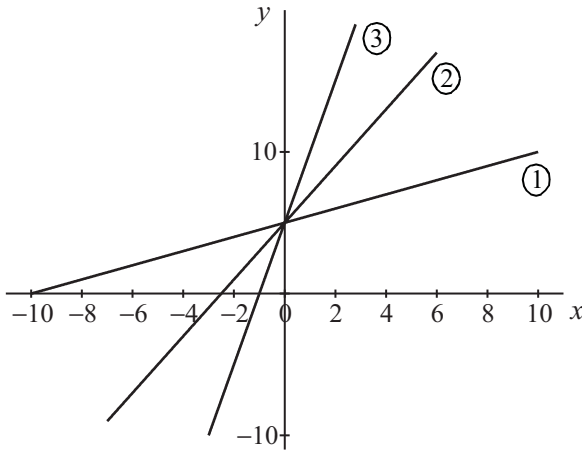
$$m = \frac{f(q_2) - f(q_1)}{q_2 - q_1} = \frac{6 - 2}{3 - (-1)} = \frac{4}{4} = 1$$

Gradient is 1.

**Activity 4.8**

- Find the gradient of a line which passes through the points with coordinates
  - (2, -1), (12, 5)
  - (2, 5), (4, 2)
- A function relates two variables,  $a$  and  $b$ , where  $b$  is dependent on  $a$ . If  $f(1) = 6$  and  $f(2) = -4$ , determine the gradient of the function.
- A function relates two variables,  $s$  and  $t$ , where  $t$  is dependent on  $s$ . If  $g(-2) = -3$  and  $g(4) = 5$ , determine the gradient of the function.
- Find the gradient of a line if its  $x$ - and  $y$ -intercepts are given respectively.
  - 3 and 2
  - $\frac{1}{2}$  and 2

You will have seen from the above activity that gradients can have a range of values. In all instances however the gradient will be the same at all parts of the graph.

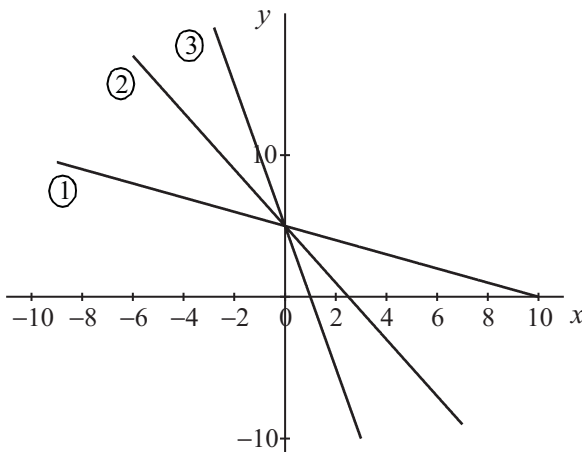


In all the graphs as  $x$  increases the value of the function ( $f(x)$ ) increases.

- Line 1: gradient is 0.5
- Line 2: gradient is 2
- Line 3: gradient is 5

The rates of change of  $y$  with respect to  $x$  are all constants.

So in line 1 the values of the function are increasing more slowly than the values of the function in line 3.

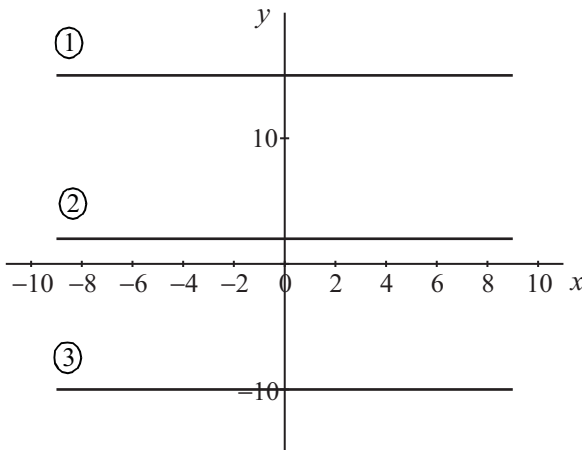


In all the graphs as  $x$  increases the value of the function ( $f(x)$ ) decreases.

- Line 1: gradient is  $-0.5$
- Line 2: gradient is  $-2$
- Line 3: gradient is  $-5$

The rates of change of  $y$  with respect to  $x$  are all constants.

So in line 1 the values of the function are decreasing more slowly than the values of the function in line 3.



In all the graphs as  $x$  increases the value of the function ( $f(x)$ ) remains constant

- Line 1: gradient is 0
- Line 2: gradient is 0
- Line 3: gradient is 0

So the rates of change of  $y$  with respect to  $x$  in all three functions is zero.

A function is **increasing** if the values of  $f(x)$  increase as  $x$  increases. The gradient will be positive.

A function is **decreasing** if the values of  $f(x)$  decrease as  $x$  increases. The gradient will be negative.

A function is **constant** if the values of  $f(x)$  remain constant as  $x$  increases or decreases. The gradient will be zero.

Recall that we can read off the gradient from its equation if it is written in the slope intercept form.

$y = mx + b$ , where  $m$  is the slope and  $b$  is the intercept on the vertical axis.

We can now revise our definition of a linear function to one in which the rate of increase or decrease is constant.

### Example

A family regularly commutes between Brisbane and Toowoomba in their Toyota Camry station wagon.  $p$  is the amount of petrol in the tank (in litres) after a number of trips and  $d$  represents the distance travelled in kilometres. If the equation representing this relationship is  $p = 70 - 0.127d$ , find the gradient and the intercepts on the vertical and horizontal axes. Explain what these quantities represent.

Using the equation  $p = -0.127d + 70$ , which is in the slope intercept form, we know that the slope is  $-0.127$  and the vertical intercept is 70. To find the horizontal intercept we have to put  $p = 0$  to get

$$0 = -0.127d + 70$$

$$d = \frac{-70}{-0.127}$$

$$\approx 551$$

The gradient is negative so this means that as the number of kilometres travelled increases the amount of petrol in the tank decreases. The value of the gradient indicates that the petrol is decreasing at a rate of 0.127 litres per kilometre or 12.7 litres per 100 kilometres (standard way in most motoring magazines).

The vertical intercept represents the original amount of petrol in the tank, 70 litres.

The horizontal intercept represents the total number of kilometres that the car could travel on one tank of petrol, 551 km.

## Activity 4.9

1. For the questions from activity 4.7 question 1 explain the meaning of the gradient, the vertical intercept and the horizontal intercept.
2. A taxi driver expects that the value of his car will decrease at a linear rate over time. When the car was first purchased it cost \$28 000. He expects that after three years, when he intends to sell his car, it will be worth \$17 500.
  - (a) For the function that represents the relationship between car value in dollars and time in years, determine the slope using the two point formula.
  - (b) Write the equation for this relationship and explain what each of the quantities mean.

### 4.2.2 The inverse – undoing a function

Have you ever wanted to reverse a function for example calculate a Fahrenheit temperature from the Centigrade temperature instead of vice versa. We have done something like this when we changed the subject of a formula in module 3. Finding the inverse function is another perspective on this issue. Let's consider the relationship between Centigrade and Fahrenheit temperatures. We could represent some of the temperatures by a set of ordered pairs.

Centigrade (C)	Fahrenheit (F)
0	32
10	50
20	68
30	86
40	104
50	122
60	140
70	158
80	176
90	194
100	212

If Fahrenheit is a function ( $f$ ) of Centigrade, for the first point we might say  $f(0) = 32$  or for the second point  $f(10) = 50$ . If we wanted to go the other way and write centigrade as a function ( $g$ ) of Fahrenheit we would write the first point as  $g(32) = 0$  and the second point as  $g(50) = 10$ .

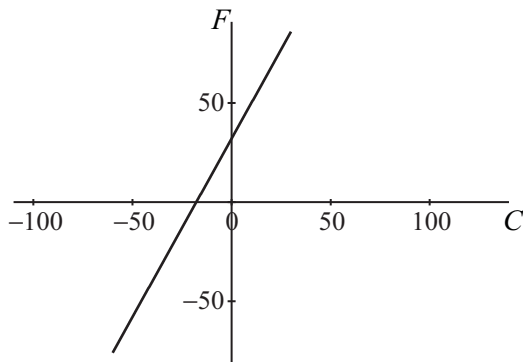
Now because the second function has reversed or 'undone' the process involved in the first function mathematicians would not just call it  $g$  but give it a special name – **the inverse function**, call it  $f^{-1}$ .

$f^{-1}$  is a special notation and is not the same as the index notation  $a^{-1} = \frac{1}{a}$  which was introduced in module 3.

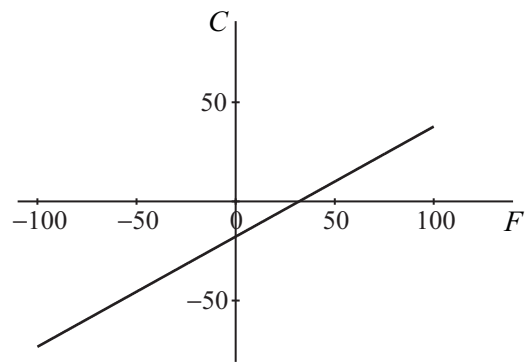
So our set of points would now look some thing like this

<b>Centigrade to Fahrenheit</b>	<b>Fahrenheit to Centigrade</b>
$f(0) = 32$	$f^{-1}(32) = 0$
$f(10) = 50$	$f^{-1}(50) = 10$
$f(20) = 68$	$f^{-1}(68) = 20$
$f(30) = 86$	$f^{-1}(86) = 30$
⋮	⋮
⋮	⋮
⋮	⋮
$f(100) = 212$	$f^{-1}(212) = 100$
<b>The Original Function</b>	<b>The Inverse Function</b>

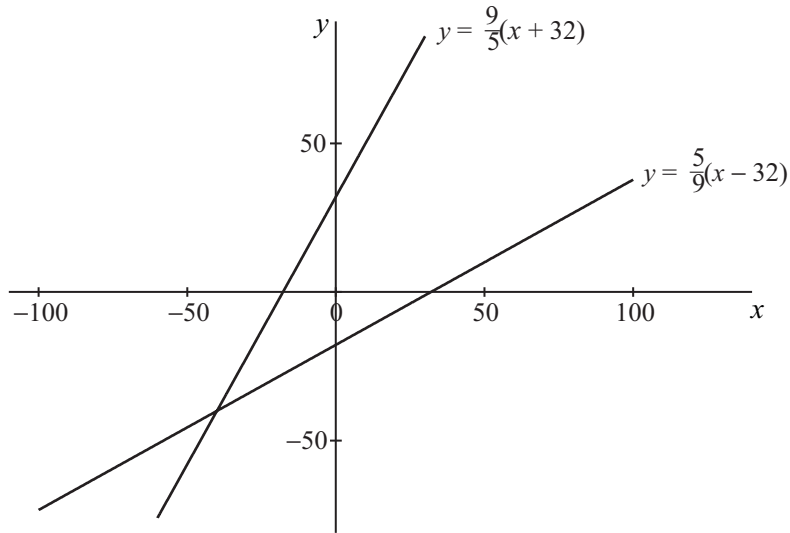
The two functions are different but have the same meaning looked at from different perspectives. If we graphed them they would look like this.



Graph when Centigrade is the independent variable and Fahrenheit the dependent variable.



Graph when Fahrenheit is the independent variable and Centigrade is the dependent variable.



When both graphs are put on the same axes we get a picture like this.

We can continue this undoing process to find the formula for the Centigrade/Fahrenheit relationship.

$F = \frac{9}{5}C + 32$ , where  $F$  was the temperature in Fahrenheit and  $C$  the temperature in Centigrade.

If we want to undo the process we have to subtract 32 then multiply by  $\frac{5}{9}$ .

So if the original function is

$$f(C) = \frac{9}{5}C + 32 = F$$

then its inverse will be

$$f^{-1}(F) = \frac{5}{9}(F - 32) = C$$

Let's have a closer look at inverses of linear functions. Note that it is not always possible to find the inverse function of functions which are not linear.

### Graphs of linear functions and their inverses.

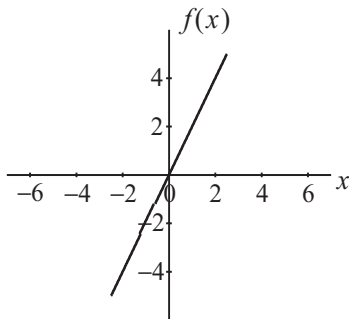
We showed the graphs of the two Centigrade/Fahrenheit functions previously, now let's look at this process in more detail using a simpler function. Consider the linear function,  $f(x) = 2x$  to find the inverse we would have to undo the multiplication by two by dividing by two. The

inverse would be  $f^{-1}(x) = \frac{1}{2}x$ . If we represented this as a set of points we would get the following:

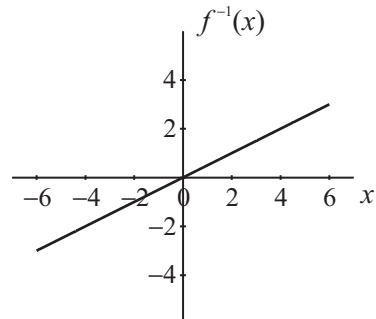
$x$	$f(x)$
1	2
2	4
3	6
4	8

$x$	$f^{-1}(x)$
2	1
4	2
6	3
8	4

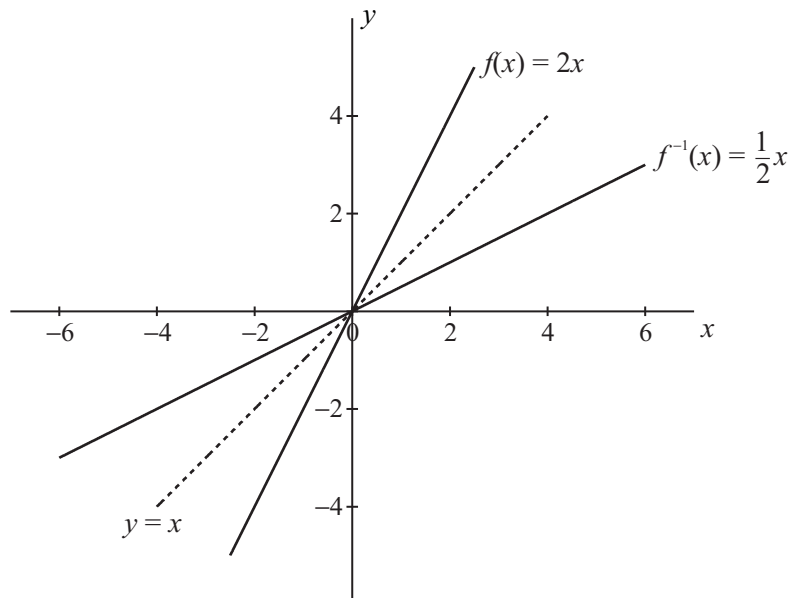
If we were to graph the original function and its inverse of the same set of axes we would get the following.



Graph of  $f(x) = 2x$



Graph of  $f^{-1}(x) = \frac{1}{2}x$



Graph of  $f(x) = 2x$  and  $f^{-1}(x) = \frac{1}{2}x$  on the same set of axes.

Notice that the original function and its inverse appear to be reflections about the line  $y = x$ . If you are not sure about this, a good experiment is to draw all of the functions on a piece of square graph paper. Then draw in the line  $y = x$  and fold the piece of paper along this line. You will notice that the two functions will line up, because they are mirror images of each other in the line  $y = x$ . The  $x$  and  $y$  coordinates are interchanged or reversed.

Further, just as the coordinates are reversed to determine the inverse function, so the domain and range will be swapped over. The domain of the original function will become the range of the inverse, while the range of the original will become the domain of the inverse.

The important things to remember about inverses are:

- If an inverse of the original function  $f(x)$  exists we call it  $f^{-1}(x)$
- If an inverse function exists then it is defined by the expression  $f^{-1}(t) = d$  meaning  $f(d) = t$
- An original function and its inverse are reflections in the line  $y = x$
- The domain of the original function becomes the range of the inverse function
- The range of the original function becomes the domain of the inverse function

### Example

The manufacturing costs of producing  $x$  products is given by the function  $f(x) = 100 + 3x$ . The government has placed quotas on production and will allow the company to produce only 1 000 of each item. What is the formula for the inverse function and how would you interpret it?

If the original function is  $f(x) = 100 + 3x$ , and valid only for the domain of  $0 \leq x \leq 1000$ . The restriction on the domain will mean that the range will also be restricted.

The range or value of the function at  $x = 1000$  is

$$f(1000) = 100 + 3 \times 1000 = 3100,$$

and at  $x = 0$  is  $f(0) = 100 + 3 \times 0 = 100$   $\therefore$  range is  $100 \leq f(x) \leq 3100$

This function will allow you to calculate the cost of producing a given number of products.

To get the inverse we must reverse the processes by subtracting 100 and then dividing by three to get  $f^{-1}(x) = \frac{x-100}{3}$ .

The domain will also be affected with the domain of the inverse function being now equivalent to the range of the original function. Domain of the inverse function is  $100 \leq x \leq 3100$ .

The range or the value of the inverse function at  $x = 3100$  is

$$f^{-1}(3100) = \frac{3100 - 100}{3} = 1000$$

The range of the inverse function is the domain of the original function.

The inverse function allows you to calculate the number of products produced at a given cost.

**Example**

The price of some goods in dollars,  $P$ , is a function of number of items sold,  $n$ , so that  $P = f(n)$ . Explain what is meant by  $f(30)$  and  $f^{-1}(25)$  in terms of prices and quantities sold.

$f(30)$  means that if the number of items sold is 30 then the price of goods will be  $f(30)$  – the value of the function at 30.

$f^{-1}(25)$  means that if the price of the goods is \$25 then the number of items will be  $f^{-1}(25)$  – the value of the inverse function at 25.

**Activity 4.10**

1. A function is given as ordered pairs. Write the inverse of the function and state its domain and range. State clearly if the inverse is not a function.
  - (a)  $(-7,4), (-2,1), (4,5), (5,7)$
  - (b)  $(-1,2), (0,-4), (3,2), (4,5)$
2. Find the inverse relation of each of the following functions. State whether or not each inverse is a function. If the inverse is a function state the domain.
  - (a)  $g(x) = 2x + 1$
  - (b)  $h(x) = 3 - 2x$
  - (c)  $f(x) = \frac{x}{2} - 3$
  - (d)  $g(x) = 2(x - 3)$

**4.2.3 When two linear functions meet**

In module 3 we looked at problems associated with equations with two unknown variables. Let's revisit that section now and see the graphical approach to the solution. The original problem in module 3 was

*A city bakery sold 1 500 bread rolls on Sunday, with sales receipts of \$1 460. Plain rolls sold for 90 cents each while gourmet rolls sold for \$1.45 each. How many of each type of roll were sold?*

Previously we solved the problem by generating two equations and then solving them algebraically.

The equations were  $P + G = 1500$  and  $0.9P + 1.45G = 1460$ , the final solution was number of gourmet rolls ( $G$ ) was 200 and number of plain rolls ( $P$ ) was 1300.

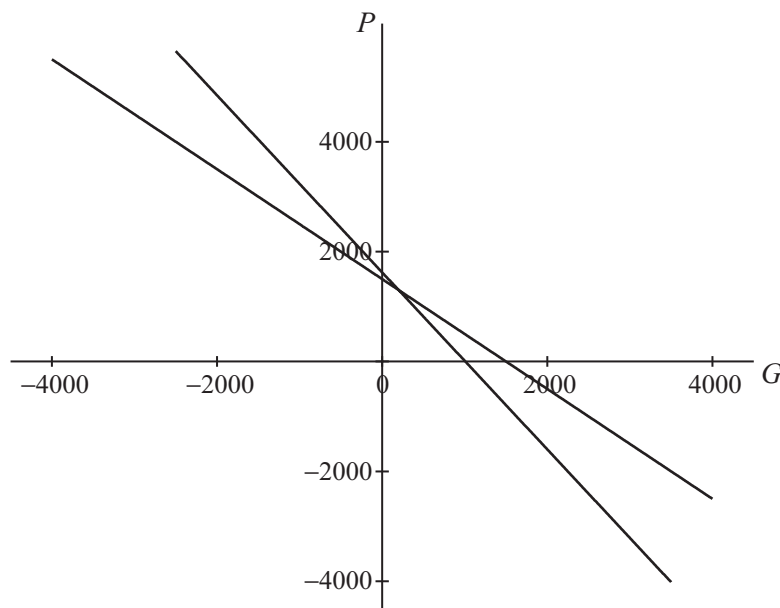
Each of the equations can be rewritten as a linear function and drawn on a set of coordinate axes. Rearranging the equations we will get:

$$P + G = 1500 \quad \text{will become} \quad P = 1500 - G$$

$$0.9P + 1.45G = 1460 \quad \text{will become} \quad P = \frac{1460 - 1.45G}{0.9}$$

Note we could have just as easily made  $G$  the subject of the formula and graphed it as the dependent variable.

Graphing the two equations we will get



So the point of intersection of two straight lines is the same as the solution to two simultaneous equations with two unknowns. The graphical approach to solving simultaneous equations like the one above is often the first step in solving more complex problems in which the algebra is time consuming or impossible.

### Example

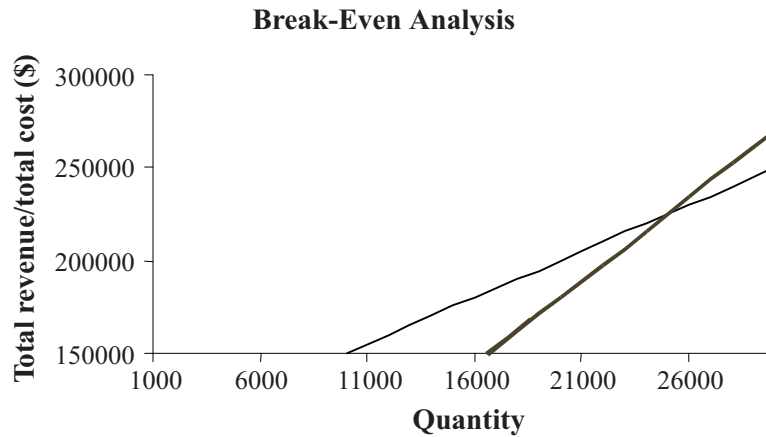
Economics uses the graphical and algebraic approach in a type of analysis called break-even analysis.

The break-even point for a business is the point at which total revenue equals total costs. To determine the break-even point for a business a company needs to consider two functions. For a specific company this might be:

The Cost function  $c(x) = 100000 + 5x$

The Revenue function  $r(x) = 9x$ ,

Where  $x$  is the quantity of goods. Note in break-even analysis we are interested in only finding value of the  $x$ -coordinate i.e. the quantity of goods that will result in the equality of the total revenue and total costs.



The  $x$ -coordinate of the point of intersection or the break-even value of the quantity from the graph is 25 000 units.

To test this algebraically substitute 25 000 into both equations and it should give the same value for cost and revenue.

$$c(x) = 100000 + 5x, \quad c(25000) = 100000 + 5 \times 25000 = 225000$$

$$r(x) = 9x, \quad r(25000) = 9 \times 25000 = 225000$$

So the break-even value of the quantity is 25 000 units.

### Activity 4.11

- Graph the pair of linear functions for the given domain. Write as an ordered pair  $(x, y)$ , the graphical solution of the simultaneous equations. Prove your solution is correct algebraically.
  - $$y = x + 1$$

$$y = 2x - 3$$
  - $$y - x = -6$$

$$y - 2x = -8$$
  - $$-4.5x + 3y = 18$$

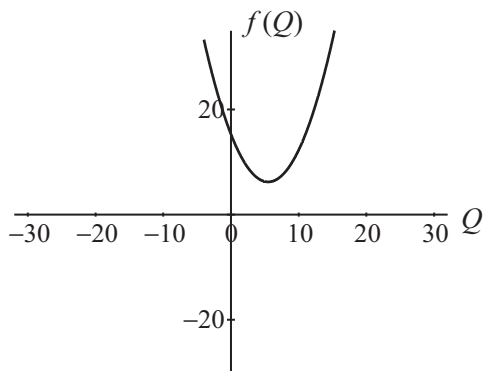
$$3x - 2y = 12$$
  - $$x + y = 6$$

$$2x + y = 9$$

2. (a) A local rent-a-car firm charges a fixed daily rate plus a fixed amount per kilometre. A total charge for the first week was \$830 with 1250 km on the clock. The 2nd week's bill was for \$710 with another 850 on the clock. Determine graphically the rental charge and the cost per kilometre.
- (b) After analysis of a company's cost structure the accountant found that the fixed cost was \$550 000 per annum with a constant variable cost of \$9.50 per unit. If the product is selling for \$15 per unit, graphically determine the break even point.

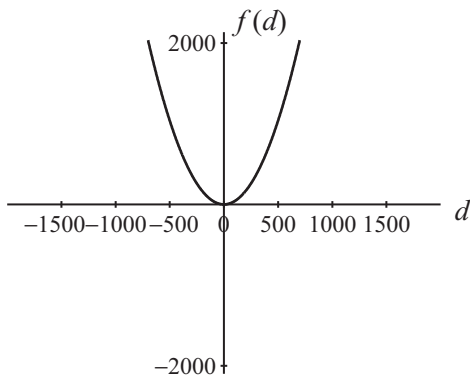
## 4.3 The quadratic function

Functions come in all shapes and forms. Some are easy to represent and interpret others are more difficult. Linear functions represent some of the easiest functions to understand. Other functions that are not linear, not too complex and occur frequently in business and science are **quadratics**. Let's return to the examples from section 3.3.3 in module 3 and look at quadratic equations from a 'function' perspective. All of the equations presented there are in fact functions... 'For every value of the independent (input) variable there is only one value of the dependent (output) variable'. The graph of a quadratic function is called a parabola (pronounced pah-rab-o-la). Let's quickly look and see what shape they are before we go into details. If you have a graphing package you may want to sketch them yourself first.



In economics, the average cost equation for a particular production plant is

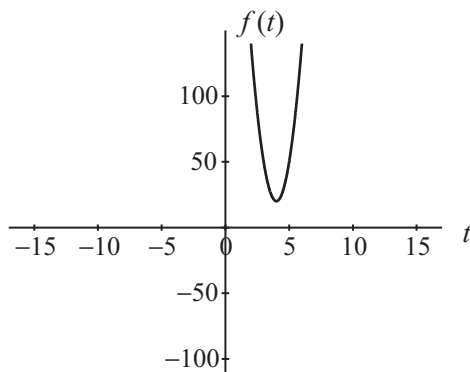
$C = f(Q) = 0.3Q^2 - 3.3Q + 15.3$  where  $C$  is the average cost in dollars of producing each unit and  $Q$  is the quantity produced.



In engineering, the positioning of a suspension bridge cable can be approximated by the

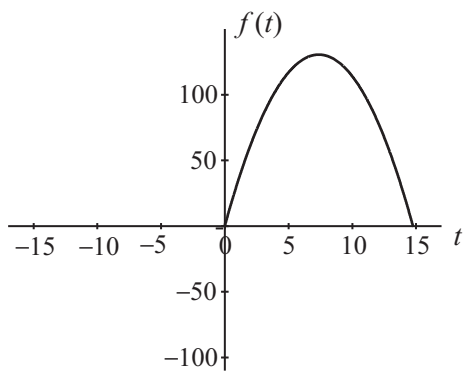
quadratic equation  $H = f(d) = \frac{3d^2}{722}$  where  $H$ ,

is height above the road in metres and  $d$  is the distance from the centre of the cable in metres.



In applied biology, the relationship

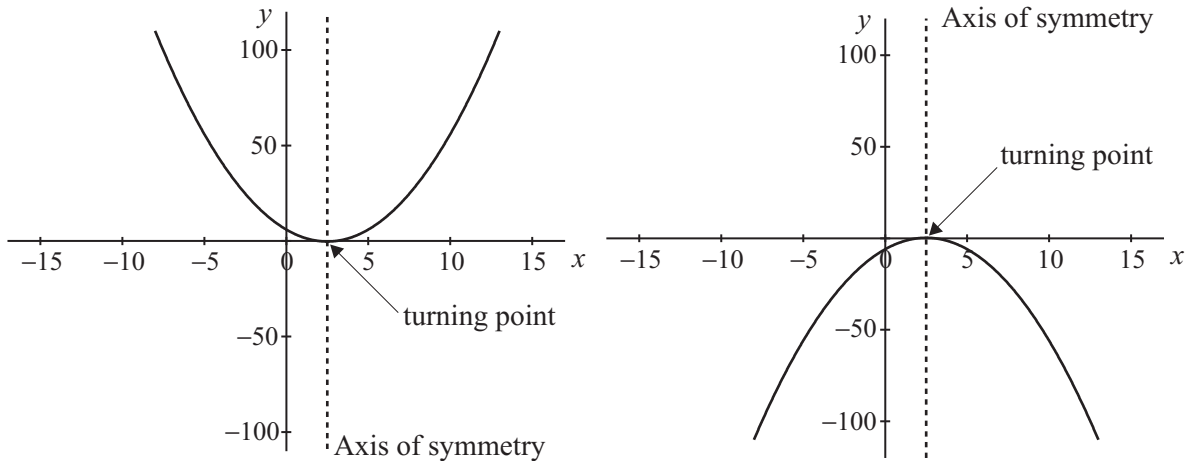
$C = f(t) = 30t^2 - 240t + 500$  is used to predict the number of bacteria ( $C$ ) in a swimming pool, measured as count per  $\text{cm}^3$ , given the time in days from treatment.



In physics, the height of a small object above the ground can be predicted from the function.

$H = f(t) = -4.9t^2 + 4.2t$ , where  $t$  is the amount of time in the air and  $H$  is the height above the ground.

Recall then that a quadratic function has a characteristic ‘satellite dish’ shape. It can be folded in half at its vertex which is called the turning point and is thus said to be symmetrical. The straight line which bisects the parabola through its turning point is called the axis of symmetry.



The general equation of this **upward opening** quadratic function is  $f(x) = ax^2 + bx + c$ , where  $a$  is positive.

The general equation of this **downward opening** quadratic function is  $f(x) = ax^2 + bx + c$ , where  $a$  is negative.

### 4.3.1 Sketching parabolas

Previously you will have drawn quadratic functions by plotting points from a table of values. Although this is a useful way to get an image of the curve it is often difficult to get specific details like the turning point or the points of intersection with the axes. Let's now follow the procedure we developed for linear functions to help us quickly sketch an accurate parabola. This technique will only work if first you recognize that the function you have will have a parabolic shape. Once you recognize this then you can continue by first finding the coordinates of the intercepts then the coordinates of the turning point. Return to section 3.3.3 if you are unsure how to pick a quadratic expression from other expressions.

#### Where does the parabola cut the axes?

Recall that on a standard Cartesian plane, where  $x$  is the independent variable and  $y$  the dependent variable, along the  $y$ -axis all values of  $x$  are  $x=0$ , while on the  $x$ -axis all values of  $y$  are  $y=0$ .

So if we want to determine the value of the  $y$ -intercept we would first put  $x=0$  and if we wanted to find the  $x$ -intercept we would put  $y=0$ . Let's look at an example from before.

*A signwriting company wants to manufacture a series of signs which their product suppliers (timber and paint) say must have a perimeter of 10 metres and an area of 6 square metres. What length and width should they make these signs?*

In this problem for the signwriting company we had to solve the equation  $l^2 - 5l + 6 = 0$  where  $l$  represents the length of the sign. Let's write the quadratic part of the equation as a function and graph it. This means that we would graph the function  $f(l) = l^2 - 5l + 6$ .

In this function  $l$  is the independent variable and would be placed on the horizontal axis and the value of the function  $f(l)$  on the vertical axis.

Just as we did for linear functions, to find the vertical intercept for a quadratic function we would put  $l = 0$ , so  $f(0) = 0^2 - 5 \times 0 + 6 = 6$ . The vertical intercept is six. Note that 6 is the constant term in the quadratic function. It will always be the case that the value of the constant term,  $c$ , in the general quadratic function  $f(x) = ax^2 + bx + c$  is the value of the vertical intercept.

To find the horizontal intercept we would put  $f(l) = 0$ , producing  $l^2 - 5l + 6 = 0$ . This is of course the quadratic equation that we spent so much time on in module 3. Quickly refresh that topic before moving on. From this point we assume that you are familiar with factorization and the use of the quadratic formula.

To solve  $l^2 - 5l + 6 = 0$  we could factorize the expression on the left hand side to get  $(l - 3)(l - 2)$  and then solve the equation to find the horizontal intercepts of  $l = 3$  and  $l = 2$ . If we had used the quadratic formula we would have evaluated:

$$l = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 1 \times 6}}{2 \times 1} = 3 \text{ or } 2.$$

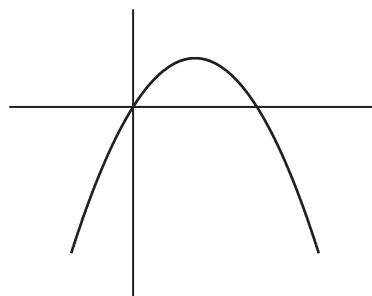
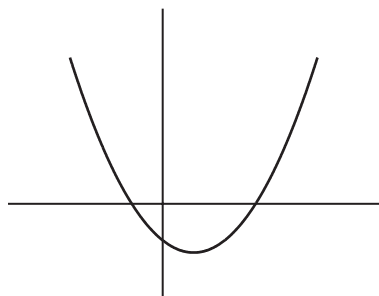
Although we wouldn't normally use the quadratic formula to find the solutions to a quadratic equation that is easy to factorize, it is useful because part of it ( $b^2 - 4ac$ , commonly called the **discriminant**) will tell us the number of solutions we would expect. This number of solutions is directly related to the number of times the curve will intersect the horizontal axis. Remember the table from module 3? Let's extend it to include this concept.

Quadratic equation	Value of $b^2 - 4ac$	Number of solutions	Number of horizontal intercepts
$x^2 + x - 3 = 0$	13	2	2
$3q^2 - 2q - 4 = 0$	52	2	2
$x^2 - 8x + 16 = 0$	0	1	1
$t^2 + 2t + 3 = 0$	-8	0	0

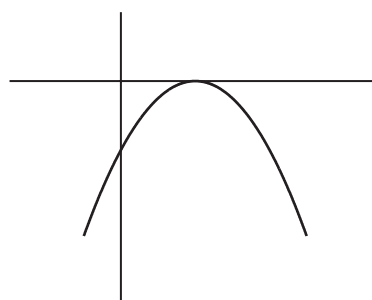
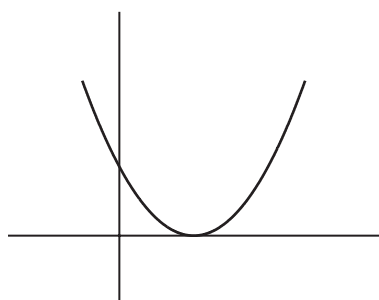
So in terms of intersection with the horizontal axis quadratic functions can be divided into three groups depending on the value of the discriminant,  $b^2 - 4ac$ .

**If  $b^2 - 4ac$  is positive ( $b^2 - 4ac > 0$ ) then the equation will have two solutions and will cut the horizontal axis twice.**

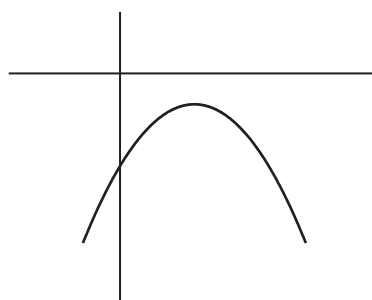
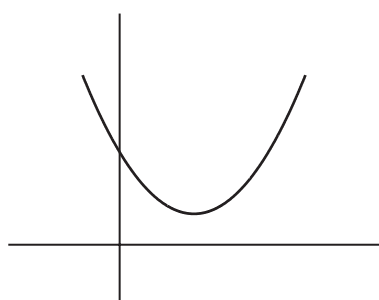
or



**If  $b^2 - 4ac$  is zero ( $b^2 - 4ac = 0$ ) then the equation will have one solution and will only touch the horizontal axis at one point.**

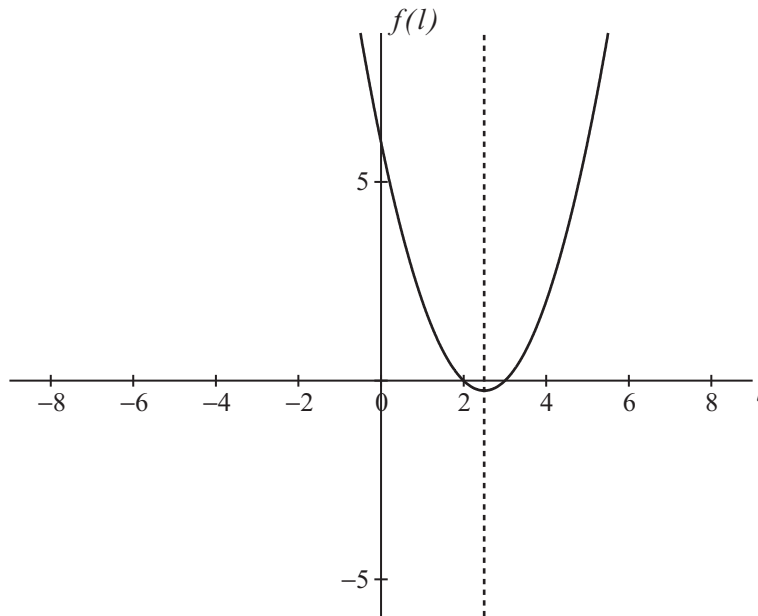


**If  $b^2 - 4ac$  is negative ( $b^2 - 4ac < 0$ ) then the equation has no solutions, cannot be solved and will not ever intersect the horizontal axis.**



### Where is the turning point?

If we return to the signwriting function  $f(l) = l^2 - 5l + 6$ , we know it is a quadratic function and so must be symmetrical about the axis of symmetry. We also know it will intersect the horizontal axis at 2 and 3. Thus the value of  $l$  midway between 2 and 3 must be the  $l$  coordinate of the turning point. Thus the turning point will be  $(2.5, -0.25)$  (we get the 2nd coordinate by evaluating  $f(2.5)$ ).



The above system works well when we have a quadratic function that touches or cuts the horizontal axis but proves difficult when the curve lies above or below the axis. In this case it is useful to know that the first coordinate of the turning point is  $-\frac{b}{2a}$  (note we will not derive the formula at this stage and will return to it in the next level of Tertiary Preparation Mathematics).

### Example

Sketch  $f(x) = -x^2 - x + 2$  indicating clearly the intercepts and the coordinates of the turning point.

$f(x) = -x^2 - x + 2$  is a quadratic function with a negative coefficient of  $x^2$  so will be a downward opening shaped graph.  $\cap$

To find the vertical intercept evaluate  $f(0)$ ,

$$f(0) = -0^2 - 0 + 2 = 2, \text{ the intercept is } 2.$$

To find the horizontal intercept solve  $f(x) = 0$

$$f(x) = 0$$

$$-x^2 - x + 2 = 0$$

$$-(x^2 + x - 2) = 0$$

$$x^2 + x - 2 = 0$$

$$(x + 2)(x - 1) = 0$$

$$x = -2 \text{ or } x = 1$$

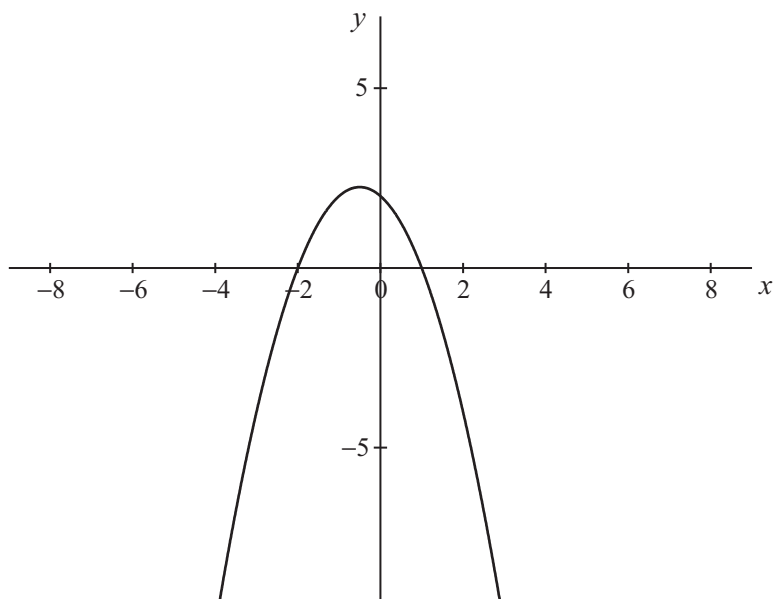
Note that  $b^2 - 4ac$  is positive so we would expect the function to have two horizontal intercepts.

To find the turning point evaluate  $x = -\frac{b}{2a} = -\frac{-1}{2 \times -1} = -\frac{1}{2}$

The value of the function at this point is  $f(-\frac{1}{2}) = -(-\frac{1}{2})^2 - (-\frac{1}{2}) + 2 = 2\frac{1}{4}$

Coordinates of the turning point are  $(-\frac{1}{2}, 2\frac{1}{4})$

We can now use these three points to get a quick sketch of the parabola.



**Example**

Sketch the graph of  $f(x) = x^2 - 2x + 3$ . Use your graph to determine the domain and range of the function.

$f(x) = x^2 - 2x + 3$  is a quadratic function with a positive coefficient of  $x^2$  so will be an upward opening shaped graph.  $\cup$

To find the vertical intercept evaluate  $f(0)$ ,

$$f(0) = 0^2 - 2 \times 0 + 3 = 3, \text{ the intercept is } 3.$$

To find the horizontal intercept solve  $f(x) = 0$

$$\begin{array}{ll} f(x) = 0 & \text{Note that } b^2 - 4ac \text{ is negative so we would expect the function to} \\ x^2 - 2x + 3 = 0 & \text{have no horizontal intercepts.} \end{array}$$

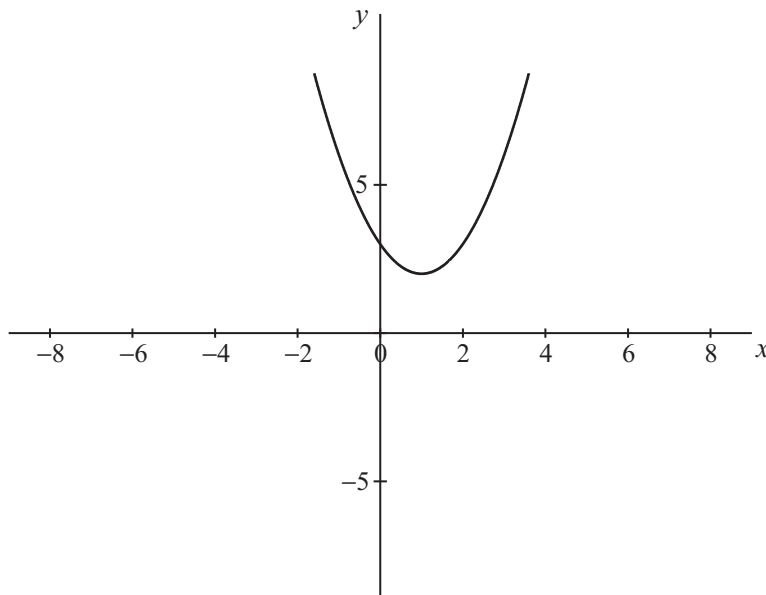
There are no solutions to this equation.

$$\text{To find the turning point evaluate } x = -\frac{b}{2a} = -\frac{-2}{2 \times 1} = 1$$

$$\text{The value of the function at this point is } f(1) = (1)^2 - 2 \times (1) + 3 = 2$$

Coordinates of the turning point are  $(1, 2)$

Put these two points together to sketch the curve.



From the graph we can see that the extent of the  $x$  values is infinite so the domain is all real values of  $x$ . The range however, is restricted in that it has a minimum value at the turning point. So the range is  $f(x) \geq 2$ .

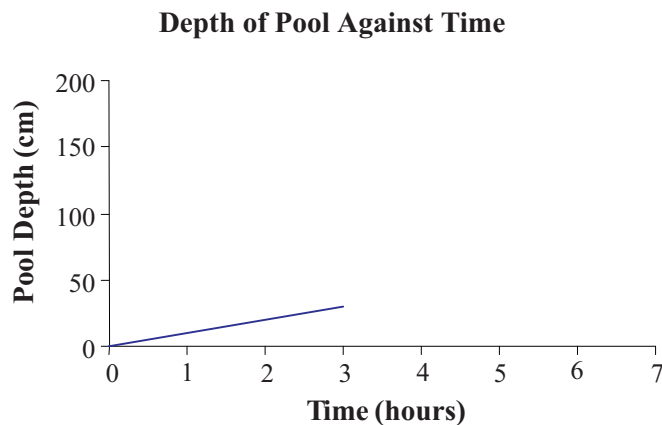
## Activity 4.12

- The following are parabolic curves. Examine the discriminant of each of the following functions to determine how many times (if any) they intersect the  $x$ -axis.
  - $y = x^2 + x - 12$
  - $y = -x^2 - x + 6$
  - $y = (x - 2)^2 + 1$
  - $y + 4 = x^2$
- Sketch the graphs of the above functions and clearly indicate the turning point, the vertical and horizontal intercepts.
- A tug-of-war team can produce a tension on a rope following the function  $F = 290(8t - 0.5t^2 - 1.4)$ , where  $t$  is time in seconds and  $F$  is the tension force. Sketch this graph and estimate when the tension is at its maximum value.

### 4.3.2 Rate of change of a quadratic function

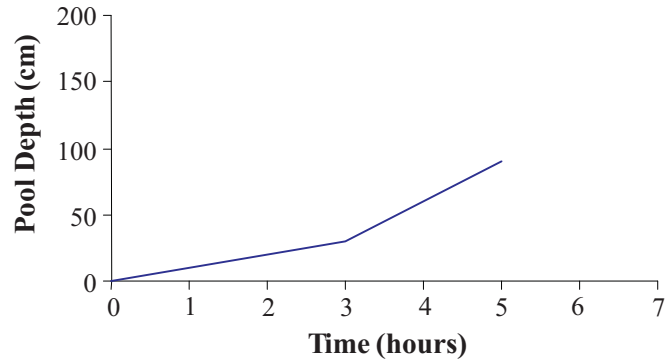
It's easy to determine the gradient or rate of change of a function if it is a linear function because linear functions always have a constant gradient or rate of change. Quadratic functions are not so easy because the rate of change of one variable with respect to the other is always changing. Let's look at part of a parabolic function step by step...you may have seen this before if you have studied the previous Tertiary Preparation Program Mathematics unit.

A straight sided backyard pool starts to be filled at a rate of 10 cm per hour. The graph of that part would be something like this with a gradient or rate of change of 10.



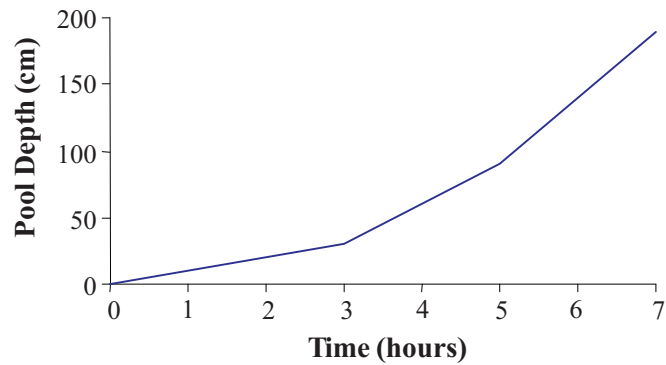
If after 3 hours the rate of filling is increased to 30 cm per hour, the gradient would be 30.

**Depth of Pool Against Time**



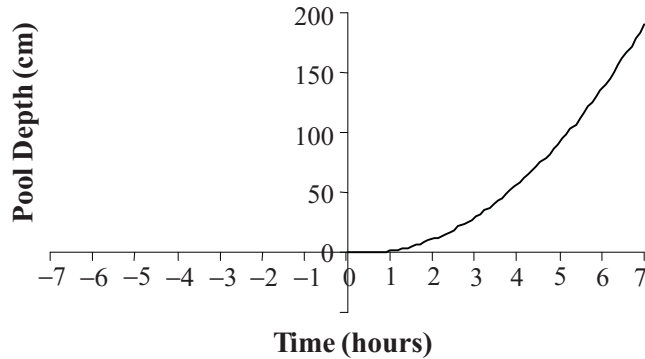
If we increased the rate at which water was added again to 50 cm per hour the gradient would be 50.

**Depth of Pool Against Time**



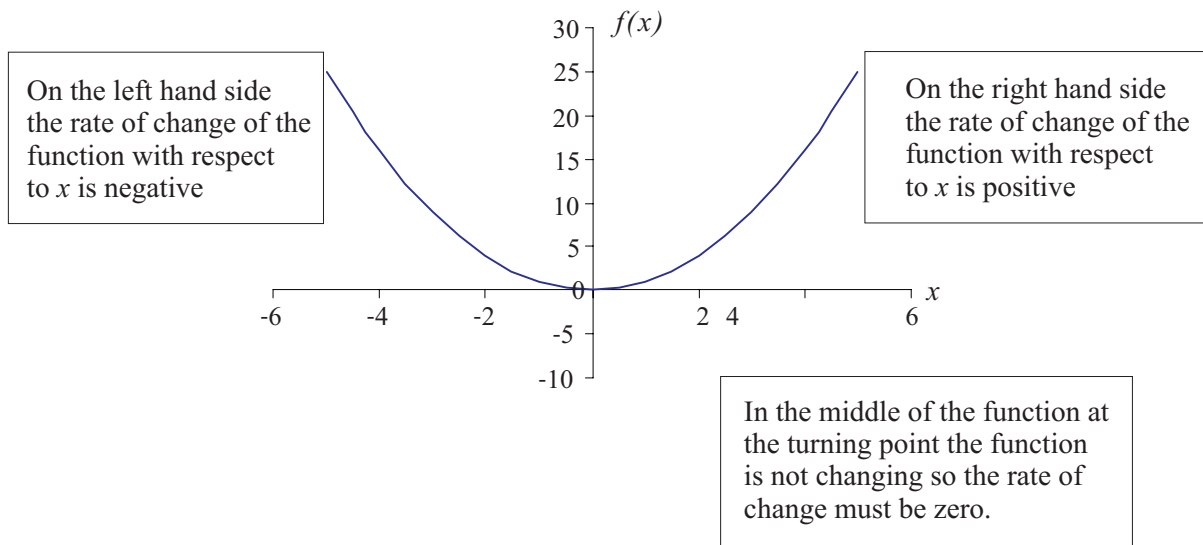
If we put all the small linear functions together in one graph, as above, the graph begins to look more like part of a quadratic function. The depth of water in the pool over time can be approximated by a quadratic function of the form  $f(t) = 4.4t^2 - 3.7t + 0.3$ , with a domain of  $t \geq 0$ . It looks like this.

**Depth of Pool Against Time**



So in this quadratic function the rate of change of depth with respect to time is increasing from 10 to 50 cm per hour.

Let's look at this in the general case of  $f(x) = x^2$



In linear functions we can determine the rate of change of the function by calculating the gradient or slope, with curves this is more difficult because the rate of change is continually changing. But we could approximate the rate of change by calculating the average rate of change between two points on a curve. Recall that in the pool problem above this is exactly what we did, finding the rate of filling changed from 10 between the values  $t = 0$  and  $t = 3$  to 30 between the values  $t = 3$  and  $t = 5$  to 50 between the values  $t = 5$  and  $t = 7$ .

The average rate of change of part of a parabola can be approximated by calculating the gradient of the straight line connecting two points on the part of the parabola of interest.

At this stage we cannot measure the rate of change of a curve any more accurately, but mathematicians have developed a method by which we can determine the rate of change of most functions any point. You might have heard of it...It's called **Differential Calculus**. It is used extensively in science, engineering and economics and will be introduced in the next Tertiary Preparation Mathematics unit.

### Example

Given the function  $f(x) = x^2 - 6x + 8$ , find the values of  $f(5)$ ,  $f(6)$ ,  $f(1)$  and  $f(0)$ . Evaluate

$\frac{f(6) - f(5)}{6 - 5}$  and  $\frac{f(1) - f(0)}{1 - 0}$  and write in your own words what these quantities represent.

$$f(x) = x^2 - 6x + 8$$

$$f(5) = 5^2 - 6 \times 5 + 8 = 3 \quad \text{This represents a point on the function, (5,3)}$$

$$f(6) = 6^2 - 6 \times 6 + 8 = 8 \quad \text{This represents a point on the function, (6,8)}$$

$$f(1) = 1^2 - 6 \times 1 + 8 = 3 \quad \text{This represents a point on the function, (1,3)}$$

$$f(0) = 8 \quad \text{This represents a point on the function, (0,8)}$$

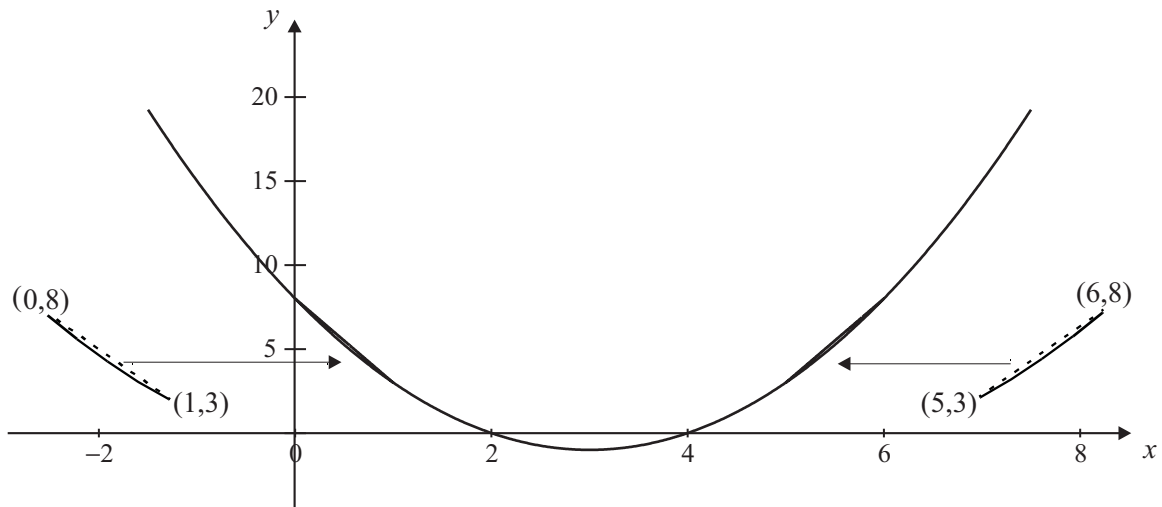
$$\frac{f(6) - f(5)}{6 - 5} = \frac{8 - 3}{1} = 5 \quad \text{This represents the difference between the function values of the two points (6,8) and (5,3) divided by the difference in the } x \text{ values. It is the gradient between (6,8) and (5,3).}$$

$$\frac{f(1) - f(0)}{1 - 0} = \frac{3 - 8}{1} = -5 \quad \text{This represents the difference between the function values of the two points (1,3) and (0,8) divided by the difference in the } x \text{ values. It is the gradient between (1,3) and (0,8).}$$

First sketch a graph of  $f(x) = x^2 - 6x + 8$  and plot the points we are asked to evaluate. If you connect the two adjacent points you will see that they are straight lines. The quantity

$\frac{f(6) - f(5)}{6 - 5}$  is the gradient of that line connecting  $x = 5$  and  $x = 6$  on the parabola. The

quantity  $\frac{f(1) - f(0)}{1 - 0}$  is the gradient of the straight line connecting  $x = 1$  and  $x = 0$ .



**Note:** Slope of the dotted lines above represent the average rate of change between the two points.

The average rate of change of the parabola between  $x = 5$  and  $x = 6$  is 5, the function is increasing.

The average rate of change of the parabola between  $x = 1$  and  $x = 0$  is  $-5$ , the function is decreasing.

At the turning point  $x = 3$  the rate of change would be zero.

### Example

If a stone is thrown and the height of trajectory over time is represented by the function  $f(t) = 25t - 5t^2$  where height is measured in metres and time in seconds. Find the average rate of change of the height with time between 2 and 4 seconds. Explain what this quantity means in physical terms.

Average rate of change can be approximated by calculating the gradient of the straight line between the two points where  $t = 4$  and  $t = 2$ .

$$\begin{aligned} \text{Average rate of change} &= \frac{f(4) - f(2)}{4 - 2} \\ &= \frac{(25 \times 4 - 5 \times 4^2) - (25 \times 2 - 5 \times 2^2)}{2} \\ &= \frac{20 - 30}{2} \\ &= -5 \end{aligned}$$

The function relates height with time so the rate of change would be the change in height over time... a measure of speed. A rate of  $-5$  means that between 2 and 4 seconds the stone is descending at 5 metres per second (on average).

### Activity 4.13

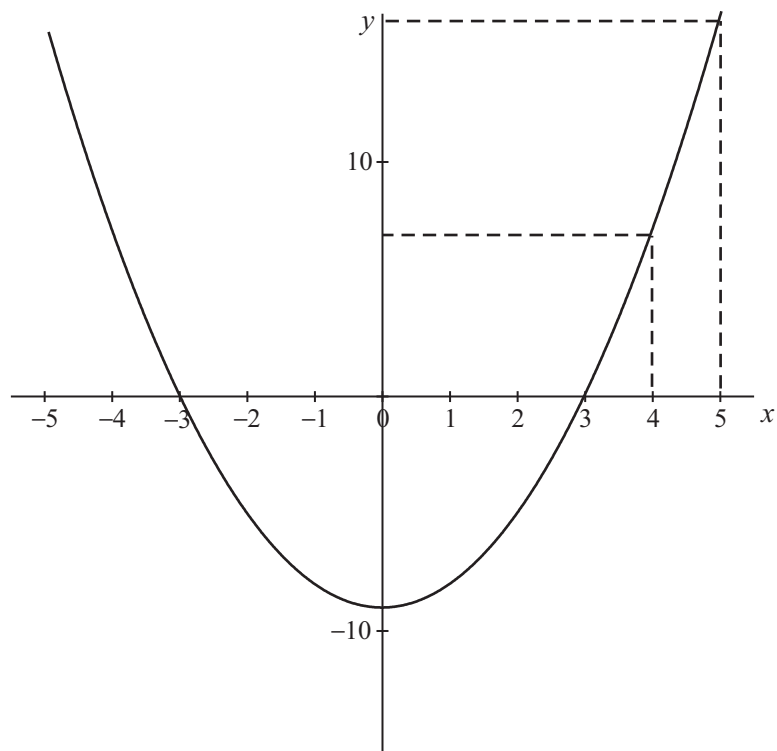
1. (a) Using the graph below find the average rate of change between

$$x = 1 \text{ and } x = 3$$

$$x = 4 \text{ and } x = 5$$

$$x = -1 \text{ and } x = -3$$

$$x = -4 \text{ and } x = -5$$



- (b) Compare the average rate of change at different parts of the parabola.
2. Using the following functions examine how the gradient changes between  $x = 0$  and  $x = 1$  and  $x = 1$  and  $x = 2$ .

(a)  $f(x) = x^2 - x - 6$

(b)  $g(x) = 2x^2 - 3x + 4$

(c)  $h(x) = -x^2 + x + 6$

3. If a teenager throws a piece of fruit into the air the following heights (cm) can be observed over a short period of time (seconds)

$t$ (sec)	0	1	2	3	4	5	6
$h$ (cm)	180	2700	4260	4860	4500	3180	900

- (a) Sketch the graph of the function represented by these data.
- (b) Recalling that velocity is the rate of change of distance with time, determine the average rate of change between each of the points in the table.
- (c) What is the significance of the different values found in (b)?
4. A computer program simulates a mountain 1 km high and 2 km across to a parabolic shape of  $H = -b^2 + 1$ , where  $H$  is the height of the mountain in kilometres and  $b$  the breadth of the base of the mountain in kilometres.
- (a) Sketch a graph of the mountain.
- (b) If steepness is the ratio between height and breadth walked what is the average steepness when you first start to climb the mountain (between  $b = 0.75$  and  $b = 1$  km)?
- (c) What is the steepness near the peak (between  $b = 0$  and  $b = 0.25$ )?
- (d) What is the significance of the sign associated with steepness?

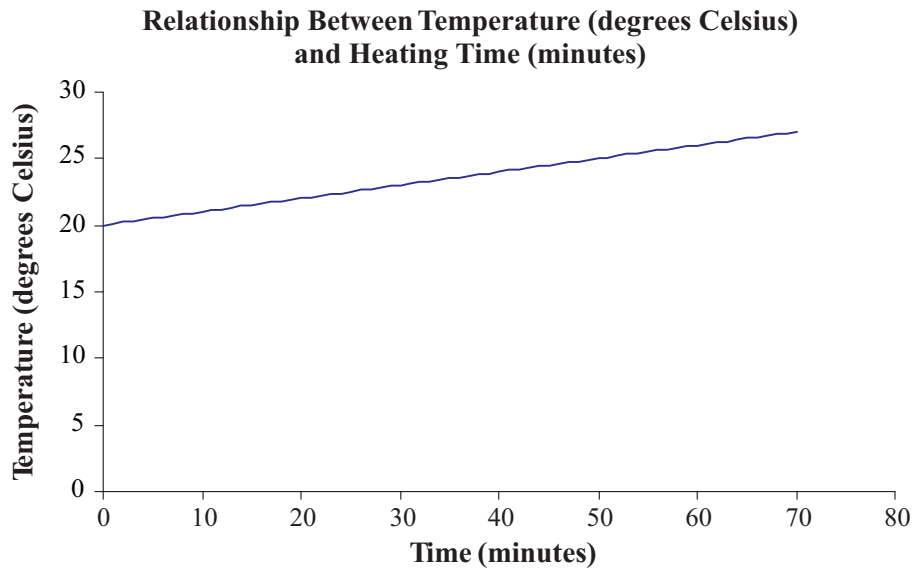
## 4.4 Other functions

This module has concentrated only on simple functions like the linear and quadratic functions. But functions exist in all shapes and forms. Later in the unit you will examine three more types of functions in detail – the exponential, logarithmic and trigonometric functions. In many branches of study functions do not follow simple linear or quadratic forms but are rather combinations of a number of different functions.

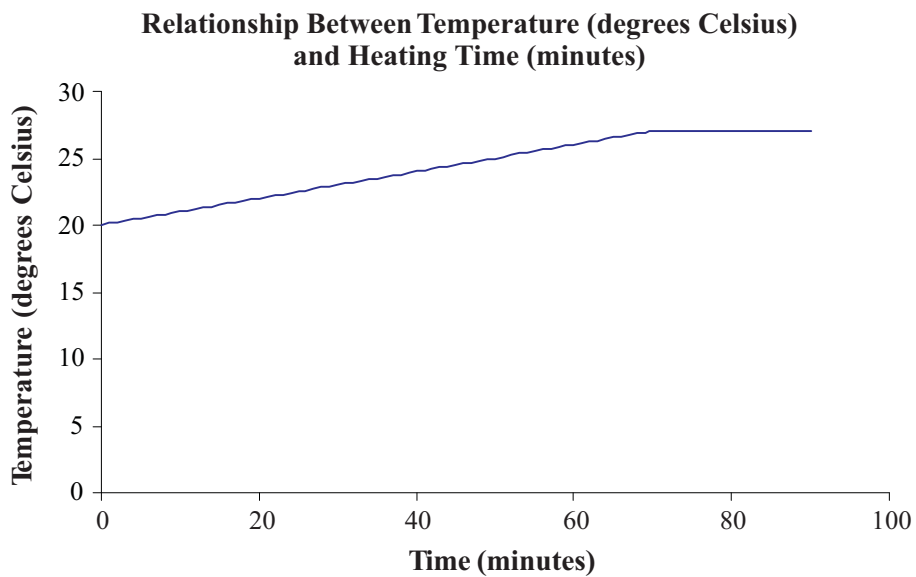
Let's return to the heating swimming pool.

*The temperature in a swimming pool cools down each night to 20°C. When the owner puts the heater on in the morning the temperature increases at 0.1°C per minute until it reaches 27°C, the temperature set on the thermostat of the pool heater.*

We sketched a graph that looked like this.



But what happens to the graph once we reach the temperature set on the thermostat (assuming that the thermostat maintains absolutely perfect temperature control). Because the temperature is now constant we might now suspect that the graph is made up of two different straight lines.



This type of function is made up of a number of parts and would be written like this.

$$f(t) = 0.1t + 20, \quad 0 \leq t \leq 70$$

$$= 27, \quad t > 70$$

(Note that in real life most thermostats would not be able to maintain the temperature so constantly.)

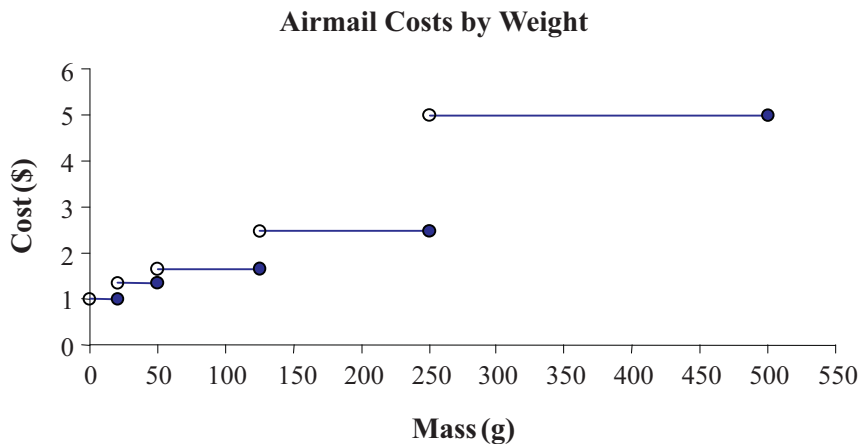
**Example**

The following list shows airmail rates for Spain

Mass (g)	Cost (\$)
Up to 20 g	1.00
Over 20 g up to 50 g	1.35
Over 50 g up to 125 g	1.65
Over 125 g up to 250 g	2.50
Over 250 g up to 500 g	5.00

Sketch a graph of the cost function, giving the domain and the rules that define it.

The first step is to identify the independent and dependent variables. In this case the independent variable is the weight of the mail (call it  $m$ ) while the dependent variable is the cost (call it  $C$ ).



The domain for the function is different in different parts of the function. We would write the rules for the function as follows:

$$\begin{aligned}
 C(m) &= 1.00, & 0 < m \leq 20 \\
 &= 1.35, & 20 < m \leq 50 \\
 &= 1.65, & 50 < m \leq 125 \\
 &= 2.50, & 125 < m \leq 250 \\
 &= 5.00, & 250 < m \leq 500
 \end{aligned}$$

In the graph above the solid black dot indicates  $\leq$  (or up to), the unfilled circle indicates  $<$  (or over).

## Activity 4.14

1. Sketch a graph of the function

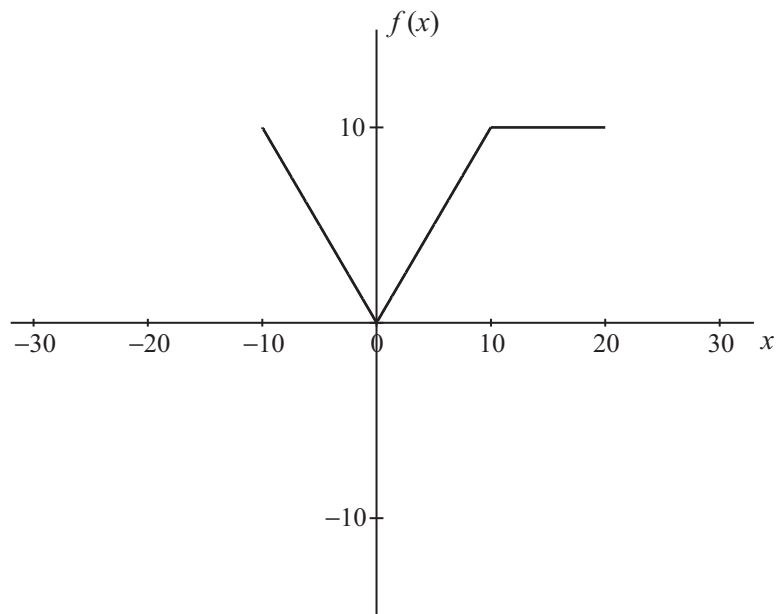
$$f(x) = 2, \quad -3 < x < 2$$

$$= x, \quad 2 \leq x \leq 4$$

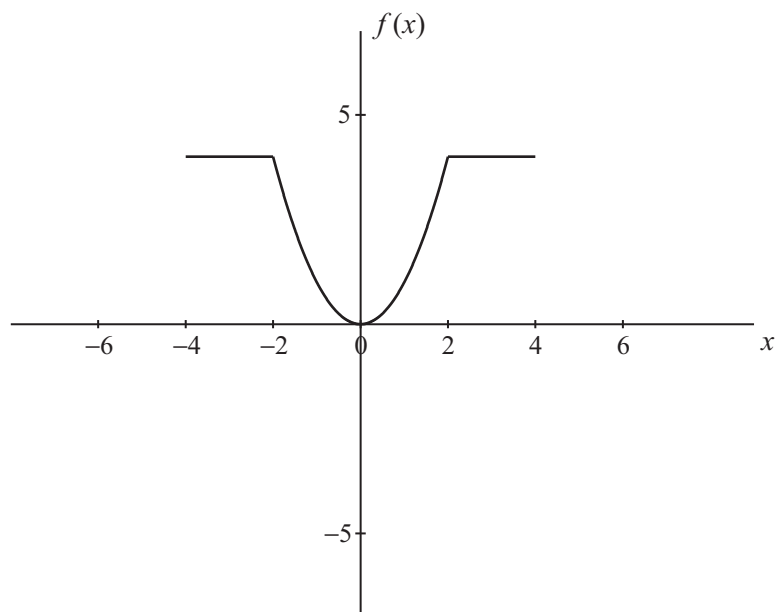
$$= 4, \quad x > 4$$

2. Specify the function represented by each of the graphs below:

(a)



(b)



That's the end of this module. Although the definition of a function is relatively young, historically speaking, it is a very important concept in modern mathematics and its applications. You will hear a lot more about functions in your future studies.

But before you are really finished you should do a number of things:

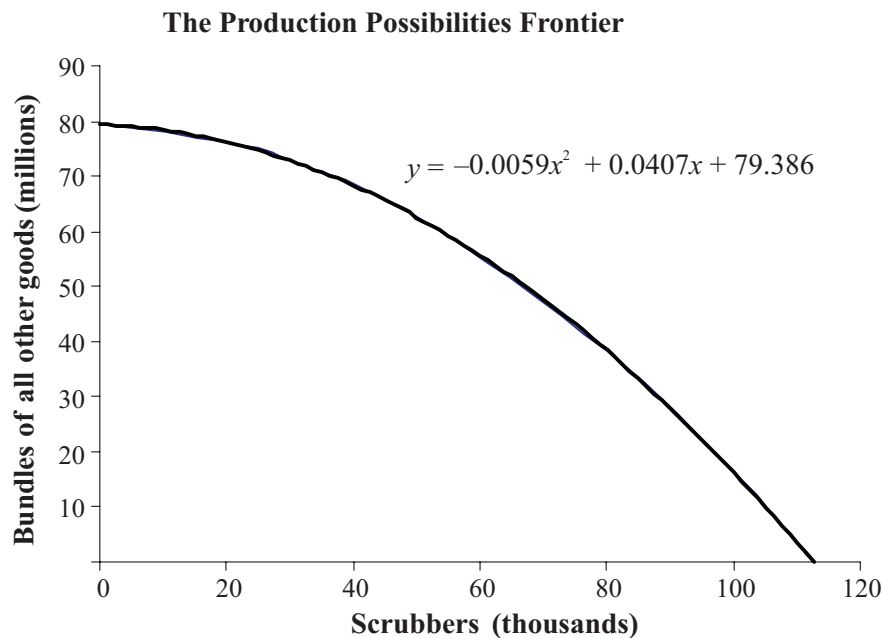
- Have a close look at your action plan for study. Are you on schedule? Or do you need to restructure your action plan or contact your tutor to discuss any delays or concerns?
- Make a summary of the important points in this module noting your strengths and weaknesses. Add any new words to your personal glossary. This will help with future revision.
- Practice some real world problems by having a go at 'A taste of things to come'
- Check your skill level by attempting the post-test.
- When you are ready, complete and submit your assignment.

## 4.5 A taste of things to come

1. A copper rod, part of an instrument, is exposed to different temperatures. Its length  $l$  is almost a linear function of the temperature  $t$  provided that  $t < 150^\circ\text{C}$ .
  - (a) Find the equation for  $l$  using the following measurements:
 
$$t_1 = 15^\circ, l_1 = 76.45\text{cm} \quad \text{and} \quad t_2 = 100^\circ, l_2 = 76.56\text{cm}$$
  - (b) What is the gradient of the function and what would it represent in terms of length of the rod and temperature?
  - (c) What is the inverse of the function and would determining it be useful to the experimenter?
2. In any economy there are balances to be considered between production of different goods when resources are limited. This is called the Production Possibilities Tradeoff. Consider the cost associated with cleaning up environmental pollution. The output of the economy can be divided into scrubbers (an anti-pollution device that removes pollution from a factory smokestack emissions) and bundles of all other goods so that the following may result in an economy where all resources are employed.

**Possible combinations of scrubbers and bundles of all other goods that may be produced in a full-employment economy (hypothetical data)**

	Scrubbers ( $x$ ) (thousands)	Bundles of all other goods ( $y$ ) (millions)
A	0	79
B	20	76
C	30	73
D	50	63
E	80	38
F	100	16
G	110	4
H	113	0



(Source: Modified from Waud, Maxwell, Hocking, Bonnici & Ward 1996, *Economics*, 3rd edn, Longman, Melbourne, pp. 37–9.)

The parabolic graph above is called the Production Possibilities Frontier.

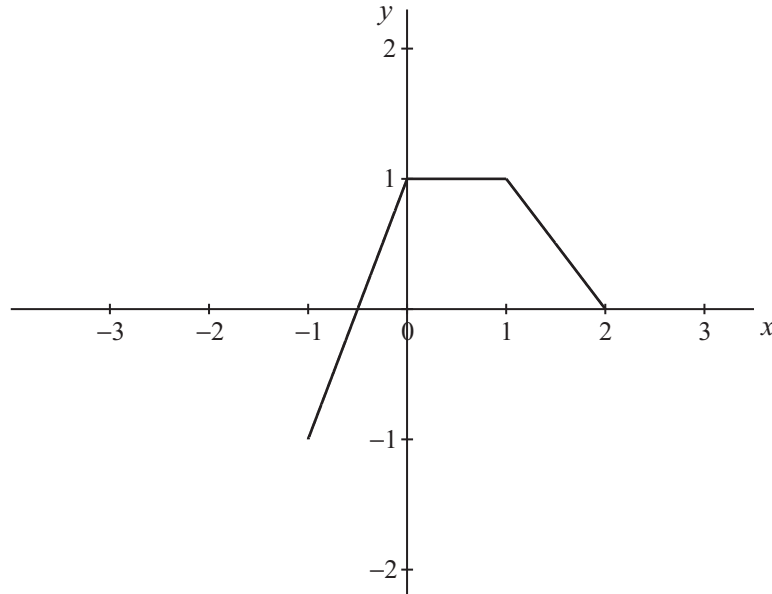
- Describe in your own words what the points A and H represent.
- Calculate the rate of change of millions of bundles with respect to thousands of scrubbers between the points A to B, C to D, E to F and G to H.
- How would you represent these figures of the graph above?
- What do these rates of change represent in terms of the scenario presented above?

## 4.6 Post-test

1. Which of the following relations are also functions? Give reasons for your decision. State the domain and range of each.

(a)  $\{(1,2), (3,4), (1,7), (6,5)\}$

(b)



(c) The rental of a community hall is \$40 for cleaning and \$150 per day or part of a day.

(d)  $y = \sqrt{1-x}$

2. (a) Graph the following functions clearly indicating  $x$  and  $y$  intercepts and turning points if appropriate.

(i)  $f(x) = -\frac{1}{2}x + 1$

(ii)  $A = 6B^2 - 5B - 4$

(b) If  $f(x) = 2(x - \frac{1}{3})$ , what is  $f^{-1}(x)$ ?

3. If  $g(x) = x^2 + 3x - 1$ , calculate

(a)  $g(1)$

(b)  $g(-1)$

(c)  $g(a+h)$

(d)  $\frac{g(1) - g(-1)}{1 - (-1)}$

Describe in your own words what the expression in part (d) represents.

4. The cost of manufacturing a certain product is related to the cost of the raw materials. The following are values of the function.

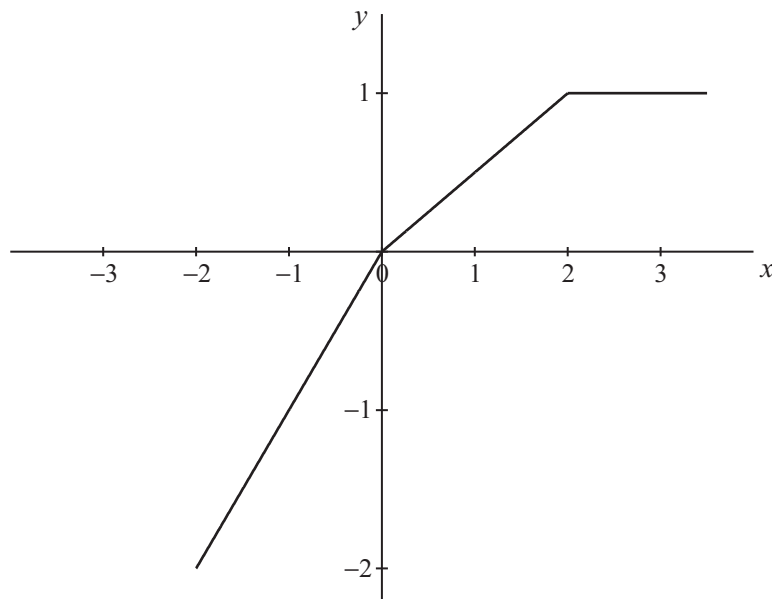
Cost of raw materials (\$)	0	20	30	40	50
Cost of manufacture (\$)	150	190	210	230	250

- (a) What is the rate of change of cost of manufacture with respect to cost of raw materials?  
 (b) What is the inverse of the function?  
 (c) What would the inverse function represent in terms of the two costs (raw materials and manufacturing).
5. (a) Solve the following equations graphically

$$2x + 3y = 4$$

$$9y + 4x = 12$$

- (b) Prove your solution correct algebraically.
6. Write the equation for the function sketched below.



7. Without solving the equation how many horizontal intercepts would each of the quadratic functions possess?
- (a)  $y = 2x^2 + 5x - 3$   
 (b)  $A = 4B^2 - 4B + 1$   
 (c)  $Z = s^2 + s - 1$   
 (d) Sketch the graph of (a)



## 4.7 Solutions

### Solutions to activities

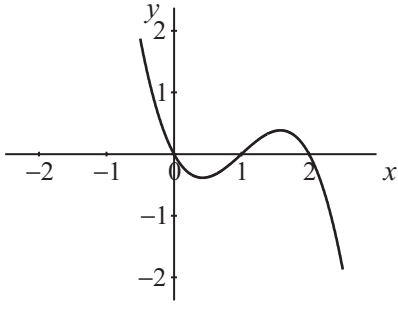
#### Activity 4.1

1.
  - (a) Independent variable is income earned.  
Dependent variable is income tax paid.
  - (b) Independent variable is age of the child.  
Dependent variable is the dosage in millilitres to be given.
2.
  - (a) The independent variable on the horizontal axis is the selling price of the water.  
The dependent variable on the vertical axis is the number of units sold.
  - (b) The independent variable on the horizontal axis is the number of frost-free days.  
The dependent variable on the vertical axis is the area covered by the plant species.
  - (c) The independent variable on the horizontal axis is the velocity of the car.  
The dependent variable on the vertical axis is the stopping distance of the car.

#### Activity 4.2

1.

Relation	Function ?	Reason
(a) $y = x + 5$	Yes	For every value of $x$ (the independent variable) there is only one value of $y$ (the dependent variable). The relation must be a function.
(b) <div style="text-align: center;"> </div>	No	It is evident from the graph that for many values of the independent variable, $x$ , there is more than one value of the dependent variable, $y$ . Therefore, this relation is not a function.
(c) $\{(-4,0), (3,2), (-1,2), (6,0)\}$	Yes	In ordered pairs the first coordinate is traditionally the independent variable and the second the dependent variable. For each of the $x$ values shown there is only one $y$ value. Therefore, the relation is a function.

<p>(d)</p> 	Yes	<p>In graphs the horizontal axis traditionally represents the independent variable and the vertical axis the dependent variable. The graph shows that for each <math>x</math> value there is only one <math>y</math> value. Therefore, the relation is a function.</p>
<p>(e) The amount of potassium absorbed by the leaf tissue of a plant when placed in darkness is graphed against time.</p>	Yes	<p>In this relation, time is the independent variable and the amount of potassium absorbed is the dependent variable. For each time there is only one amount of potassium absorbed. Therefore, the relation is a function.</p>
<p>(f) <math>\{(3,27), (-9,2), (11,27), (3,5)\}</math></p>	No	<p>In ordered pairs the first coordinate is traditionally the independent variable and the second the dependent variable. For <math>x = 3</math> there are two alternatives of <math>y = 27</math> or <math>y = 5</math>. The relation has two values of the dependent variable (<math>y</math>) for the same value of the independent variable (<math>x</math>). Therefore, the relation is not a function.</p>

### Activity 4.3

1.

- The domain of the function is all real values of  $x$  except  $x = 0$ .
- The domain of the function is all real values of  $x$  except  $x = 1$ .
- The domain of the function is all real values of  $v$  greater than or equal to 4 i.e.  $v \geq 4$ .
- The domain is all real values of  $x$  except  $x = -3$  and  $x = 4$ .
- The domain is all real values of  $n$ .

**Activity 4.4**

1.

(a) Domain =  $0 \leq x \leq 3$

Range =  $-0.5 \leq y \leq 2$

(b) Domain =  $-5 \leq x \leq 5$

Range =  $-50 \leq y \leq 50$  (answers in both questions are approximate due to difficulties reading from the graphs)

2.

(a)  $a = 2b + 3$  where the domain is all real values of  $b$

The range is all real values of  $a$ .

(b) For the domain of  $-3 \leq r \leq 8$ , the range is  $-10 \leq q \leq 23$ .

(c)  $y = 2 + 4x$  where the domain is  $0 \leq x \leq 50$

The range is  $2 \leq y \leq 202$ **Activity 4.5**

1.

(a) The maximum magnification,  $M$ , obtained when using a converging lens is a function

of the focal length,  $f$ , of the converging lens  $M = f(f) = 1 + \frac{25}{f}$

(b) The daily cost,  $C$ , of producing a product is a linear function of the number,  $q$ , of the product made  $C = f(q) = 5q + 45$ (c) The daily sales,  $S$ , of a product is a function of the time the product has been on the

market  $S = f(t) = \frac{4000}{1 + 10e^{-0.03t}}$

(d) The surface area of a sphere is a function of the radius of the sphere  $S = f(r) = 4\pi r^2$ 

2.

(a)  $f(x) = 2x + 4$

$f(3) = 2(3) + 4$

$f(3) = 6 + 4$

$f(3) = 10$

$f(-0.5) = 2(-0.5) + 4$

$f(-0.5) = -1 + 4$

$f(-0.5) = 3$

$f(0) = 2(0) + 4$

$f(0) = 0 + 4$

$f(0) = 4$

$$f(2+t) = 2(2+t) + 4$$

$$f(2+t) = 4 + 2t + 4$$

$$f(2+t) = 8 + 2t$$

$$f(a) = 2(a) + 4$$

$$f(a) = 2a + 4$$

$$(b) f(x) = \frac{4x}{2x-3}$$

$$f(3) = \frac{4(3)}{2(3)-3}$$

$$f(3) = \frac{12}{6-3}$$

$$f(3) = \frac{12}{3}$$

$$f(3) = 4$$

$$f(-0.5) = \frac{4(-0.5)}{2(-0.5)-3}$$

$$f(-0.5) = \frac{-2}{-1-3}$$

$$f(-0.5) = \frac{-2}{-4}$$

$$f(-0.5) = \frac{1}{2} = 0.5$$

$$f(0) = \frac{4(0)}{2(0)-3}$$

$$f(0) = \frac{0}{0-3}$$

$$f(0) = 0$$

$$f(2+t) = \frac{4(2+t)}{2(2+t)-3}$$

$$f(2+t) = \frac{8+4t}{4+2t-3}$$

$$f(2+t) = \frac{8+4t}{1+2t}$$

$$f(a) = \frac{4(a)}{2(a)-3}$$

$$f(a) = \frac{4a}{2a-3}$$

$$(c) f(x) = x^2 - 2x + 9$$

$$f(3) = (3)^2 - 2(3) + 9$$

$$f(3) = 9 - 6 + 9$$

$$f(3) = 12$$

$$f(-0.5) = (-0.5)^2 - 2(-0.5) + 9$$

$$f(-0.5) = 0.25 + 1 + 9$$

$$f(-0.5) = 10.25$$

$$f(0) = (0)^2 - 2(0) + 9$$

$$f(0) = 0 - 0 + 9$$

$$f(0) = 9$$

$$f(2+t) = (2+t)^2 - 2(2+t) + 9$$

$$f(2+t) = (2+t)(2+t) - 4 - 2t + 9$$

$$f(2+t) = 2(2+t) + t(2+t) - 4 - 2t + 9$$

$$f(2+t) = 4 + 4t + t^2 - 4 - 2t + 9$$

$$f(2+t) = t^2 + 2t + 9$$

$$f(a) = (a)^2 - 2(a) + 9$$

$$f(a) = a^2 - 2a + 9$$

$$(d) f(x) = \sqrt{4x+5}$$

$$f(3) = \sqrt{4(3)+5}$$

$$f(3) = \sqrt{12+5}$$

$$f(3) = \sqrt{17}$$

$$f(-0.5) = \sqrt{4(-0.5)+5}$$

$$f(-0.5) = \sqrt{-2+5}$$

$$f(-0.5) = \sqrt{3}$$

$$f(0) = \sqrt{4(0)+5}$$

$$f(0) = \sqrt{0+5}$$

$$f(0) = \sqrt{5}$$

$$f(2+t) = \sqrt{4(2+t)+5}$$

$$f(2+t) = \sqrt{8+4t+5}$$

$$f(2+t) = \sqrt{13+4t}$$

$$f(a) = \sqrt{4(a) + 5}$$

$$f(a) = \sqrt{4a + 5}$$

3. Given the functional relationship,  $g(t) = 4 - 3t - 2t^2$

$$(a) \quad g(-2) = 4 - 3(-2) - 2(-2)^2$$

$$g(-2) = 4 + 6 - 2 \times 4$$

$$g(-2) = 4 + 6 - 8$$

$$g(-2) = 2$$

$$(b) \quad g(a + b) = 4 - 3(a + b) - 2(a + b)^2$$

$$g(a + b) = 4 - 3a - 3b - 2(a + b)(a + b)$$

$$g(a + b) = 4 - 3a - 3b - 2(a^2 + 2ab + b^2)$$

$$g(a + b) = 4 - 3a - 3b - 2a^2 - 4ab - 2b^2$$

4. Find the weight of a person standing on a mountain 3200 m above sea level. This means that the person is 3.2 km from the surface of the earth, so  $d = 3.2$

$$w(3.2) = \frac{6400^2 \times 65}{(3.2 + 6400)^2}$$

$$w(3.2) \approx 64.935$$

Therefore a 65 kg person would weigh 64.935 kg standing on a mountain 3200 m above sea level.

#### Activity 4.6

1.

(a) The horizontal intercept occurs when  $y = 0$

Substituting into equation,  $4x - 3y = 24$

$$4x - 3(0) = 24$$

$$4x = 24$$

$$x = 6$$

The vertical intercept occurs when  $x = 0$  (or content in the slope intercept form)

Substituting into equation,  $4x - 3y = 24$

$$4(0) - 3y = 24$$

$$-3y = 24$$

$$y = -8$$

Therefore, the horizontal intercept is (6,0) and the vertical intercept is (0,-8).

- (b) The horizontal intercept occurs when
- $b = 0$

Substituting into equation,  $b = -2a + 5$ 

$$0 = -2a + 5$$

$$-5 = -2a$$

$$a = \frac{5}{2}$$

The vertical intercept occurs when  $a = 0$ Substituting into equation,  $b = -2a + 5$ 

$$b = -2(0) + 5$$

$$b = 5$$

Therefore, the horizontal intercept is  $(\frac{5}{2}, 0)$  and the vertical intercept is  $(0, 5)$ 

- (c) The horizontal intercept occurs when
- $t = 0$

Substituting into equation,  $\frac{4}{5}s - 8 = t$ 

$$\frac{4}{5}s - 8 = 0$$

$$\frac{4}{5}s = 8$$

$$s = 8 \times \frac{5}{4}$$

$$s = 10$$

The vertical intercept occurs when  $s = 0$ Substituting into equation,  $\frac{4}{5}s - 8 = t$ 

$$\frac{4}{5}(0) - 8 = t$$

$$t = -8$$

Therefore, the horizontal intercept is  $(10, 0)$  and the vertical intercept is  $(0, -8)$ .

- (d) The horizontal intercept occurs when
- $n = 0$

Substituting into equation,  $9 = 2m + 4n$ 

$$9 = 2m + 4(0)$$

$$9 = 2m$$

$$m = 4.5$$

The vertical intercept occurs when  $m = 0$ Substituting into equation,  $9 = 2m + 4n$ 

$$9 = 2(0) + 4n$$

$$9 = 4n$$

$$n = \frac{9}{4} = 2.25$$

Therefore, the horizontal intercept is (4.5,0) and the vertical intercept is (0,2.25).

Note: we could have also completed this with  $m$  as the independent variable.

- (e) The horizontal intercept occurs when  $y = 0$

Substituting into equation,  $3x = \frac{(x - y)}{2}$

$$3x = \frac{(x - (0))}{2}$$

$$3x = \frac{x}{2}$$

$$6x = x$$

$$5x = 0$$

$$x = 0$$

The vertical intercept occurs when  $x = 0$

Substituting into equation,  $3x = \frac{(x - y)}{2}$

$$3(0) = \frac{((0) - y)}{2}$$

$$0 = \frac{-y}{2}$$

$$y = 0$$

Therefore, the function passes through the origin (0,0).

- (f) The horizontal intercept occurs when  $q = 0$

Substituting into equation,  $q - 3 = 5(p - 2)$

$$(0) - 3 = 5(p - 2)$$

$$-3 = 5p - 10$$

$$7 = 5p$$

$$p = \frac{7}{5} = 1.4$$

The vertical intercept occurs when  $p = 0$

Substituting into equation,  $q - 3 = 5(p - 2)$

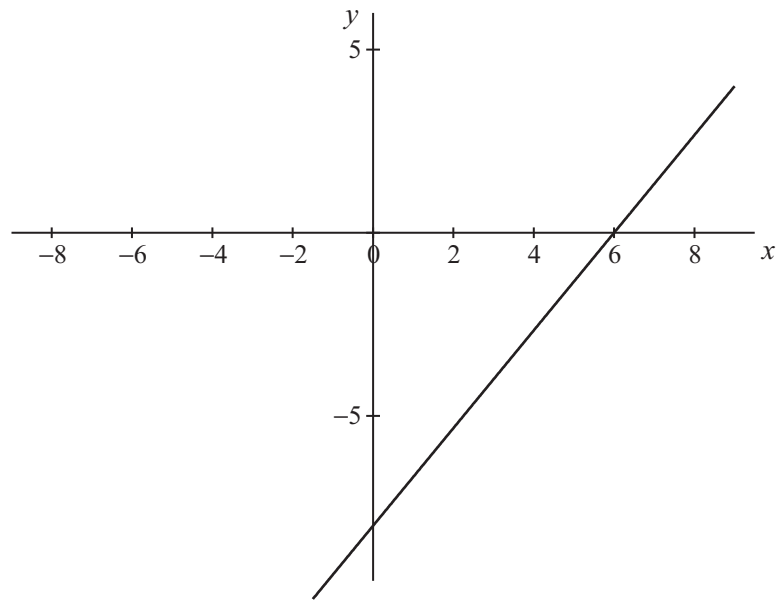
$$q - 3 = 5((0) - 2)$$

$$q - 3 = -10$$

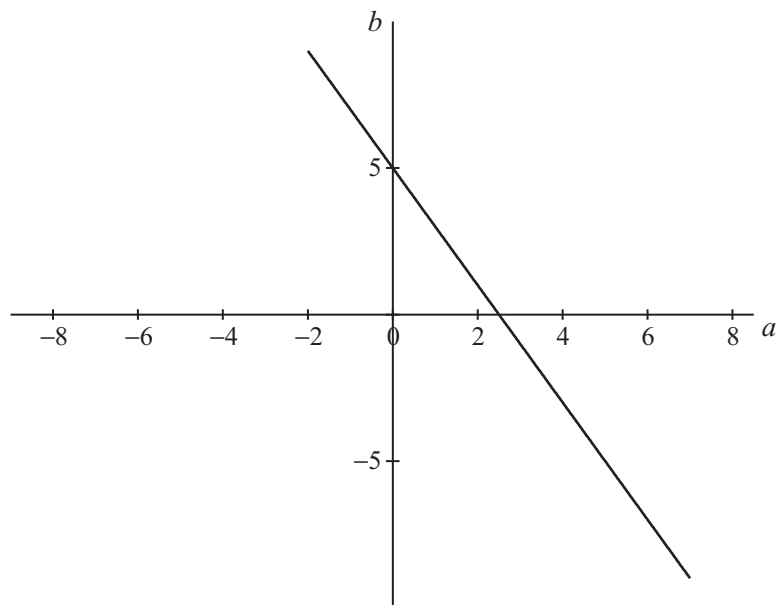
$$q = -7$$

Therefore, the horizontal intercept is (1.4,0) and the vertical intercept is (0,-7)

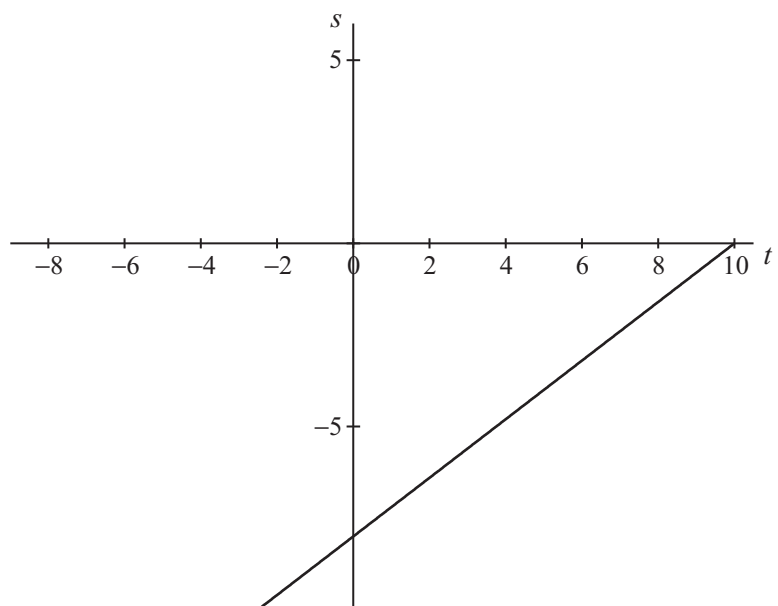
2. (a)



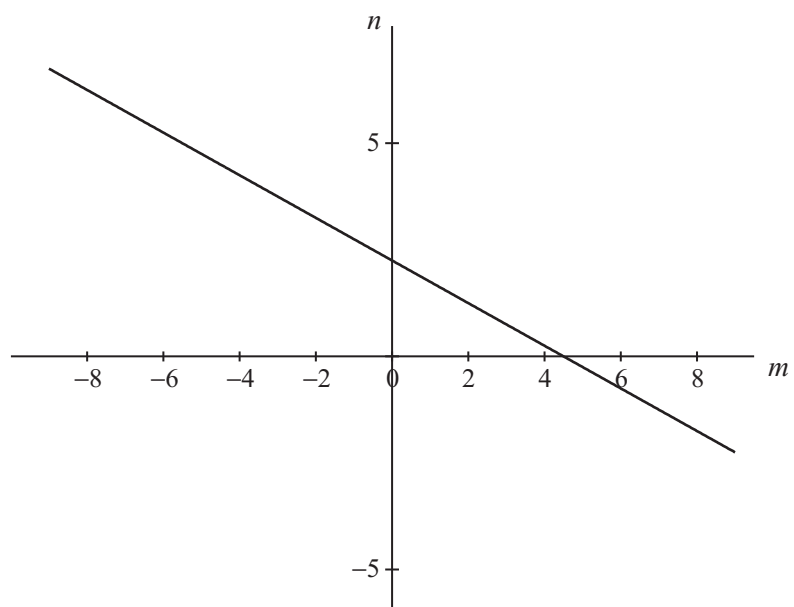
(b)



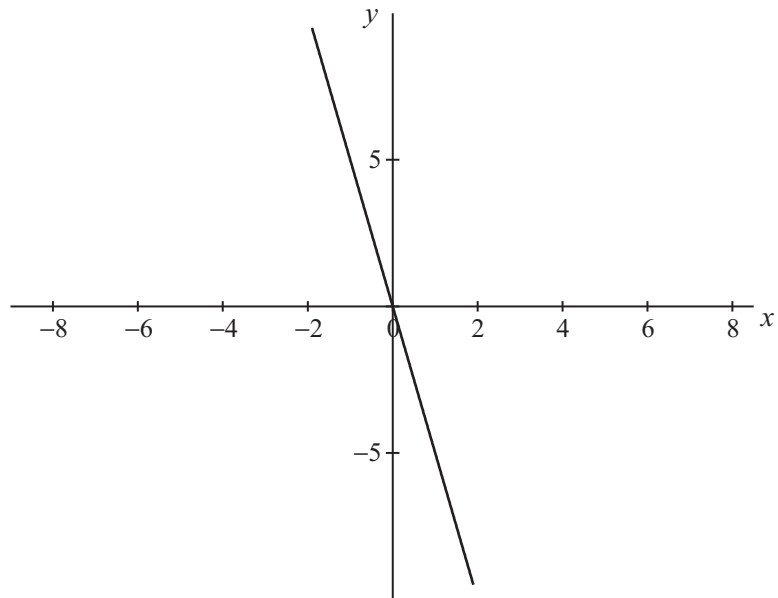
(c)



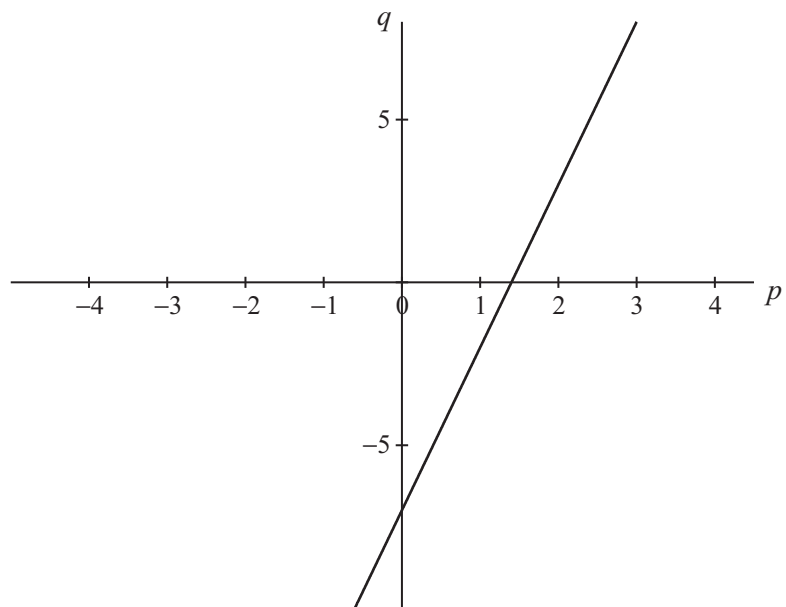
(d)



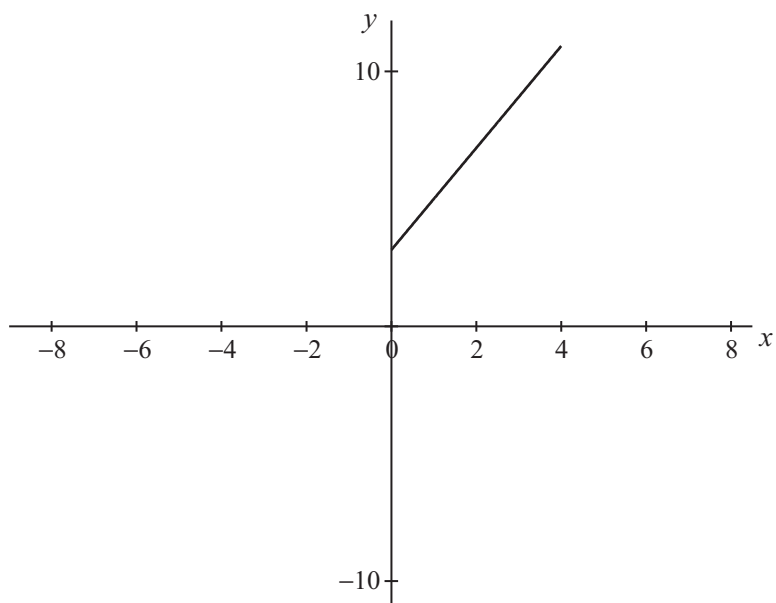
(e)



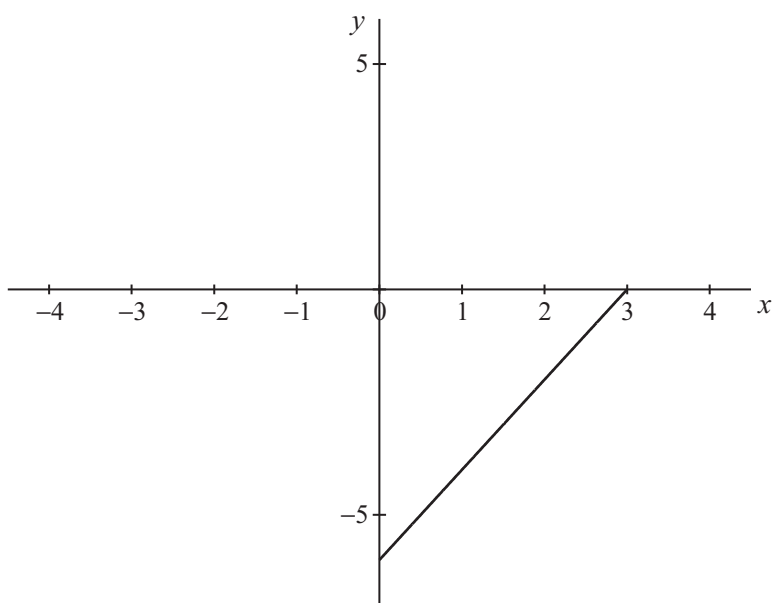
(f)



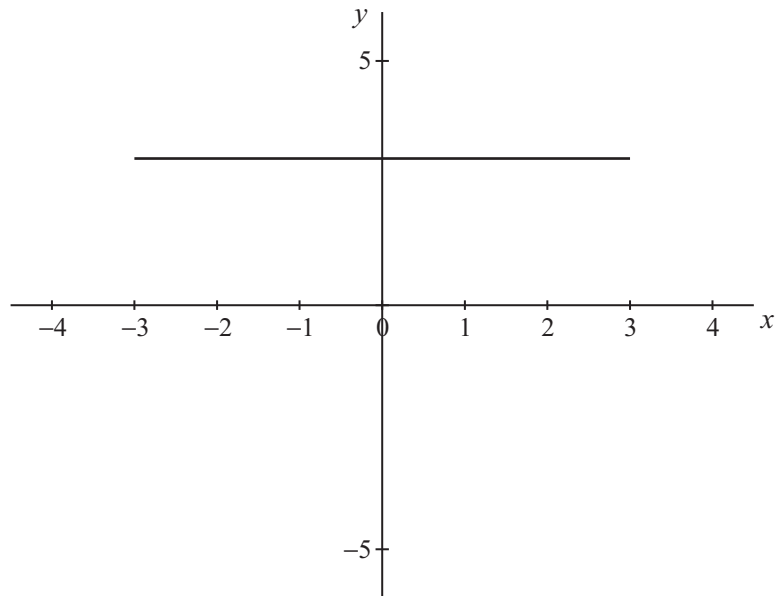
3. (a)



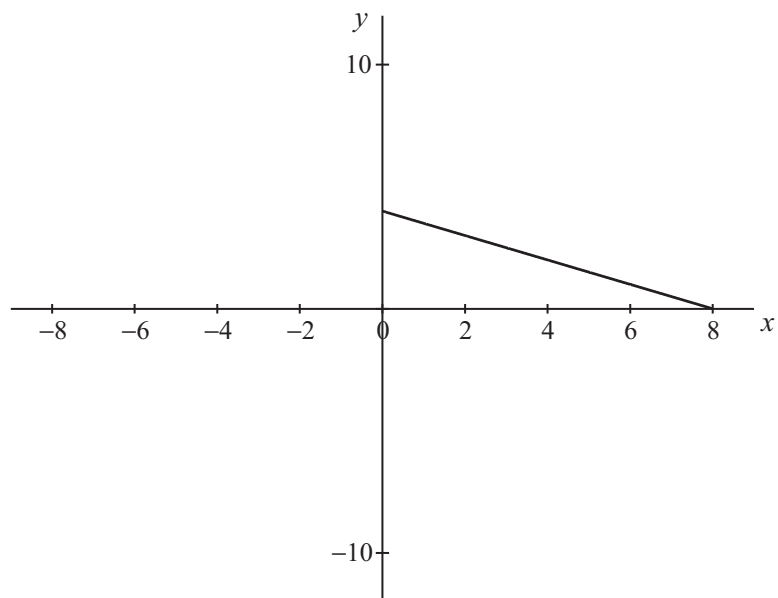
(b)



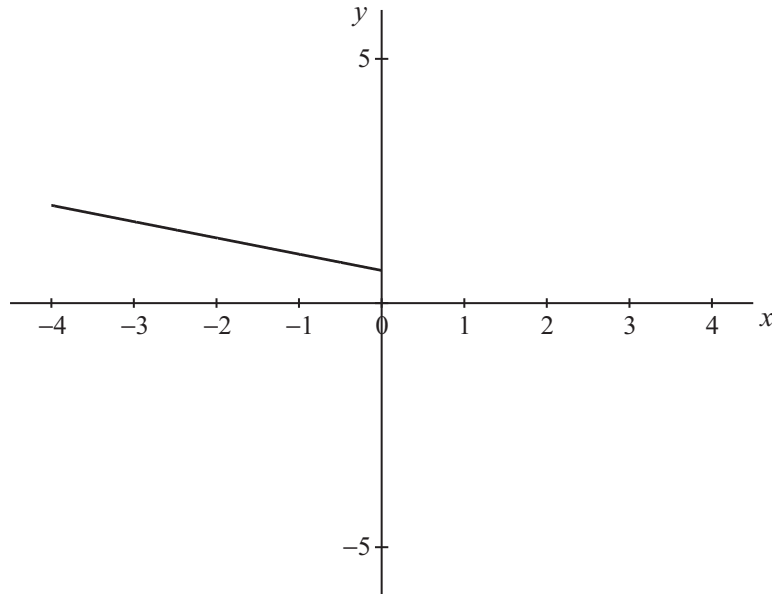
(c)



(d)



(e)



**Activity 4.7**

1.

- (a) Let  $h$  be the height of the candle in mm  
Let  $t$  be time in hours

Then,  $h = f(t) = 80 - 10t$   
 Domain  $0 \leq t \leq 8$   
 Range  $0 \leq h \leq 80$

- (b) Let  $c$  be the capacity of the tank in litres.  
Let  $t$  be time in hours.

Then,  $c = g(t) = 2000 - 20t$   
 Domain  $0 \leq t \leq 100$   
 Range  $0 \leq c \leq 2000$

- (c) Let  $c$  be the total cost of printing t-shirts in dollars.  
Let  $n$  be the number of units of 20 t-shirts printed.

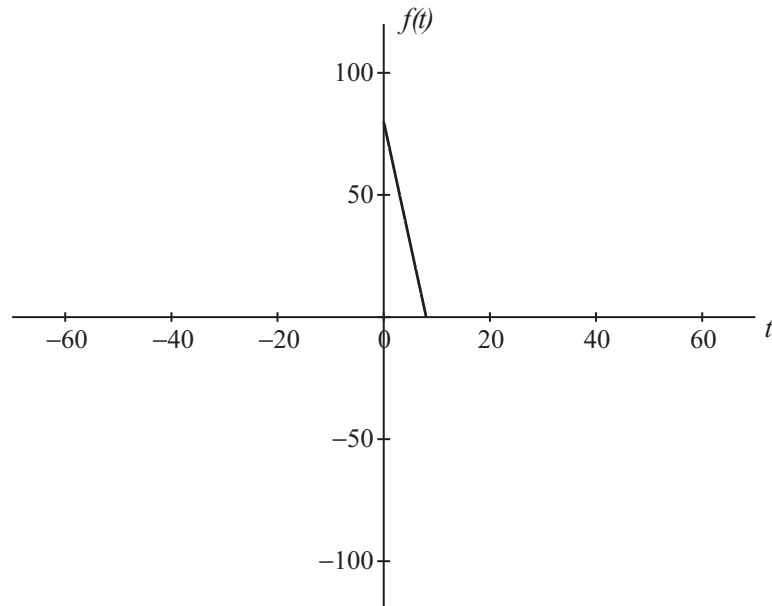
Then,  $c = f(n) = 8.50 + 140n$   
 Domain  $n \geq 0$   
 Range  $c \geq 8.50$

- (d) Let  $d$  be the distance from his holiday house in kilometres.  
Let  $t$  be the time in hours.

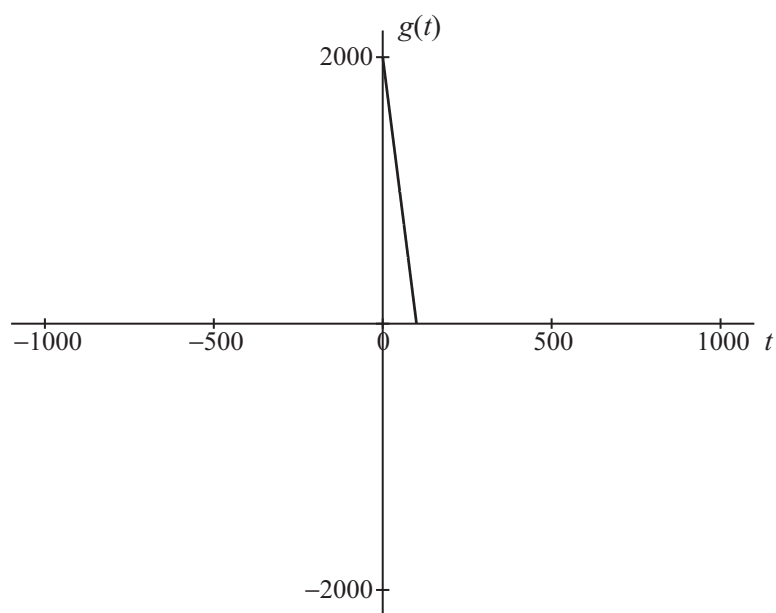
Then,  $d = f(t) = 540 - 90t$   
 Domain  $0 \leq t \leq 6$   
 Range  $0 \leq d \leq 540$

2. Choose an appropriate scale to graph the linear functions in Question 1.

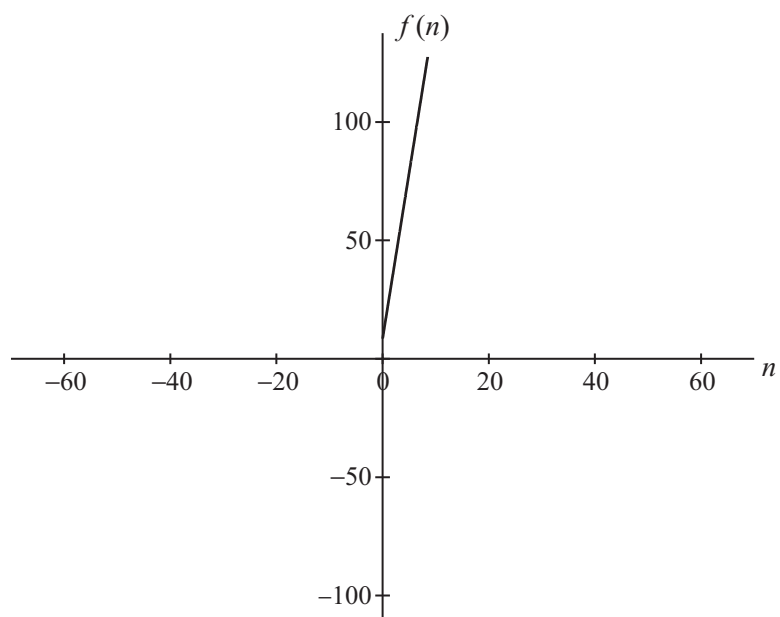
(a)



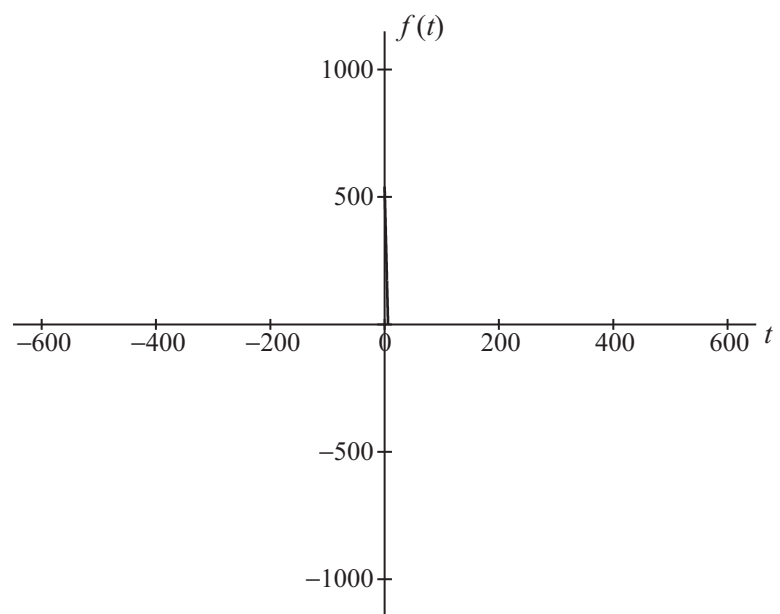
(b)



(c)



(d)



**Activity 4.8**

1.

$$(a) m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$m = \frac{5 - (-1)}{12 - 2}$$

$$m = \frac{6}{10}$$

$$m = \frac{3}{5}$$

$$(b) m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$m = \frac{2 - 5}{4 - 2}$$

$$m = -\frac{3}{2}$$

$$2. m = \frac{f(a_2) - f(a_1)}{a_2 - a_1}$$

$$m = \frac{-4 - 6}{2 - 1}$$

$$m = \frac{-10}{1}$$

$$m = -10$$

$$3. m = \frac{g(s_2) - g(s_1)}{s_2 - s_1}$$

$$m = \frac{5 - (-3)}{4 - (-2)}$$

$$m = \frac{8}{6}$$

$$m = \frac{4}{3}$$

4.

- (a) The coordinates of the  $y$ -intercept are  $(0, 2)$ , the coordinates of the  $x$ -intercept are  $(-3, 0)$

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$m = \frac{2 - 0}{0 - (-3)}$$

$$m = \frac{2}{3}$$

- (b) The coordinates of the  $y$ -intercept are  $(0, 2)$ , the coordinates of the  $x$ -intercept are  $(\frac{1}{2}, 0)$

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$m = \frac{2 - 0}{0 - \frac{1}{2}}$$

$$m = -4$$

### Activity 4.9

1.

- (a) The gradient is the rate of change of the length of the candle over time (10 mm/sec). The vertical intercept is the maximum length of the candle. The horizontal intercept is the time when there is no candle left.
- (b) The gradient is the rate of change of the volume over time (20 L/h). The vertical intercept is the maximum volume of the tank. The horizontal intercept is when there is no water left in the tank.
- (c) The gradient is the rate of change of cost of printing with the number of shirts printed (\$140/lot of 20). The vertical intercept is the value of the fixed costs, the horizontal intercept is negative and has no real practical meaning.
- (d) The gradient is his speed (90 km/h). The vertical intercept is initial distance he is from his holiday house. The horizontal intercept is the time he takes to reach his holiday house.

2.

- (a) Let  $t$  represent the time in years  
Let  $c$  represent the car value in dollars.

$$m = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

$$m = \frac{17500 - 28000}{3 - 0}$$

$$m = \frac{-10500}{3}$$

$$m = -3500$$

- (b) The equation is,  $c = 28000 - 3500t$

The gradient is negative so this means that the value of the car decreases as the number of years increases. The value of the gradient indicates that the value of the car is decreasing at the rate of \$3 500 per year.

The vertical intercept represents the original value of the car, \$28 000.

The horizontal intercept represents the time when the car will have no value, 8 years.

#### Activity 4.10

1.

- (a)  $(-7,4), (-2,1), (4,5), (5,7)$

The inverse is,  $(4, -7), (1, -2), (5,4), (7,5)$

Domain  $\{1, 4, 5, 7\}$

Range  $\{-7, -2, 4, 5\}$

The inverse is a function because there is only one value of the dependent variable for each independent variable.

- (b)  $(-1,2), (0, -4), (3,2), (4,5)$

The inverse is,  $(2, -1), (-4,0), (2,3), (5,4)$

Domain  $\{-4, 2, 5\}$

Range  $\{-1, 0, 3, 4\}$

The inverse is not a function because there are two values of the dependent variable for one independent variable ( $x = 2$ ).

2.

(a)  $g(x) = 2x + 1$

Undoing the process gives (subtract 1 and divide by 2)

$$g^{-1}(x) = \frac{x-1}{2}$$

$$g^{-1}(x) = \frac{1}{2}x - \frac{1}{2}$$

 $g^{-1}$  is a function, as for each value of  $x$  there is only one value of the functionDomain is all real values of  $x$ , range is all real values of the function

(b)  $h(x) = 3 - 2x$

Undoing the process (subtract 3 and divide by  $-2$ )

$$h^{-1}(x) = \frac{-3+x}{-2}$$

$$h^{-1}(x) = \frac{3}{2} - \frac{x}{2}$$

 $h^{-1}$  is a function as for each value of  $x$  there is only one value of the functionDomain is all real values of  $x$ . The range is all real values of the function.

(c)  $f(x) = \frac{x}{2} - 3$

Undoing the function (add 3 and multiply by 2)

$$f^{-1}(x) = 2(x+3)$$

 $f^{-1}$  is a function as for each value of  $x$  there is only one value of the functionDomain is all real values of  $x$ . The range is all real values of the function.

(d)  $g(x) = 2(x-3)$

Undoing the function (divide by 2 and add 3)

$$g^{-1}(x) = \frac{x}{2} + 3$$

 $g^{-1}$  is a function as for each value of  $x$  there is only one value of the functionDomain is all real values of  $x$ . The range is all real values of the function.

**Activity 4.11**

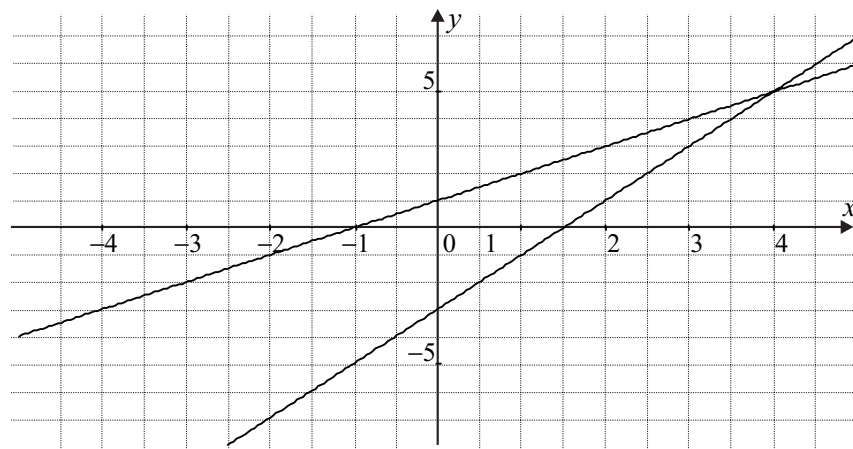
1.

$$(a) \quad y = x + 1$$

$$y = 2x - 3$$

As equations are written in slope-intercept form, from the first equation, Gradient = 1,  $y$ -intercept = 1. From the second equation, Gradient = 2,  $y$ -intercept = -3

Graphing the two functions,



The graphical solution to the linear functions is (4,5)

To prove solution correct algebraically, substitute the point into both equations.

When  $x = 4$  and  $y = 5$ ,  $y = x + 1$  is  $5 = 4 + 1$ , satisfies equation

When  $x = 4$  and  $y = 5$ ,  $y = 2x - 3$  is  $5 = 2 \times 4 - 3$ , satisfies equation.

Thus the solution is correct.

$$(b) \quad y - x = -6$$

$$y - 2x = -8$$

To graph the first equation,  $x$ -intercept occurs when  $y = 0$

$$0 - x = -6$$

$$-x = -6$$

$$x = 6$$

$y$ -intercept occurs when  $x = 0$

$$y - (0) = -6$$

$$y = -6$$

Therefore the  $x$ -intercept is (6,0) and the  $y$ -intercept is (0,-6).

For the second equation,  $x$ -intercept occurs when  $y = 0$

$$0 - 2x = -8$$

$$\frac{-2}{-2}x = \frac{-8}{-2}$$

$$x = 4$$

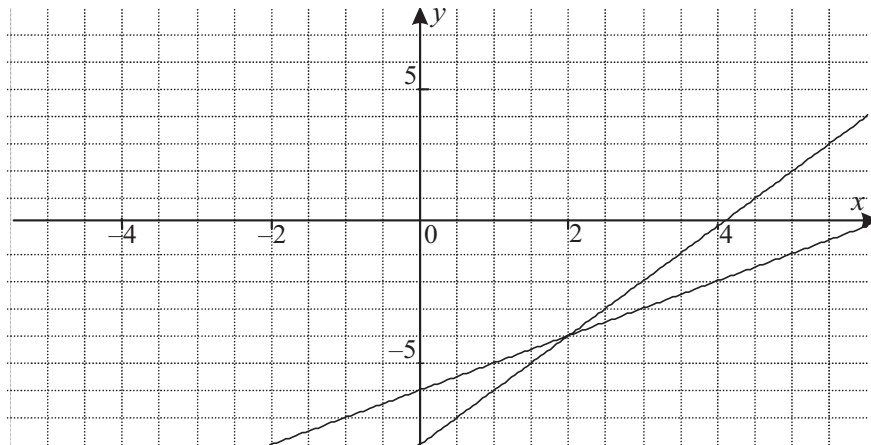
$y$ -intercept occurs when  $x = 0$

$$y - 2(0) = -8$$

$$y = -8$$

Therefore the  $x$ -intercept is  $(4,0)$  and the  $y$ -intercept is  $(0,-8)$ .

Graphing the two functions,



The graphical solution to the linear function is  $(2, -4)$ .

To prove solution correct algebraically, substitute the point into both equations.

When  $x = 2$  and  $y = -4$ ,  $y - x = -6$  is  $-4 - 2 = -6$ , satisfies equation

When  $x = 2$  and  $y = -4$ ,  $y - 2x = -8$  is  $-4 - 2 \times 2 = -8$ , satisfies equation.

Thus the solution is correct.

(c)  $-4.5x + 3y = 18$

$$3x - 2y = 12$$

To graph the first equation,  $x$ -intercept occurs when  $y = 0$ ,

$$-4.5x + 3(0) = 18$$

$$-4.5x = 18$$

$$\frac{-4.5}{-4.5}x = \frac{18}{-4.5}$$

$$x = -4$$

$y$ -intercept occurs when  $x = 0$ ,

$$-4.5(0) + 3y = 18$$

$$3y = 18$$

$$\frac{3}{3}y = \frac{18}{3}$$

$$y = 6$$

Therefore, the  $x$ -intercept is  $(-4,0)$  and the  $y$ -intercept is  $(0,6)$ .

For the second equation,  $x$ -intercept occurs when  $y = 0$ ,

$$3x - 2(0) = 12$$

$$\frac{3}{3}x = \frac{12}{3}$$

$$x = 4$$

$y$ -intercept occurs when  $x = 0$ ,

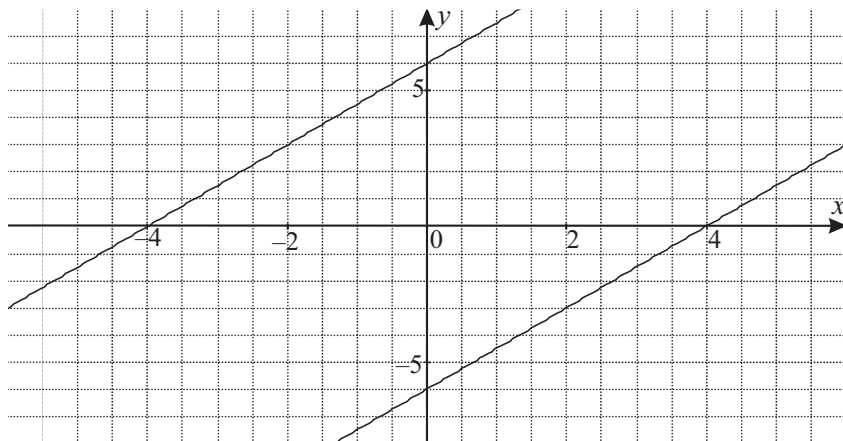
$$3(0) - 2y = 12$$

$$\frac{-2}{-2}y = \frac{12}{-2}$$

$$y = -6$$

Therefore, the  $x$ -intercept is  $(4,0)$  and the  $y$ -intercept is  $(0,-6)$ .

Graphing the two functions,



There is no simultaneous solution for these two functions as they are parallel lines.

$$(d) \begin{aligned} x + y &= 6 \\ 2x + y &= 9 \end{aligned}$$

To graph the first equation  $x$ -intercept occurs when  $y = 0$

$$\begin{aligned} x + (0) &= 6 \\ x &= 6 \end{aligned}$$

$y$ -intercept occurs when  $x = 0$

$$\begin{aligned} (0) + y &= 6 \\ y &= 6 \end{aligned}$$

Therefore the  $x$ -intercept is  $(6,0)$  and the  $y$ -intercept is  $(0,6)$ .

For the second equation,  $x$ -intercept occurs when  $y = 0$

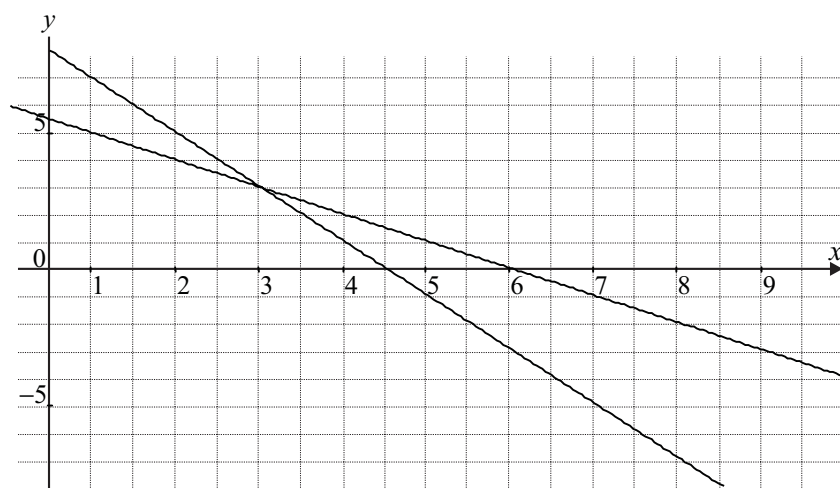
$$\begin{aligned} 2x + (0) &= 9 \\ 2x &= 9 \\ \frac{2}{2}x &= \frac{9}{2} \\ x &= 4.5 \end{aligned}$$

$y$ -intercept occurs when  $x = 0$

$$\begin{aligned} 2(0) + y &= 9 \\ y &= 9 \end{aligned}$$

Therefore the  $x$ -intercept is  $(4.5,0)$  and the  $y$ -intercept is  $(0,9)$ .

Graphing the two functions,



The graphical solution to the linear functions is  $(3,3)$   
Check your graphical solution by substituting back into the original equations.

$$x + y = 6$$

$$LHS = (3) + (3) = 6 = RHS$$

$$2x + y = 9$$

$$LHS = 2(3) + 3 = 9 = RHS$$

So the solution is correct.

2.

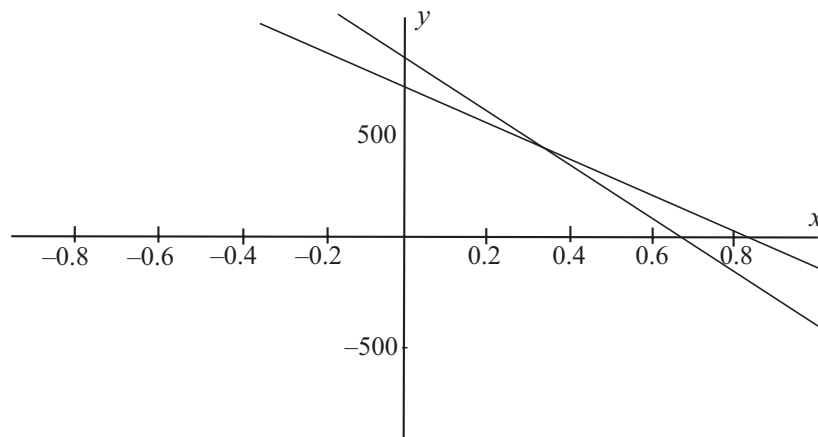
(a) Let  $y$  be the initial fixed cost and  $x$  the cost per kilometre.

The two equations developed from the data are

$$y + 1250x = 830$$

$$y + 850x = 710$$

The graphs of the two functions are below



From the graph we can read that the point of intersection is  $x = 0.3$  and  $y = 455$ .

To prove that this solution is correct we can substitute into both equations.

$$\text{For } y + 1250x = 830, 455 + 1250 \times 0.3 = 830$$

$$\text{For } y + 850x = 710, 455 + 850 \times 0.3 = 710$$

Thus the fixed cost must be \$455 per week and the cost per kilometre is 30 c.

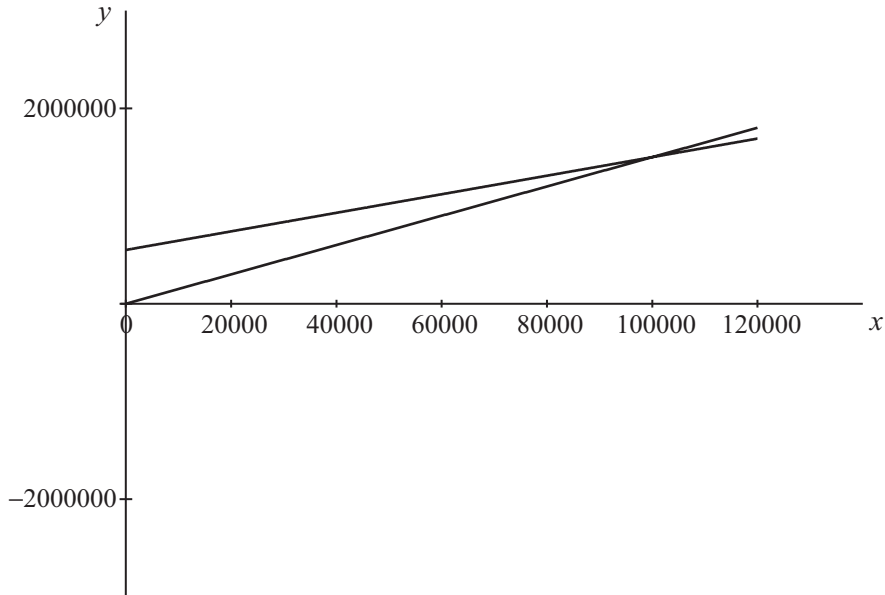
(b) Let  $y$  be the price of either production or selling (in dollars) and  $x$  the number of goods produced or sold.

$$\text{Thus the production costs are } y = 550000 + 9.50x,$$

$$\text{The selling returns are } y = 15x$$

For the break even point the dollars spent on production must be the same as the dollars received from sales. We need to graph the two functions above and solve for  $x$ , the number of units produced or sold.

The graphs of the two functions are



The selling price and the production price are the same when  $x = 100\,000$ .

To check algebraically the accuracy of this value

$$550000 + 9.50x = 15x, \text{ when } x = 100\,000.$$

$$LHS = 550000 + 9.50 \times 100000 = 1500000$$

$$RHS = 15 \times 100000 = 1500000$$

$$LHS = RHS$$

Thus the solution is correct and the break even point is 100 000 units.

#### Activity 4.12

1.

(a)  $y = x^2 + x - 12$  using the form  $y = ax^2 + bx + c$

$$a = 1, b = 1 \text{ and } c = -12,$$

$$\text{Discriminant} = b^2 - 4ac = 1^2 - 4 \times 1 \times -12 = 49$$

The discriminant is a positive number so the equation has two solutions and will cut the  $x$ -axis in two places.

(b)  $y = -x^2 - x + 6$  using the form  $y = ax^2 + bx + c$

$$a = -1, b = -1 \text{ and } c = 6,$$

$$\text{Discriminant} = b^2 - 4ac = (-1)^2 - 4 \times -1 \times 6 = 25$$

The discriminant is a positive number so the equation has two solutions and will cut the  $x$ -axis in two places.

(c)  $y = (x - 2)^2 + 1$

$$y = x^2 - 4x + 4 + 1$$

$$y = x^2 - 4x + 5$$

Using the form  $y = ax^2 + bx + c$ ,  $a = 1$ ,  $b = -4$  and  $c = 5$ 

Discriminant =  $b^2 - 4ac = (-4)^2 - 4 \times 1 \times 5 = -4$

The discriminant is a negative number so the equation has no solutions and will not cut the  $x$ -axis.

(d)  $y + 4 = x^2$

$$y = x^2 - 4$$

Using the form  $y = ax^2 + bx + c$ ,  $a = 1$ ,  $b = 0$  and  $c = -4$ 

Discriminant =  $b^2 - 4ac = (0)^2 - 4 \times 1 \times -4 = 16$

The discriminant is a positive number so the equation has two solutions and will cut the  $x$ -axis in two places.

2.

(a) You should firstly notice that this is an upward opening parabola.

To find the  $y$  intercept put  $x = 0$  in the equation  $y = x^2 + x - 12$ 

$$y = 0 + 0 - 12 = -12$$

 $y$  intercept is  $(0, -12)$ To find the  $x$ -intercept put  $y = 0$  in the equation  $y = x^2 + x - 12$ 

$$x^2 + x - 12 = 0$$

$$(x + 4)(x - 3) = 0$$

$$x = -4, x = 3$$

 $x$ -intercepts are  $(-4, 0)$  and  $(3, 0)$ The turning point will be midway between the  $x$ -intercepts or at the point  $x = -\frac{b}{2a}$  or $x = -\frac{1}{2}$ . The  $y$ -coordinate of the turning point is found by substituting  $x = -\frac{1}{2}$  into the original equation

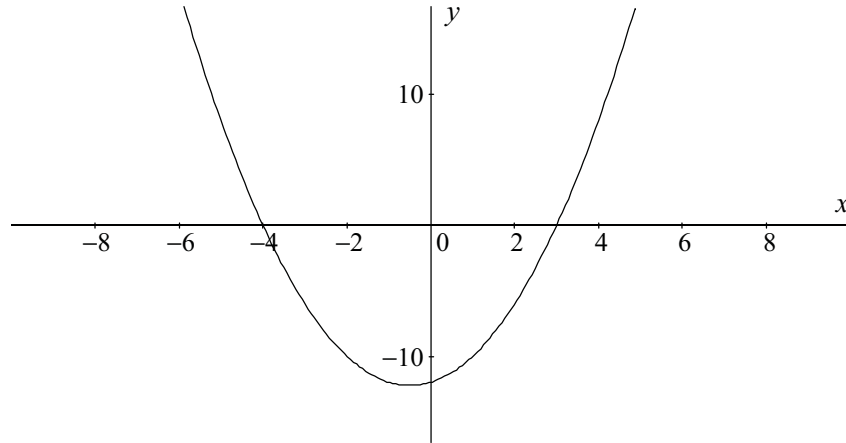
$$y = x^2 + x - 12$$

$$y = \left(-\frac{1}{2}\right)^2 + -\frac{1}{2} - 12$$

$$= -12\frac{1}{4}$$

The turning point is  $\left(-\frac{1}{2}, -12\frac{1}{4}\right)$

Using these four points we can now sketch the curve,  $y = x^2 + x - 12$



(b) You should notice this is a downward opening parabola.

To find the  $y$  intercept put  $x = 0$  in the equation  $y = -x^2 - x + 6$

$$y = -0 - 0 + 6 = 6$$

$y$ -intercept is  $(0, 6)$

To find the  $x$ -intercept put  $y = 0$  in the equation  $y = -x^2 - x + 6$

$$-x^2 - x + 6 = 0$$

$$-(x^2 + x - 6) = 0$$

$$x^2 + x - 6 = 0$$

$$(x + 3)(x - 2) = 0$$

$$x = -3, x = 2$$

$x$ -intercepts are  $(-3, 0)$  and  $(2, 0)$

The turning point will be midway between the  $x$ -intercepts or at the point  $x = -\frac{b}{2a}$  or

$x = -\frac{-1}{2 \times -1} = -\frac{1}{2}$ . The  $y$ -coordinate of the turning point is found by substituting

$x = -\frac{1}{2}$  into the original equation

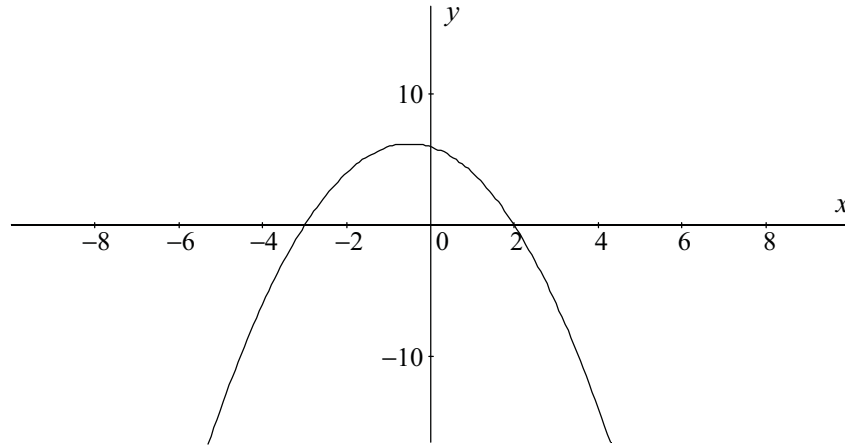
$$y = -x^2 - x + 6$$

$$y = -\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) + 6$$

$$= 6\frac{1}{4}$$

The turning point is  $\left(-\frac{1}{2}, 6\frac{1}{4}\right)$

Using these four points we can now sketch the curve,  $y = -x^2 - x + 6$



(c) This will be an upward opening parabola.

To find the  $y$  intercept put  $x = 0$  in the equation  $y = (x - 2)^2 + 1$

$$y = (0 - 2)^2 + 1 = 5$$

$y$ -intercept is  $(0, 5)$

To find the  $x$ -intercept put  $y = 0$  in the equation  $y = (x - 2)^2 + 1$

$$(x - 2)^2 + 1 = 0$$

$$x^2 - 4x + 4 + 1 = 0$$

$$x^2 - 4x + 5 = 0$$

Since the discriminant = 0 there are no solutions and the graph does not cut the  $x$ -axis.

The turning point will be at the point  $x = -\frac{b}{2a}$  or  $x = -\frac{-4}{2 \times 1} = 2$ . The  $y$ -coordinate of

the turning point is found by substituting  $x = 2$  into the original equation

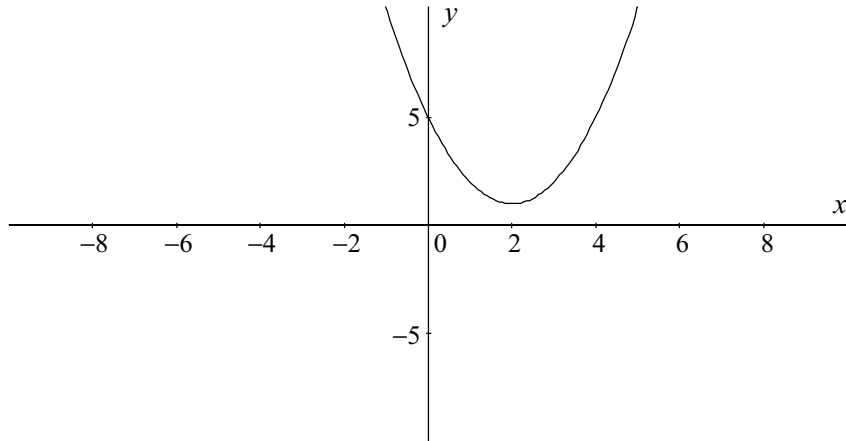
$$y = x^2 - 4x + 5$$

$$y = (2)^2 - 4 \times 2 + 5$$

$$= 1$$

The turning point is  $(2, 1)$

Using these two points we can now sketch the curve.



- (d) This will be an upward opening parabola.  
To find the  $y$ -intercept put  $x = 0$  in the equation  $y + 4 = x^2$

$$y + 4 = (0)^2$$

$$y = -4$$

$y$  intercept is  $(0, -4)$

To find the  $x$ -intercept put  $y = 0$  in the equation  $y + 4 = x^2$

$$y + 4 = x^2$$

$$0 + 4 = x^2$$

$$x^2 = 4$$

$$x = \pm 2$$

$x$  intercepts are  $(-2, 0)$  and  $(2, 0)$

The turning point will be midway between the  $x$ -intercepts or at the point  $x = -\frac{b}{2a}$  or

$x = -\frac{0}{2 \times 1} = 0$ . The  $y$ -coordinate of the turning point is found by substituting  $x = 0$

into the original equation

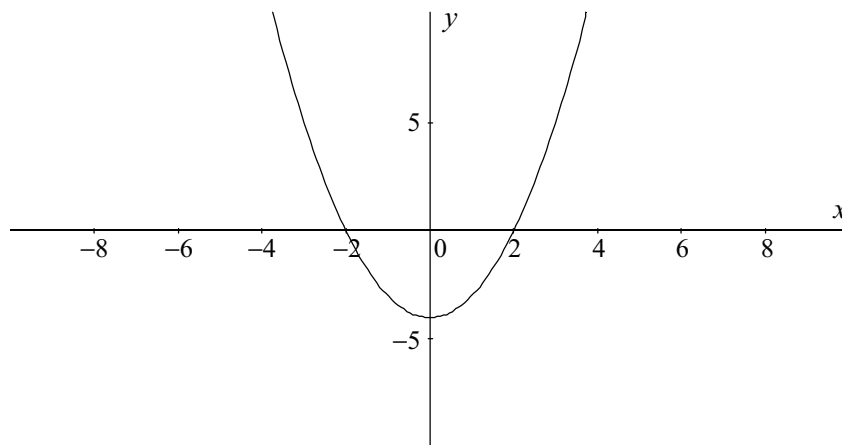
$$y + 4 = x^2$$

$$y + 4 = (0)^2$$

$$y = -4$$

The turning point is  $(0, -4)$ . This is the same point as the  $y$ -intercept.

Using these three points we can now sketch the curve.



3. Notice that this will be a downward opening parabola. The  $F$  variable will be on the vertical axis and  $t$  on the horizontal axis. To find the  $F$ -intercept put  $t = 0$  in the equation
- $$F = 290(8t - 0.5t^2 - 1.4)$$

$$F = 290(8t - 0.5t^2 - 1.4)$$

$$F = 290(8 \times 0 - 0.5 \times 0^2 - 1.4)$$

$$F = -406$$

$F$ -intercept is  $(0, -406)$

To find the  $t$ -intercept put  $F = 0$  in the equation  $F = 290(8t - 0.5t^2 - 1.4)$

$$290(8t - 0.5t^2 - 1.4) = 0$$

$$8t - 0.5t^2 - 1.4 = 0$$

$$-0.5t^2 + 8t - 1.4 = 0$$

Using the quadratic formula we can solve this equation to get  $t \approx 0.2$  or  $t \approx 15.8$   
 $t$ -intercepts are  $(0.2, 0)$  and  $(15.8, 0)$

The turning point will be midway between the  $t$ -intercepts or at the point  $t = -\frac{b}{2a}$  or

$t = -\frac{8}{2 \times -0.5} = 8$ . The  $F$ -coordinate of the turning point is found by substituting  $t = 8$  into the original equation

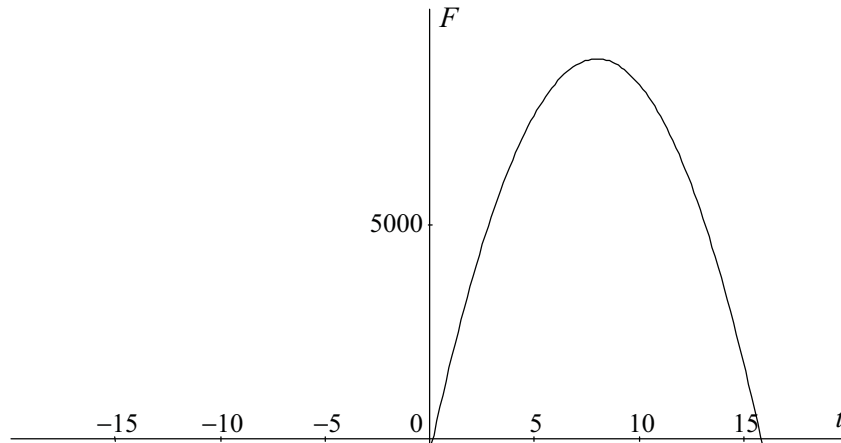
$$F = 290(8t - 0.5t^2 - 1.4)$$

$$F = 290(8 \times 8 - 0.5 \times 8^2 - 1.4)$$

$$= 8874$$

The turning point is  $(8, 8874)$

Using these four points we can now sketch the curve.



From the graph it appears that the maximum tension occurs after 8 seconds.

#### Activity 4.13

1.

(a) The average rate of change is the same as finding the gradient between two points.

(i) Between  $x = 1$  and  $x = 3$ , gradient  $= \frac{0 - -8}{3 - 1} = 4$

(ii) Between  $x = 4$  and  $x = 5$ , gradient  $= \frac{16 - 7}{5 - 4} = 9$

(iii) Between  $x = -1$  and  $x = -3$ , gradient  $= \frac{0 - -8}{-3 - -1} = -4$

(iv) Between  $x = -4$  and  $x = -5$ , gradient  $= \frac{16 - 7}{-5 - -4} = -9$

(b) On the right hand side of the parabola the rate of change is positive and increasing (getting steeper) as the function increases. On the left hand side of the parabola the rate of change is negative and increasing (getting steeper) as the function decreases.

2.

- (a) For
- $f(x) = x^2 - x - 6$
- , calculate the values at the three points indicated.

$$f(0) = -6$$

$$f(1) = -6$$

$$f(2) = -4$$

$$\frac{f(1) - f(0)}{1 - 0} = \frac{-6 - -6}{1} = 0$$

$$\frac{f(2) - f(1)}{2 - 1} = \frac{-4 - -6}{1} = 2$$

The gradient increases from 0 to 2. When the average gradient is zero it must be in close vicinity to the turning point.

- (b) For
- $g(x) = 2x^2 - 3x + 4$
- , calculate the values at the three points indicated

$$g(0) = 4$$

$$g(1) = 3$$

$$g(2) = 6$$

$$\frac{g(1) - g(0)}{1 - 0} = \frac{3 - 4}{1} = -1$$

$$\frac{g(2) - g(1)}{2 - 1} = \frac{6 - 3}{1} = 3$$

Between the first two points the gradient is negative and the function is decreasing. Between the next two point the gradient is positive and the function is increasing. A turning point must lie between 0 and 2.

- (c) For
- $h(x) = -x^2 + x + 6$
- , calculate the values at the three points indicated.

$$h(0) = 6$$

$$h(1) = 6$$

$$h(2) = 4$$

$$\frac{h(1) - h(0)}{1 - 0} = \frac{6 - 6}{1} = 0$$

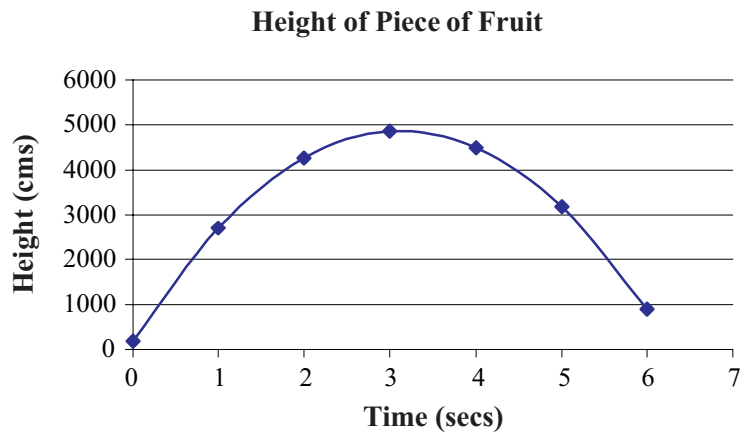
$$\frac{h(2) - h(1)}{2 - 1} = \frac{4 - 6}{2 - 1} = -2$$

Between the first two points the gradient is zero and we must thus be close to a turning point.

Between the next two points the gradient is negative and the function is decreasing.

3.

(a)



(b) The gradients between each of the points are detailed below

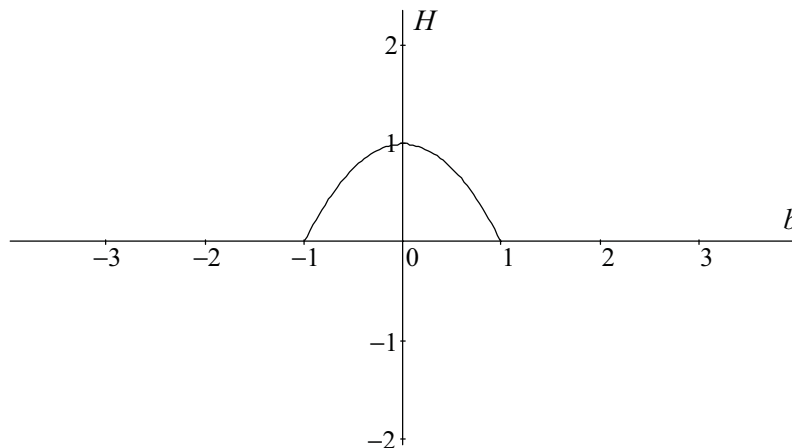
Between points	Gradient
t = 0, and t = 1	2520
t = 1, and t = 2	1560
t = 2, and t = 3	600
t = 3, and t = 4	-360
t = 4, and t = 5	-1320
t = 5, and t = 6	-2280

(c) Between 0 and 3 the function is increasing and has a positive rate of change or gradient.

Between 3 and 6 the function is decreasing and has a negative gradient. The turning point must occur in the close vicinity of three.

4.

(a)



(b) Let  $f(b) = H = -b^2 + 1$

Steepness or the gradient is

$$\begin{aligned} & \frac{f(1) - f(0.75)}{1 - 0.75} \\ &= \frac{0 - 0.4375}{0.25} \\ &= -1.75 \end{aligned}$$

This means that for every kilometre you walk in horizontal distance you go up 1.75 km.

(c) At the peak the steepness is

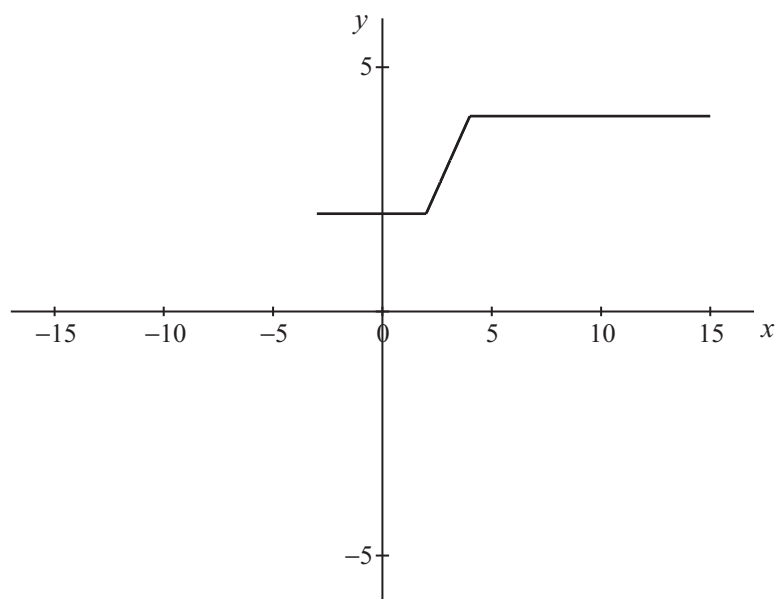
$$\begin{aligned} & \frac{f(0.25) - f(0)}{0.25 - 0} \\ &= \frac{0.9375 - 1}{0.25} \\ &= -0.25 \end{aligned}$$

For every kilometre you walk in the horizontal you go up 0.25 km.

(d) If you look at the graph the mountain is steeper at the base than at the peak. Usually functions are viewed from the origin out. If we look at this function from this perspective then it is decreasing hence the negative sign to the gradients on the right hand side of the parabola.

**Activity 4.14**

1.



2.

$$\begin{aligned} \text{(a) } f(x) &= -x, x \leq 0 \\ &= x, 0 < x \leq 10 \\ &= 10, x > 10 \end{aligned}$$

$$\begin{aligned} \text{(b) } f(x) &= 4, x \leq -2 \\ &= x^2, -2 < x < 2 \\ &= 4, x \geq 2 \end{aligned}$$

## Solutions to a taste of things to come

1.

- (a) Using the point slope form of a straight line,  $y = mx + b$ , we can substitute the two sets of points into this equation to determine the values of  $m$  and  $b$  by solving the equation simultaneously.

When  $(15, 76.45)$  the equation is  $76.45 = 15m + b$

When  $(100, 76.56)$  the equation is  $76.56 = 100m + b$

Write the first equation as  $b = 76.45 - 15m$

Write the second equation as  $b = 76.56 - 100m$

Thus

$$76.45 - 15m = 76.56 - 100m$$

$$85m = 0.11$$

$$m \approx 0.00129$$

Substituting back into one of the original equation we get,

$$b = 76.45 - 15 \times 0.00129 = 76.43$$

So the equation of the straight line is  $l = 0.00129t + 76.43$

- (b) The gradient of the function is 0.00129. It is the rate of change of length with respect to temperature, so that when the temperature changes by 1 degree the length of the copper rod changes by 0.00129 cm.

- (c) If the original function is  $l = f(t) = 0.00129t + 76.43$  then the inverse will be

$$f^{-1}(l) = \frac{l - 76.43}{0.00129}. \text{ This would be useful as you could use it to determine the}$$

temperature if a change in the length of the copper rod was detected... possibly another type of thermometer.

2.

- (a) A is when all resources are put into producing goods and nothing is put into pollution protection by producing scrubbers. B is when all resources are put into producing pollution protection and no goods are produced.

- (b) The rate represents the gradients of the straight lines between each of the points as follows,

Points	Gradient
A to B	-0.15
C to D	-0.50
E to F	-1.1
G to H	-1.3

- (c) The rate represents the change in the bundles of good produced for a change in scrubbers produced. So that when the rate is  $-1.1$  for each thousand scrubbers introduced we would have to produce 1.1 million bundles of goods if all resources were to be utilized.
- (d) The gradient is negative because the function relating scrubbers to bundles of goods is a decreasing function at those points.

## Solutions to post-test

1.

- (a) This is not a function because when  $x = 1$  it can take two possible values of  $y$ ,  $y = 2$  and  $y = 7$ .

$$\text{Domain: } -1 \leq x \leq 2$$

$$\text{Range: } -1 \leq y \leq 1$$

- (b) This is a function because for each value of  $x$  on the graph there is only one value of  $y$ .

$$\text{Domain: } \text{all real values of } x$$

$$\text{Range: } y \leq 1$$

- (c) This is a function because for each value of number of days (the independent variable) there is only one possible cost.

$$\text{Domain: } \text{number of days} \geq 0$$

$$\text{Range: } \text{costs} \geq \$40$$

- (d) This is a function because for each value of  $x$  there is only one value for  $y$ .

$$\text{Domain: } x \leq 1$$

$$\text{Range: } y \geq 0$$

2.

(a)

$$(i) f(x) = -\frac{1}{2}x + 1$$

To find the  $f(x)$ -intercept put  $x = 0$ ,  $f(0) = 1$

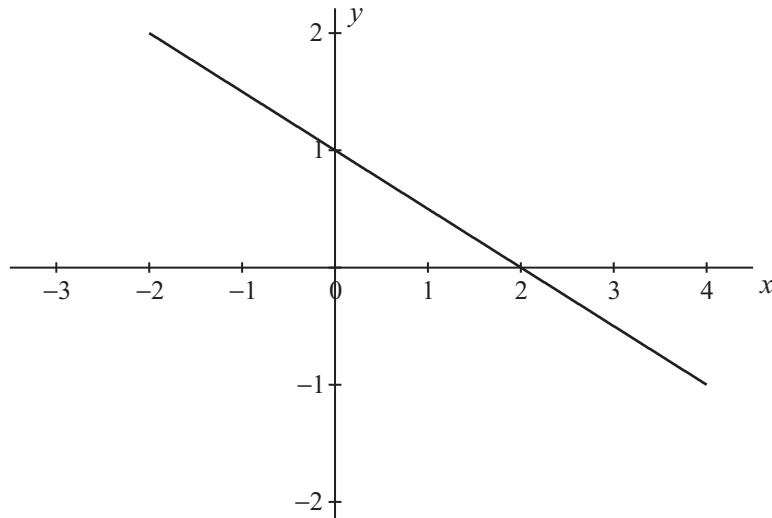
To find the  $x$ -intercept put

$$f(x) = 0$$

$$-\frac{1}{2}x + 1 = 0$$

$$x = 2$$

This is a straight line and has no possible turning points. The graph is below.



(ii)  $A = 6B^2 - 5B - 4$

To find the  $A$ -intercept put  $B = 0$ ,  $A = -4$

To find the  $B$ -intercept put  $A = 0$

giving the point  $(0; -4)$

$$A = 6B^2 - 5B - 4$$

$$0 = 6B^2 - 5B - 4$$

$$0 = (3B - 4)(2B + 1)$$

giving points  $(\frac{4}{3}, 0)$  and  $(-\frac{1}{2}, 0)$

$$3B - 4 = 0 \text{ and } 2B + 1 = 0$$

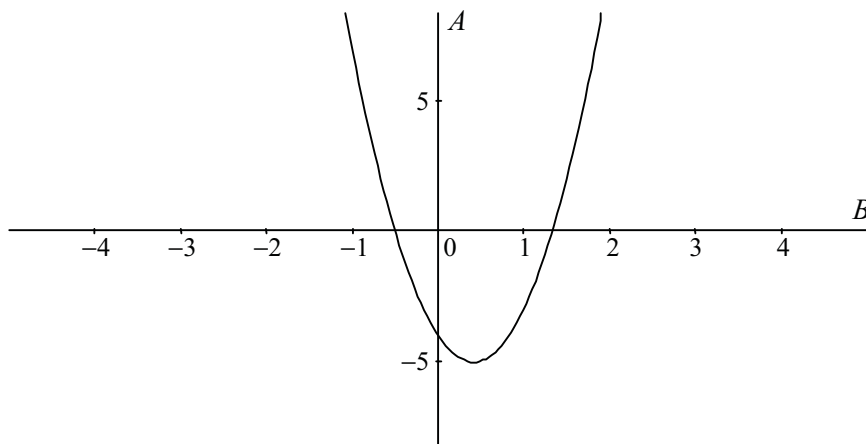
$$B = \frac{4}{3}, B = -\frac{1}{2}$$

The turning point will occur midway between the  $x$ -intercepts or at

$$B = -\frac{b}{2a} = -\frac{-5}{12} = \frac{5}{12}$$

The  $A$ -coordinate of the turning point is  $6 \times (\frac{5}{12})^2 - 5(\frac{5}{12}) - 4 \approx -5.04$

We use these four points to sketch the graph.



(b) To find the inverse of the function undo the processes, that is divide by 2 and add a third. The inverse function is  $f^{-1}(x) = \frac{x}{2} + \frac{1}{3}$ .

3.

(a)  $g(1) = 1 + 3 - 1 = 3$

(b)  $g(-1) = 1 - 3 - 1 = -3$

(c)  $g(a + h) = (a + h)^2 + 3(a + h) - 1$

(d)  $\frac{g(1) - g(-1)}{1 - (-1)} = \frac{3 + 3}{2} = 3$

The last expression is the average rate of change of the function between  $x = 1$  and  $x = -1$ .

4.

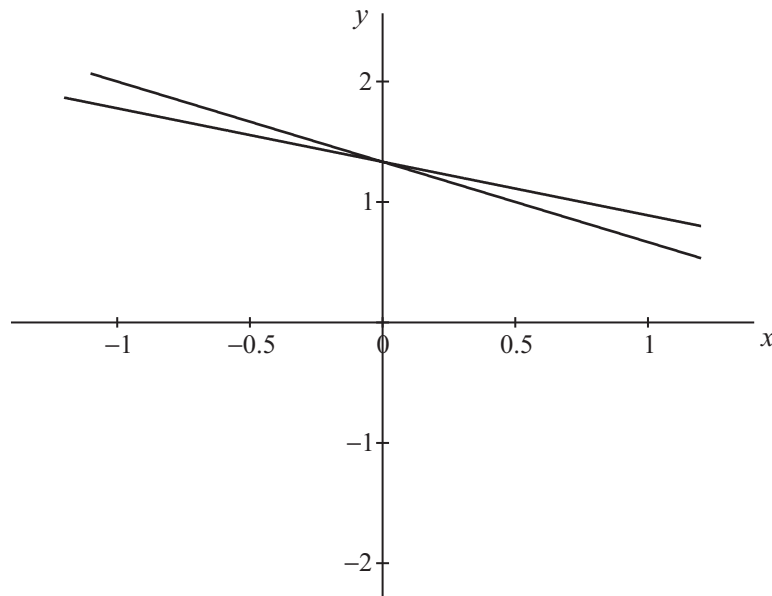
(a) The rate of change at all points in the function is 2.

(b) The original function is  $f(r) = 150 + 2r$ , where  $r$  is the cost of the raw materials in dollars. The inverse of this function is  $f^{-1}(m) = \frac{m - 150}{2}$ , where  $m$  is the cost of manufacturing the goods in dollars.

(c) The original function determined the cost of manufacturing the goods given the cost of the raw materials. The inverse function determines the cost of the raw materials if we know the cost of manufacturing the goods.

5.

(a)



The point of intersection is  $(0, \frac{4}{3})$ .

(b) To prove this is correct algebraically substitute it into both equations.

$$LHS = 2x + 3y = 2 \times 0 + 3 \times \frac{4}{3} = 4 = RHS$$

$$LHS = 9y + 4x = 9 \times \frac{4}{3} + 4 \times 0 = 12 = RHS$$

The solution satisfies both equation so must be correct.

6. The equation of the function is

$$\begin{aligned} f(x) &= x, \quad x < 0 \\ &= 0.5x, \quad 0 \leq x \leq 2 \\ &= 1, \quad x > 2 \end{aligned}$$

7. To determine if a parabolic function crosses the horizontal axis we examine the value of the discriminant,  $b^2 - 4ac$ .

(a) For  $y = 2x^2 + 5x - 3$ ,  $b^2 - 4ac = 25 - 4 \times 2 \times -3 = 49$ . This is a positive non zero number so the function must cut the horizontal axis twice.

(b) For  $A = 4B^2 - 4B + 1$ ,  $b^2 - 4ac = 16 - 4 \times 4 \times 1 = 0$ . This is zero so the function must cut the horizontal axis once.

(c) For  $Z = s^2 + s - 1$ ,  $b^2 - 4ac = 1 - 4 \times 1 \times -1 = 5$ . This is a positive number so the function must cut the horizontal axis twice.

(d) To sketch a parabola it is useful to know the  $x$  and  $y$  intercepts and the turning point.

To find the  $y$ -intercept put  $x = 0$  to give us  $y = -3$  giving the point  $(0, -3)$

To find the  $x$ -intercepts put  $y = 0$

$$2x^2 + 5x - 3 = 0$$

$$(2x - 1)(x + 3) = 0$$

$$x = \frac{1}{2}, x = -3$$

giving the points  $(\frac{1}{2}, 0)$  and  $(-3, 0)$

The turning point is midway between the  $x$  intercepts at  $-1\frac{1}{4}$ , the corresponding

$$y\text{-coordinate of the turning point is } 2 \times \left(-\frac{5}{4}\right)^2 + 5 \times -\frac{5}{4} - 3 = -6\frac{1}{8}$$

The graph is

