Module 1

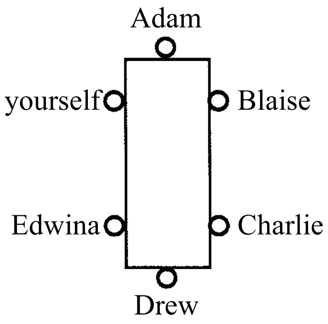
DISCRETE MATHEMATICS

Introduction

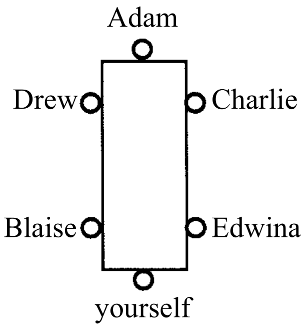
Discrete mathematics is concerned with functions and relationships which are defined only for the positive integers.

1.1 Factorials

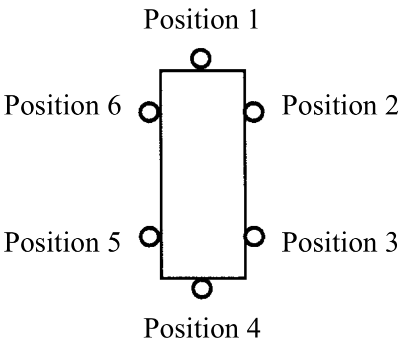
Suppose you are organising a dinner party for six people and you are interested in the number of different ways you can arrange the seating order of the guests (and yourself). One order may be:



while another could be



Obviously there are many ways to fill the six seats at the table. Consider the positions at the table numbered as shown below:



For your dinner party there are 6 different ways to fill Position 1 (Adam or Blaise or Charlie or Drew or Edwina or yourself). Once a person has been allocated to Position 1, there are five people from which an allocation to Position 2 can be made. Then there are four people left to make a choice for Position 3 etc. until there is only one person left to be allocated to Position 6.

The number of different seating arrangements for six people is:

$$\underset{\substack{\uparrow\\\text{six people to}\\\text{choose from}\\\text{for Position 1}}}{6} \times \overset{\substack{\text{five people to}\\\text{choose from for}\\\text{Position 2}\\\downarrow}}{5} \times 4 \times 3 \times 2 \times \underset{\substack{\uparrow\\\text{1 person to}\\\text{‘choose from’}\\\text{for Position 6}}}{1}$$

i.e. there are 720 different ways to seat the six people at the dinner table.

The mathematical expression for this sort of counting i.e. counting the number of ways to arrange six objects **without** replacement is factorial six.

The symbol used is 6!

In general for any positive integer n,

$$n! = n \times (n - 1) \times (n - 2) \times \ldots \times (1).$$

Note that 0! is defined to be 1 and 1! is defined to be 1.

On the ‘Casio *fx* – 82 LB fraction’ calculator the factorial key is the back function on the **M**emory **R**ecall key.

Make sure you can find the factorial key on your calculator.

When dealing with problems which involve counting you must use common sense. Often, very simple mathematics is all that is required.

✪ How many different six digit telephone numbers are there? (Assume zero cannot occupy the first position.)

Your estimate? . . . .

Answer:

One way of doing this is to recognise that we are dealing with permutations and replacement is allowed.

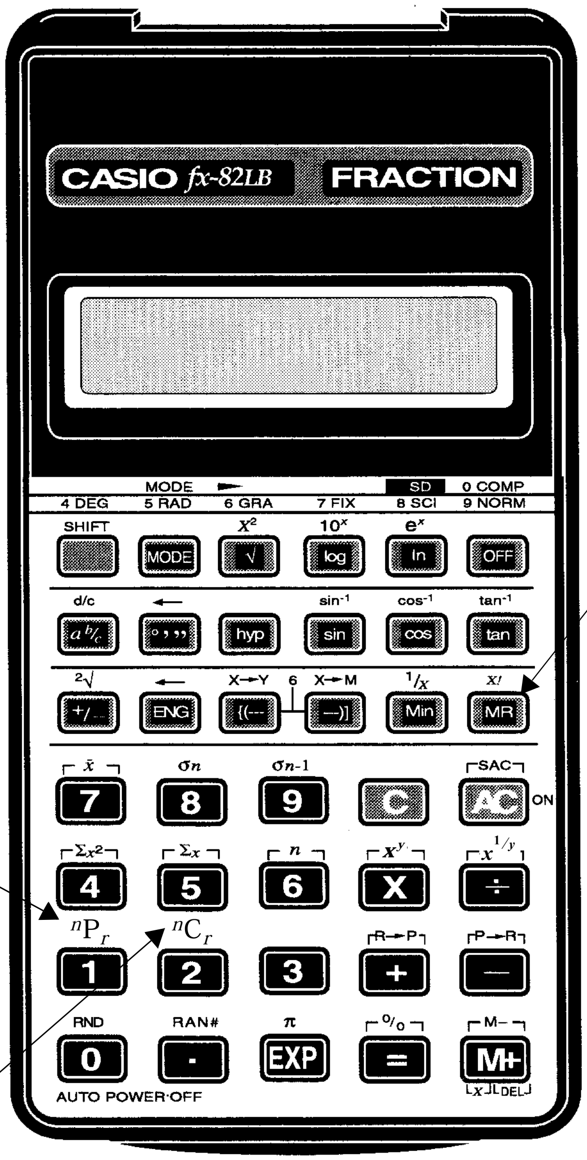
No. of ways of selecting first digit $= 9$ $\quad \begin{Bmatrix} 1, 2, 3, 4, 5, 6, 7, 8 \text{ or } 9 \\ \text{zero is excluded} \end{Bmatrix}$

No. of ways of selecting second digit $= 10$

$\vdots \qquad \vdots$

No. of ways of selecting sixth digit $= 10$

$$\therefore \text{ Total number of permutations} = 9 \times 10 \times 10 \times 10 \times 10 \times 10$$
$$= 9 \times 10^5 = 900\,000$$



Factorial key

$\boxed{x!} = x \times (x - 1) \times (x - 2) \ldots \times (1$

No. of Permutations

key $\boxed{^nP_r} = \frac{n!}{r!}$

No. of Combinations

key $\boxed{^nC_r} = \frac{n!}{r!(n-r)!}$

Using factorials is very convenient for many instances which involve counting (generally called **combinatorics**).

Combinations

✪ Suppose you only want three of your five friends to come to dinner how many different combinations of guests can you get? (As you will always be there don't include yourself in your calculation.)

Your estimate is: ....

How did you do it? ....

....

....

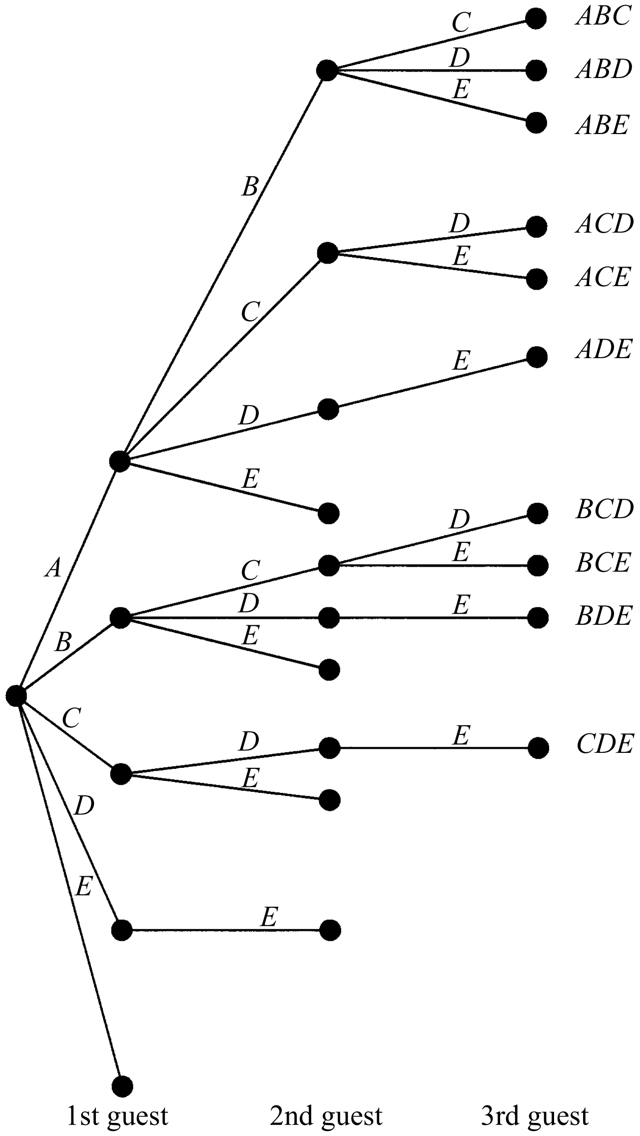
Answer:

One way to determine the answer is to go through your friends and make up various dinner party lists e.g.

Adam, Blaise, Charlie (*ABC*)
Adam, Blaise, Drew (*ABD*)
⋮
Charlie, Drew, Edwina (*CDE*)

This can be represented in a tree diagram as:

See Note 1



There are ten different possible combinations of 5 people taken 3 at a time.

■

Notes

1. Note how doing things systematically (i.e. following a strategy or pattern) is beneficial.

The number of possible combinations of 5 things taken 3 at a time when order is not important is written as ${}^5C_3$ or $\binom{5}{3}$.

$${}^5C_3 = \binom{5}{3} = \frac{5!}{(3!)(5-3)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1) \times (2 \times 1)} = 10$$

(as found above by listing all the possible combinations)

In general, the number of possible combinations of n objects taken r at a time without replacement is ${}^nC_r = \binom{n}{r} = \frac{n!}{(r!)(n-r)!}$

Find the ${}^nC_r$ key on your calculator. It is the back function on the 2 digit on the Casio shown on p. 1.3

When considering combinations the order of selection is not important e.g. Charlie, Drew, Edwina make the same list of dinner guests as Drew, Charlie, Edwina or Edwina, Drew, Charlie etc. However it is often the case that the order is important e.g. number plates, telephone numbers, preferential voting, word building games. When order is important there are many more ways of selecting a subset of objects than when order is unimportant.

Permutations

Consider again five objects from which three are to be selected but this time the order is important. If no replacement is allowed there are 5 ways of filling Position 1; 4 ways of filling Position 2; and 3 ways of filling Position 3.

i.e. $5 \times 4 \times 3 = 60$ different ways

The total number of different ways when order of selection is important is called the number of permutations. The number of permutations of 5 things taken 3 at a time when order is important is written as ${}^5P_3$

$${}^5P_3 = \frac{5!}{(5-3)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(2 \times 1)} = 60$$

and in general, the number of permutations of n objects taken r at a time without replacement is ${}^nP_r = \frac{n!}{(n-r)!}$

Find the ${}^nP_r$ key on your calculator. It is the back function on the 1 digit on the Casio shown on p. 1.3

Permutations and combinations also occur when replacement or repetition is allowed.

e.g. in six digit telephone numbers

292 654 is a different number to 296 254

✪ How many different six digit telephone numbers are there? (Assume zero cannot occupy the first position.)

Your estimate? . . . .

Answer:

One way of doing this is to recognise that we are dealing with permutations and replacement is allowed.

No. of ways of selecting first digit = 9

No. of ways of selecting second digit = 10

$\vdots$ $\qquad\qquad$ $\vdots$

No. of ways of selecting sixth digit = 10

$$\therefore \text{ Total number of permutations} = 9 \times 10 \times 10 \times 10 \times 10 \times 10$$
$$= 9 \times 10^5 = 900\,000$$

Example 1.1:

A club committee consists of 12 members. Under the constitution of the club, a minimum of eight members is required for a quorum.

(i) In how many ways can a minimum quorum occur?

(ii) In how many ways can a quorum occur?

(iii) If a president, secretary and treasurer are to be elected from the committee members, how many ways can this be achieved?

Solution:

(i) Order is unimportant, so we want the number of ways of selecting 8 members from 12 members.

i.e. $\binom{12}{8} = 495$ **See Note** 1

A quorum is 8, 9, 10, 11 or 12 members. Again, the order is unimportant, so the total number of ways a quorum can occur is

$$\binom{12}{8} + \binom{12}{9} + \binom{12}{10} + \binom{12}{11} + \binom{12}{12} = 495 + 220 + 66 + 12 + 1$$
$$= 794$$

(ii) We will assume that no committee member can hold more than one executive position. The number of ways three people can be selected from 12 is

${}^{12}P_3 = 1\,320$ **See Note** 2

Notes

1. The key strokes for $\binom{12}{8}$ are $\boxed{12}$ $\boxed{{}^nC_r}$ $\boxed{8}$ $\boxed{=}$ and the display should read 495.

   (You can check this result by finding $\frac{12!}{8!4!}$).

2. The keystrokes for ${}^{12}P_3$ are $\boxed{12}$ $\boxed{{}^nP_r}$ $\boxed{3}$ $\boxed{=}$ and the display should read 1 320.

   (You can check this result by finding $\frac{12!}{9!}$).

Exercise Set 1.1

1. How many three letter permutations of MAP are there if
   (i) Repetitions are allowed
   (ii) Repetitions are not allowed

2. In some states of Australia car license plates are in the form of three single digit numbers followed by three letters of the alphabet. How many different number plates are possible using this system?

3. If there are ten applicants for four similar jobs in a company, how many different ways can the positions be filled.

4. Six people are running for election for two positions on a committee. In how many ways can the positions be filled?

5. Show that $\binom{10}{4} = \binom{10}{6}$

6. A laboratory cage contains eight white mice and six brown mice. Find the number of ways of choosing five mice from the cage if
   (i) They can be of either colour
   (ii) They must be of the same colour
   (iii) Three must be white and two must be brown

1.2 Binomial Theorem

Now that you have mastered factorials and combinations we can proceed to utilise them for expansions of algebraic expressions.

Example 1.2:

Expand $(x + 3)^2$

> *Now use your graphing package to draw the graph of $y = f(x) = (x + 3)^2$ and on the same screen draw the graph of your expansion. If you have the correct expansion, your graphs will be coincidental.*

Solution:

You should have got $(x + 3)(x + 3) = x^2 + 3x + 3x + 3^2$ {using the Distributive Law}

$= x^2 + 6x + 3^2$

■

Example 1.3:

Expand $(x + a)^4$

Solution:

You should have got $(x + a)(x + a)(x + a)(x + a)$

$= x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$

■

The RHS of this equation can be written as

$$1x^4a^0 + 4x^3a^1 + 6x^2a^2 + 4x^1a^3 + 1x^0a^4$$

What is the pattern in the powers of x? ....

What is the pattern in the powers of a? ....

What do the indices of x and a sum to in each term? ....

Answer: The power of x decreases by one for each term, while the power of a increases by one for each term. The sum of the indices is 4 for every term.

Examine the coefficient of each term. What is the pattern? [**Hint:** Think in terms of combinations.] ....

....

....

....

Answer:

Term	**Coefficient**	**Combination**	
$1x^4a^0$	1	$\begin{pmatrix}4\\4\end{pmatrix}$	4 things taken 4 at a time
$4x^3a^1$	4	$\begin{pmatrix}4\\3\end{pmatrix}$	4 things taken 3 at a time
$6.^2a^2$	6	$\begin{pmatrix}4\\2\end{pmatrix}$	4 things taken 2 at a time
$4x^1a^3$	4	$\begin{pmatrix}4\\1\end{pmatrix}$	4 things taken 1 at a time
$1x^0a^4$	1	$\begin{pmatrix}4\\0\end{pmatrix}$	4 things taken 0 at a time

■

Example 1.4:

Expand $(x + y)^3$. Find patterns in the coefficients and powers of x and y in each term

Solution:

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = 1x^3y^0 + 3x^2y^1 + 3xy^2 + 1x^0y^3$$

Term	**Coefficient**	**Combination**
x^3	1	$\begin{pmatrix}3\\3\end{pmatrix}$
$3x^2y$	3	$\begin{pmatrix}3\\2\end{pmatrix}$
$3xy^2$	3	$\begin{pmatrix}3\\1\end{pmatrix}$
y^3	1	$\begin{pmatrix}3\\0\end{pmatrix}$

Coefficients of each term are combinations of 3 things taken 3, 2, 1, 0 at a time

Power of x decreases by one for each term
Power of y increases by one for each term
The sum of the indices of x and y is 3 for each term

■

Example 1.5:

Write an expression for $(x + \mathrm{a})^5$ using the same pattern as shown in Example 1.4 above.

. . . .

. . . .

Solution:

$$(x + \mathrm{a})^5 = \binom{5}{5} x^5\mathrm{a}^0 + \binom{5}{4} x^4\mathrm{a}^1 + \binom{5}{3} x^3\mathrm{a}^2 + \binom{5}{2} x^2\mathrm{a}^3 + \binom{5}{1} x^1\mathrm{a}^4 + \binom{5}{0} x^0\mathrm{a}^5$$

$$= x^5 + 5x^4\mathrm{a} + 10x^3\mathrm{a}^2 + 10x^2\mathrm{a}^3 + 5x\mathrm{a}^4 + \mathrm{a}^5$$

■

Note that in each expansion e.g. $(x + \mathrm{a})^4$ and $(x + \mathrm{a})^5$ there is another pattern in the coefficients.

- When the power of the expansion is **odd**, there is an even number of terms in the expanded form and the coefficients of the two central terms are the same value $\left\{\binom{5}{3} = \binom{5}{2}\right\}$ and the coefficients of the terms on either side of the central terms are the same value $\left\{\binom{5}{4} = \binom{5}{4} \quad \text{and} \quad \binom{5}{5} = \binom{5}{0}\right\}$
- When the power of the expansion is **even**, there is an odd number of terms in the expanded form and the coefficient of the central term forms the 'pivot' for the symmetry of the coefficients on either side.

  $$\left\{\binom{4}{3} = \binom{4}{1} \quad \text{and} \quad \binom{4}{4} = \binom{4}{0}\right\}$$

  In general $\binom{n}{r} = \binom{n}{n-r}$

The general expansion of $(x + \mathrm{a})^n$ where n is a positive integer is given by:

$$(x + \mathrm{a})^n = \binom{n}{n} x^n\mathrm{a}^0 + \binom{n}{n-1} x^{n-1}\mathrm{a}^1 + \binom{n}{n-2} x^{n-2}\mathrm{a}^2 + \binom{n}{n-3} x^{n-3}\mathrm{a}^3 + \ldots$$

$$+ \binom{n}{3} x^3\mathrm{a}^{n-3} + \binom{n}{2} x^2\mathrm{a}^{n-2} + \binom{n}{1} x^1\mathrm{a}^{n-1} + \binom{n}{0} x^0\mathrm{a}^n$$

which is exactly the same as

$$(x + \text{a})^n = \binom{n}{0} x^n \text{a}^0 + \binom{n}{1} x^{n-1}\text{a}^1 + \binom{n}{2} x^{n-2}\text{a}^2 + \binom{n}{3} x^{n-3}\text{a}^3 + \ldots$$

$$\binom{n}{n-3} x^3\text{a}^{n-3} + \binom{n}{n-2} x^2\text{a}^{n-2} + \binom{n}{n-1} x^1\text{a}^{n-1} + \binom{n}{n} x^0\text{a}^n$$

This expression can be summarised succinctly using Sigma Notation

$$(x + \text{a})^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r}\text{a}^r$$

This is known as the Binomial Theorem. **There is no need for you to learn the expanded form of the Theorem.** Think about the patterns and remember the Sigma Notation form only.

The first term in the summation i.e. $(r = 0)$ will be $\binom{n}{0} x^n\text{a}^0$; the other terms then follow the pattern.

The second term (i.e. $r = 1$) is

$$\underset{\substack{\uparrow\\ \text{increase of 1}}}{\binom{n}{1}} \; x^{n-\overset{\substack{\text{decrease of 1}\\ \downarrow}}{1}}\text{a}^1 \longleftarrow \text{increase of 1}$$

Check that indices of x and a add to n.

$(n - 1) + 1 = n$ ✓

The third term is

$$\underset{\substack{\uparrow\\ \text{increase of 1}}}{\binom{n}{2}} \; x^{n-\overset{\substack{\text{decrease of 1}\\ \downarrow}}{2}}\text{a}^2 \longleftarrow \text{increase of 1}$$

$(n - 2) + 2 = n$ ✓

and so on until the last term is $\binom{n}{n} x^0\text{a}^n$

$n + 0 = n$ ✓

Exercise Set 1.2

1.

(a) Write down the first term in the expansion of $(x + 4)^6$

(b) Write down the remaining terms in the form of the Binomial Theorem following the pattern from (a)

(c) Calculate the coefficients on the calculator using the $\boxed{{}^nC_r}$ key

(d) Simplify the expanded terms

(e) Use your graphing package to draw the graph of $(x + 4)^6$ and your expansion. [If the graphs are not coincidental check your work.]

2.

(a) Write down the first term in the expansion of $(x - 4)^5$ [**Hint:** Care is needed with the negative 4]

(b) Write down the remaining terms in the form of the Binomial Theorem following the pattern from (a)

(c) Calculate the coefficients

(d) Simplify the expanded terms

(e) Draw the graph of $(x - 4)^5$ and your expansion. [If the graphs are not coincidental check your work]

3. Expand

(a) $(x + {}^1\!/\!_y)^4$

(b) $(2x + y^2)^3$

(c) (i) $(x^2 + {}^1\!/\quad x^2)^5$ (ii) Draw the graphs of $(x^2 + {}^1\!/\!_{x^2})^5$ and your expansion

(d) (i) $(p - 2p^3)^6$

4. Find the coefficient of x^2 and x^{-1} in the following expansions

(a) $(x^2 + {}^1\!/\!_x)^4$

(b) $({}^1\!/\!_x - x^2)^4$

5. Use the Binomial Theorem to find $(1.01)^4$ [**Hint:** Express 1.01 as $(1 + 0.01)$]

1.3 Sequence and Series

Sequences

In the Revision Module we focussed on continuous functions that generally were defined for all values of x. i.e. the set of real numbers was the domain of the function. In sequences, we are dealing with functions which are defined only for the positive integer values of x. Sequences take many forms and the one you are probably familiar with is the Fibonacci sequence which occurs commonly in nature e.g. in the breeding of rabbits, the arrangement of pine cones.

The Fibonacci sequence is obtained from an iterative process (i.e. one which builds upon itself in a regular manner). Iterative processes are used widely in computing and mathematics and often depend upon the definition of a recurrence relation. A sequence can be defined by its recurrence relation i.e. the relationship between a term and its preceding terms or by its functional relationship. Sequences are usually written as $s(n)$ to avoid confusion with $f(x)$ which is usually used for continuous functions.

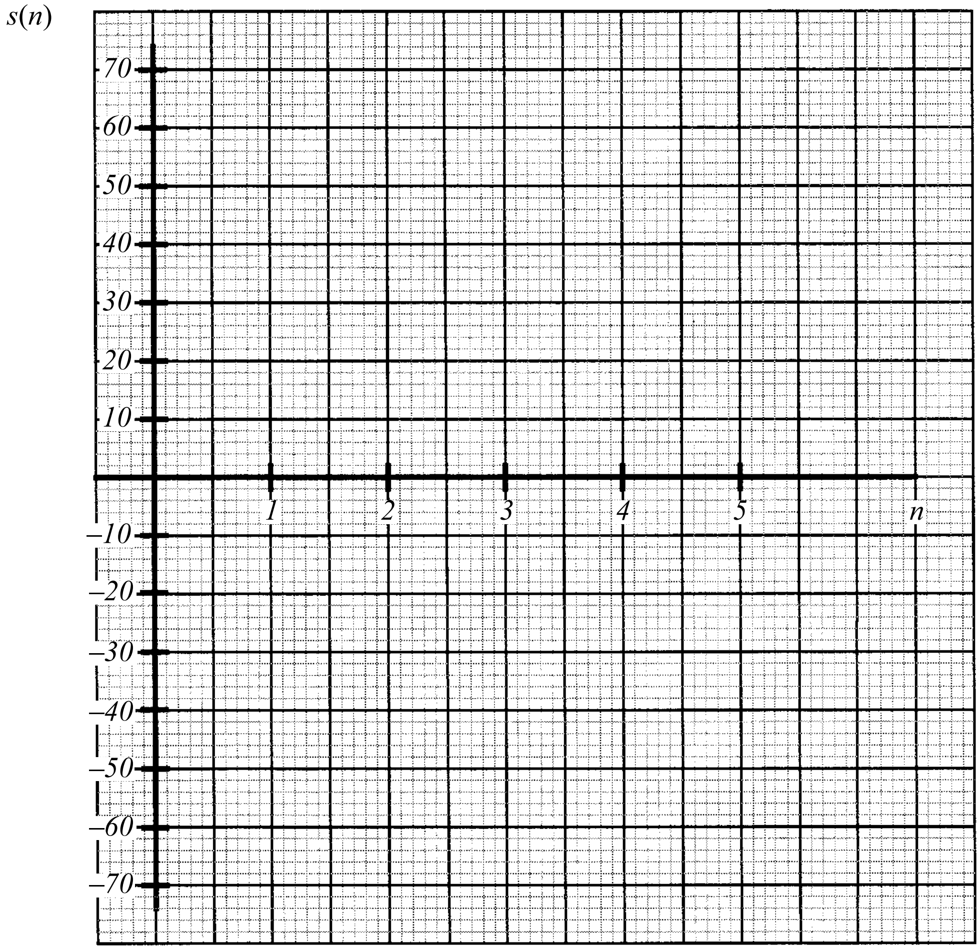
To start our work on sequences complete the following example.

Example 1.6:

Complete the following table for the sequence $s(n) = 2^n, \quad n = 1, 2, 3$

n	1	2	3	4	5	…	10	…	20	…	100
$s(n) = 2^n$	$2^1 = 2$	$2^2 = 4$	$3^3 = 8$			…		…		…	

Draw the graph of the sequence, $s(n) = 2^n$ for $1 \le n \le 5$ on the graph paper below.



You can check your graph by drawing $f(x) = 2^x$ for say $1 \le x \le 5$ on your computer. But remember that for the sequence the points at the integer x values are not joined and that it is defined for all the positive integers.

■

The sequence $s(n) = 2^n$ is an example of a divergent sequence i.e. There is no limit to the size of $s(n)$. (Look back at the values of $s(n)$ for $n = 5, 10, 20, 100$)

A divergent sequence is one where, as $n \to \infty$, $s(n) \to \infty$ (or $-\infty$).

A convergent sequence is one where, as $n \to \infty$, $s(n) \to$ some finite fixed number.

Sometimes sequences do not converge to a unique fixed number but oscillate between two values. These sequences are not convergent.

Exercise Set 1.3

1. A sequence is defined by $s(n) = (-1)^n (n^2 - 3)$. Find the first four terms and the 10th and 20th term of the sequence.
2. Draw the graph of $s(n) = (^1\!/_2)^n$

   Does this sequence converge? If so, what is the limit of $s(n)$ as $n \to \infty$?
3. Consider $s(n) = (-1)^n$.

Does this sequence converge? If so what is the limit of $s(n)$ as $n \to \infty$?

The Fibonacci sequence is obtained by starting with the numbers 1 and 1 and obtaining each new term as the sum of the two preceding terms.

If we let the nth term in this sequence be u_n, complete the following:

$u_1 = 1$ $u_2 = 1$ $u_3 = 2$ $u_4 = 3$ $u_5 = \ldots$

$u_6 = \ldots$ $u_{12} = \ldots$ $u_{15} = \ldots$

The **general term** for the Fibonacci sequence is u_n and the **recurrence relation** is

See Note 1

$$u_n = u_{n-1} + u_{n-2} \quad \text{(given that } u_1 = 1 \text{ and } u_2 = 1)$$

Notes

1. The general term for any sequence is usually called u_n.

Exercise Set 1.4

1. (a) Find the next 4 numbers in the following sequences

   (i) 2, 4, 6, 8, 10, …

   (ii) 0, 3, 8, 15, …

   (iii) 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, …

   Write down u_n, the general term for each sequence in (a)

   Find u_{20}, u_{100} and $u_{1\,000}$ for each sequence in (a)

2. (a) Find a recurrence relation for u_n for each of the following sequences.

   (i) 1, 2, 3, 4, …

   (ii) $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, …

   (iii) 1, –1, –3, –5, –7, …

   (b) Determine if each sequence is convergent or divergent. If convergent find the limit of $s(n)$ as $n \to \infty$.

Geometric Sequence

Consider the sequence whose general term u_n is given by

$$u_n = \tfrac{1}{2}\, u_{n-1}$$

Then if

$$\begin{aligned}
u_1 &= 1 \\
u_2 &= \tfrac{1}{2} \times u_1 &&= \tfrac{1}{2} \times 1 &&= \tfrac{1}{2} \\
u_3 &= \tfrac{1}{2} \times u_2 &&= \tfrac{1}{2} \times \tfrac{1}{2} &&= (\tfrac{1}{2})^2 \\
u_4 &= \tfrac{1}{2} \times u_3 &&= \tfrac{1}{2} \times (\tfrac{1}{2})^2 &&= (\tfrac{1}{2})^3 \\
&\vdots \\
u_n &= \tfrac{1}{2} \times u_{n-1} &&= \tfrac{1}{2} \times (\tfrac{1}{2})^{n-2} &&= (\tfrac{1}{2})^{n-1}
\end{aligned}$$

and we can write the function as $s(n) = (\tfrac{1}{2})^{n-1}$

This type of sequence is known as a geometric sequence and its general form is

$$\begin{aligned} u_1 &= a \\ u_2 &= ar \\ u_3 &= ar^2 \\ u_4 &= ar^3 \\ &\vdots \\ u_n &= ar^{n-1} \end{aligned}$$

where a is the first term and r is the common ratio between succeeding terms. So in the example above $a = 1$ and $r = \frac{1}{2}$.

Geometric sequences are convergent if $-1 < r < 1$

If we add say the first four terms of a sequence the result is called the fourth **partial sum** and is denoted by S_4.

So the fourth partial sum for our example above is

$$S_4 = 1 + \tfrac{1}{2} + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 = 1.875$$

In general, the nth partial sum is

$$S_n = 1 + \tfrac{1}{2} + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + \ldots + (\tfrac{1}{2})^{n-2} + (\tfrac{1}{2})^{n-1}$$

Obviously if we want to find S_{20} the calculations would be very tedious. Fortunately, for geometric sequences we can use a simple rule to find partial sums. **See Note** 1

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Let's try the rule on our example to find the fourth partial sum and the twentieth partial sum

$$\begin{aligned} S_4 &= \frac{1((\tfrac{1}{2})^4 - 1)}{\tfrac{1}{2} - 1} \\ &= \frac{1(\tfrac{1}{16} - 1)}{-\tfrac{1}{2}} \\ &= 1 \times \frac{-15}{16} \times \frac{-2}{1} \\ &= 1.875 \\ S_{20} &= \frac{1((\tfrac{1}{2})^{20} - 1)}{\tfrac{1}{2} - 1} \\ &= 1.999998093 \end{aligned}$$

Notes

1. This rule was proved in Unit 11083.

Now we know that $s(n) = (^1\!/_2)^{n-1}$ is convergent because it is a geometric sequence with $r = {}^1\!/_2$.

From examination of the partial sums above and the rule for S_n, it seems fairly reasonable to say that the 'infinite' partial sum i.e. $S_\infty = 2$.

Generally we say the 'sum to infinity of the geometric series' and we write

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(r^n - 1)}{r-1} = S_\infty = \frac{a}{1-r}$$

Note that we use the term series when we are considering the sum of an infinite number of terms of a sequence.

> *You need to be careful to use the correct symbols for the functional relationship for a sequence i.e. lower case s in s(n) and the partial sums i.e. upper case s in S_n.*

Exercise Set 1.5

1. Find the required partial sum for each sequence.

   (a) S_4 for $s(n) = 0.1^n$

   (b) S_6 for $s(n) = \dfrac{1}{2^n}$

   (c) S_{10} for $s(n) = e^{3n}$

   (d) S_5 for $s(n) = (-1)^n$

2. For each of the sequences in question 1 above

   (a) determine if it is convergent

   (b) if it is convergent find its sum to infinity if possible

---

Series

The Binomial Theorem for expanding $(x + a)^n$ that you met earlier can be extended to the case where n is not positive integer. The result of such a binomial explansion is a **series** (i.e. the sum of the terms in a sequence).

Consider the expansion of $(a + b)^n$ where a and b are real numbers and n is a positive integer.

✪ Write out the binomial expansion. ....

....

Answer:

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \ldots + \binom{n}{n} a^0 b^n$$

✪ Express the binomial coefficients as products and quotients, simplify and rewrite the expansion

. . . .

Answer:

$$(a+b)^n = a^n + na^{n-1}b^1 + \frac{n(n-1)}{2} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{6} a^{n-3}b^3 + \ldots + b^n$$

✪ Express the divisors of each term, except the first and last, as factorials (Look for a pattern)

Answer:

$$(a+b)^n = a^n + \frac{na^{n-1}b}{1!} + \frac{n(n-1)}{2!} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3}b^3 + \ldots + b^n \quad \_\_\_\_\_ \ ①$$

■

This is now a **finite** series with $(n+1)$ terms. ________________________________ **See Note** 1

If we rewrite $(a+b)$ as $a\left(1+\frac{b}{a}\right)$, **See Note** 2

$$\text{then } (a+b)^n = \left\{a\left(1+\frac{b}{a}\right)\right\}^n$$

$$= a^n\left(1+\frac{b}{a}\right)^n$$

See Note 3

So to find $(a+b)^n$ we need only find $\left(1+\frac{b}{a}\right)^n$ and then multiply by a^n.

Notes

1. 'finite' because you can count the exact number of terms.
2. Taking a as a common factor out of $(a+b)$.
3. Using the index rule $(ab)^m = a^m b^m$.

Let $x = \frac{b}{a}$, then $\left(1+\frac{b}{a}\right)^n = (1+x)^n$

and by using the form of equation ① we get

$$(1+x)^n = 1^n + \frac{n\,.\,1^{n-1}x}{1!} + \frac{n(n-1)\,.\,1^{n-2}x^2}{2!} + \frac{n(n-1)(n-2)\,.\,1^{n-3}x^3}{3!} + \dots + x^n$$

$$\therefore\ (1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots + x^n \qquad \underline{\hspace{4cm}}$$ ②

So, in general, when n is a **positive integer** $(1+x)^n$ can be written as a **finite series** and thus has a finite sum.

There are interesting outcomes when n is not a positive integer. We will consider first when n is a positive fraction, say $n = 1/2$. When $n = 1/2$. Equation ② becomes

$$(1+x)^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}x}{1!} + \frac{\frac{1}{2}(\frac{1}{2}-1)x^2}{2!} + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)x^3}{3!} + \dots$$

$$= 1 + \frac{\frac{1}{2}x}{1!} \times \frac{\frac{1}{2}\times -\frac{1}{2}x^2}{2!} + \frac{\frac{1}{2}\times -\frac{1}{2}\times -\frac{3}{2}x^3}{3!} + \dots$$

$$= 1 + \frac{1}{2}x - \frac{\frac{1}{4}x^2}{2!} + \frac{\frac{3}{8}x^3}{3!} + \dots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots$$

This is an **infinite** series, whose sum is finite providing $-1 < x < 1$ **See Note** 1

The condition $-1 < x < 1$ ensures that the later terms in the series get progressively smaller.

Now consider $(1+x)^n$ when $n = -1$.

In this case, Equation ② becomes

$$(1+x)^{-1} = 1 + \frac{(-1)x}{1!} + \frac{(-1)(-1-1)x^2}{2!} + \frac{(-1)(-1-1)(-1-2)x^3}{3!} + \dots \qquad \underline{\hspace{4cm}}$$ ③

i.e. $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$

again we can see that if $-1 < x < 1$, this infinite series will have a finite sum.

Notes

1. infinite because you cannot count the exact number of terms.

Example 1.7:

Use Equation ③ to write $(1-x)^{-1}$ as an infinite series in simplest form using sigma notation.

Solution:

$$\begin{aligned}(1-x)^{-1} &= (1+-x)^{-1}\\ &= 1+\frac{(-1)(-x)}{1!}+\frac{(-1)(-1-1)(-x)^2}{2!}+\frac{(-1)(-1-1)(-1-2)(-x)^3}{3!}+\ldots\\ &= 1+x+x^2+x^3+\ldots\\ &= \sum_{n=0}^{\infty} x^n\end{aligned}$$

■

$(1-x)^{-1}$ is so common it is called **THE** geometric series. It occurs very frequently in engineering, statistics etc. **See Note** 1

It has a finite sum $\frac{1}{x-1}$ when $-1 < x < 1$.

So when $x = \frac{1}{2}$, $1 + x + x^2 + x^3 + \ldots = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

when $x = \frac{1}{4}$, $1 + x + x^2 + x^3 + \ldots = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$

Notes

1. Compare this geometric series with the geometric sequence on page 1.17.

1.4 Mathematical Induction

The principle of mathematical induction is used widely to prove statements where there are no suitable theorems to support the proof. We can use it to prove the sum of a series is a certain finite value, once we know the general form of the series.

Suppose you are asked to add the numbers from 1 to 1 000 – obviously a tedious task!

$1 + 2 + 3 + 4 + 5 + \ldots + 997 + 998 + 999 + 1\,000 = ?$

It turns out that this series sums to $\frac{1\,000 \times 1\,001}{2} = 500\,500$

In general the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

To prove this statement (or any other) using induction we use the following procedure

1. **Show** that the statement is true for $n = 1$

2. **Assume** that the statement is true for any value of n, say k

3. **Verify** that, under the assumption in Step 2, the statement is true for the next value of n i.e. for $n = (k + 1)$

If Step 3 is successful, then the statement must be true for $n = 2$, $n = 3$, $n = k + 2$, $n = k + 3, \ldots$ etc (because k could be any number). i.e. we can conclude that the statement is true for all n.

Example 1.8:

Prove that $S_n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ **See Note** 1

Solution:

Step 1: **Show** statement is true for $n = 1$

When $n = 1$, $S_n = S_1 = 1$ and $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = 1$

$\therefore$ statement is true for $n = 1$

Step 2: **Assume** statement is true for $n = k$

When $n = k$, $S_n = S_k = 1 + 2 + 3 + \ldots + k$ and $\frac{n(n+1)}{2} = \frac{k(k+1)}{2}$

$\therefore$ the assumption is: $1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$ ______________ ①

Notes

1. Again do not confuse $s(n)$ and S_n. $s(n)$ refers to the functional relationship which shows how each s value depends on an n value while S_n is the sum of the first n terms of the sequence.

Step 3: **Verify** the statement is true for $n = (k + 1)$, using the assumption in Step 2

When $n = (k + 1)$

$S_n = S_{k+1} = 1 + 2 + 3 + \ldots + k + (k + 1)$ and

$$\frac{n(n+1)}{2} = \frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_} ②$$

Now $1 + 2 + 3 + \ldots + k = \dfrac{k(k+1)}{2}$ (From assumption in Step 2 i.e. equation ①)

$\therefore$ We can substitute in equation ② for most of its LHS, (for the first k terms) and get

$$S_{k+1} = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$ (Taking the common factor (RH) out)

$= \text{RHS of equation } ②$

Thus, $S_n = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$ is true for any positive integer value of n.

■

Instead of starting at $n = 1$, some induction arguments start at another positive integer. The steps involved are the same, except in Step 1 you show that the statement is true for your selected starting point.

Example 1.9:

Show that $S_n = \dfrac{2}{3^1} + \dfrac{2}{3^2} + \dfrac{2}{3^3} + \ldots + \dfrac{2}{3^n} = 1 - \dfrac{1}{3^n}$

Solution:

Step 1: **Show** statement is true for $n = 1$

When $n = 1$, $S_n = S_1 = \dfrac{2}{3^1} = \dfrac{2}{3}$ and $1 - \dfrac{1}{3^n} = 1 - \dfrac{1}{3^1} = \dfrac{2}{3}$

$\therefore$ statement is true for $n = 1$

Step 2: **Assume** statement is true for $n = k$

When $n = k$, $S_n = S_k = \dfrac{2}{3} + \dfrac{2}{3^2} + \dfrac{2}{3^3} + \ldots + \dfrac{2}{3^k}$ and $1 - \dfrac{1}{3^n} = 1 - \dfrac{1}{3^k}$

$\therefore$ the assumption is:

$$\frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \ldots + \frac{2}{3^k} = 1 - \frac{1}{3^k} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_} ①$$

Step 3: **Verify** the statement is true for $n = (k + 1)$, using the assumption in Step 2.

When $n = (k + 1)$

$$S_n = S_{k+1} = \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \ldots + \frac{2}{3^k} + \frac{2}{3^{(k+1)}} \quad \text{and} \quad 1 - \frac{1}{3^n} = 1 - \frac{1}{3^{(k+1)}} \quad \_\_\_\_ ②$$

Using the assumption from Step 2, gives $\frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \ldots + \frac{2}{3^k} = 1 - \frac{1}{3^k}$

∴ Substituting for most of LHS of equation ② gives

$$\begin{aligned} S_{k+1} &= 1 - \frac{1}{3^k} + \frac{2}{3^{(k+1)}} \\ &= 1 - \frac{3}{3^{(k+1)}} + \frac{2}{3^{(k+1)}} \\ &= 1 - \frac{1}{3^{(k+1)}} \\ &= \text{RHS of equation ②} \end{aligned}$$

See Note 1

Thus, $S_n = \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \ldots \frac{2}{3^n} = 1 - \frac{1}{3^n}$ for all positive integer values of n.

Notes

1. Make sure you can show that $\frac{1}{3^k} = \frac{3}{3^{(k+1)}}$.

Exercise Set 1.6

Use the technique of mathematical induction to prove the following

1. $\frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots + (\frac{1}{2})^n \quad = 1 - (\frac{1}{2})^n$
2. $1 + x + x^2 + x^3 + \ldots + x^n \quad = \dfrac{1 - x^{n+1}}{1 - x} \quad$ (if $x \neq 1$)

   (Compare this result with the formula for the partial sum of a geometric sequence)
3. The sum of the squares of the first n integers is $\dfrac{n(n+1)(2n+1)}{6}$
4. $(s + t) + (s + 2t) + (s + 3t) + \ldots + (s + nt) \quad = n\left(s + \dfrac{n+1}{2}t\right)$

Solutions to Exercise Sets

Solutions Exercise Set 1.1 page 1.8

1. (i) If repetitions are allowed there are 3 letters available for the first position, 3 letters available for the second position and 3 letters available for the third position

   $\therefore$ no. of permutations $= 3 \times 3 \times 3 = 3^3 = 27$

   (ii) If repetitions are not allowed there are 3 letters available for the first position, 2 letters available for the second position and 1 letter available for the third position.

   $\therefore$ no. of permutations $= 3 \times 2 \times 1 = 6$

   Note: ${}^nP_r = {}^3P_3 = \dfrac{3!}{(3-3)!} = \dfrac{3!}{0!} = \dfrac{3!}{1} = 6$

2. There are 10 single digits (0, 1, 2, 3, …, 9). There are 10 ways to fill the first position of the numbers, 10 ways to occupy the second position and 10 ways to occupy the third position.

   $\therefore$ The numbers on the plates can be chosen in $10 \times 10 \times 10 = 10^3 = 1\,000$ ways.

   Each of the sets of numbers can be followed by 3 letters from a possible $26 \times 26 \times 26 = 26^3 = 17\,576$ arrangement of letters.

   $\therefore$ The total number of different license plates $= 1\,000 \times 17\,576 = 17\,576\,000$

3. As the jobs are similar it does not matter which job goes to which person (i.e. order is not important).

   $\therefore$ The positions can be filled in ${}^nC_r = {}^{10}C_4 = \dfrac{10!}{4!6!} = 210$

4. If we assume order is important e.g. say the two positions are Secretary and Treasurer, then Person 1 → Secretary and Person 2 → Treasurer will give a different committee from Person 2 → Secretary and Person 1 → Treasurer etc.

   So the number of different committees $= {}^nP_r = {}^6P_2 = \dfrac{6!}{4!} = 30$

   - If we assume order is not important e.g. the two positions on the committee are as ‘regular’ committee members then the number of different committees $= {}^nC_r = {}^6C_2 = \dfrac{6!}{2!4!} = 15$

Solutions Exercise Set 1.1 cont.

5. $\binom{10}{4} = \frac{10!}{4!(10-4)!} = \frac{10!}{4!6!}$ and $\binom{10}{6} = \frac{10!}{6!(10-6)!} = \frac{10!}{6!4!}$

$\therefore \binom{10}{4} = \binom{10}{6}$ and in general $\binom{n}{r} = \binom{n}{n-r}$

6. (i) There are 14 mice altogether $\therefore$ no. of ways of choosing any five
$= {}^nC_r = {}^{14}C_5 = 2\,002$

(ii) There are 8 white mice $\therefore$ no. of ways of choosing five
$= {}^nC_r = {}^8C_5 = 56$

There are 6 brown mice $\therefore$ no. of ways of choosing five
$= {}^nC_r = {}^6C_5 = 6$

$\therefore$ no. of ways of choosing five mice of the same colour
$= 56 + 6 = 62$

(iii) There are ${}^8C_3$ ways of picking the three white mice and each of these can be associated with one selection of two brown mice. There are ${}^6C_2$ ways of getting two brown mice.

$\therefore$ Total no. of ways of choosing five white mice and two brown mice

$= {}^8C_3 \times {}^6C_2 = 56 \times 15 = 840$

Solutions Exercise Set 1.2 page 1.13

1. $(x + 4)^6$

(a) First term is ${}^6C_0 x^6 4^0 = x^6$

(b) $(x + 4)^6 = {}^6C_0 x^6 4^0 + {}^6C_1 x^5 4^1 + {}^6C_2 x^4 4^2 + {}^6C_3 x^3 4^3 + {}^6C_4 x^2 4^4 + {}^6C_5 x^1 4^5 + {}^6C_6 x^0 4^6$

(c) ${}^6C_0 = 1 \;;\; {}^6C_1 = 6 \;;\; {}^6C_2 = 15 \;;\; {}^6C_3 = 20;$

${}^6C_4 = 15 \;;\; {}^6C_5 = 6 \;;\; {}^6C_6 = 1$

(d) $(x + 4)^6 = x^6 + 6x^5.4^1 + 15x^4.4^2 + 20x^3.4^3 + 15x^2 4^4 + 6x^1 4^5 + 4^6$

$= x^6 + 24x^5 + 240x^4 + 1\,280x^3 + 3\,840x^2 + 6\,144x + 4\,096$

2. (a) $(x - 4)^5 = {}^5C_0 x^5(-4)^0$

(b) $(x - 4)^5 = {}^5C_0 x^5(-4)^0 + {}^5C_1 x^4(-4)^1 + {}^5C_2 x^3(-4)^2 + {}^5C_3 x^2(-4)^3 + {}^5C_4 x^1(-4)^4 + {}^5C_5 x^0(-4)^5$

(c) ${}^5C_0 = 1 \;;\; {}^5C_1 = 5 \;;\; {}^5C_2 = 10 \;;\; {}^5C_3 = 10;$

${}^5C_4 = 5 \;;\; {}^5C_5 = 1$

(d) $(x - 4)^5 = x^5 + 5x^4(-4)^1 + 10x^3(-4)^2 + 10x^2(-4)^3 + 5x^1(-4)^4 + (-4)^5$

$= x^5 - 20x^4 + 160x^3 - 640x^2 + 1\,280x - 1\,024$

3. (a) $(x + {}^1\!/_y)^4 = {}^4C_0 x^4({}^1\!/_y)^0 + {}^4C_1 x^3({}^1\!/_y)^1 + {}^4C_2 x^2({}^1\!/_y)^2 + {}^4C_3 x^1({}^1\!/_y)^3 + {}^4C_4 x^0({}^1\!/_y)^4$

$$= x^4 + \frac{4x^3}{y} + \frac{6x^2}{y^2} + \frac{4x}{y^3} + \frac{1}{y^4}$$

(b) $(2x + y^2)^3 = {}^3C_0(2x)^3(y^2)^0 + {}^3C_1(2x)^2(y^2)^1 + {}^3C_2(2x)^1(y^2)^2 + {}^3C_4(2x)^0(y^2)^3$

$= (2x)^3 + 3(2x)^2y^2 + 3(2x)(y^2)^2 + (y^2)^3$

$= 8x^3 + 3.4x^2.y^2 + 3.2x.y^4 + y^6$

$= 8x^3 + 12x^2y^2 + 6xy^4 + y^6$

Solutions Exercise Set 1.2 cont.

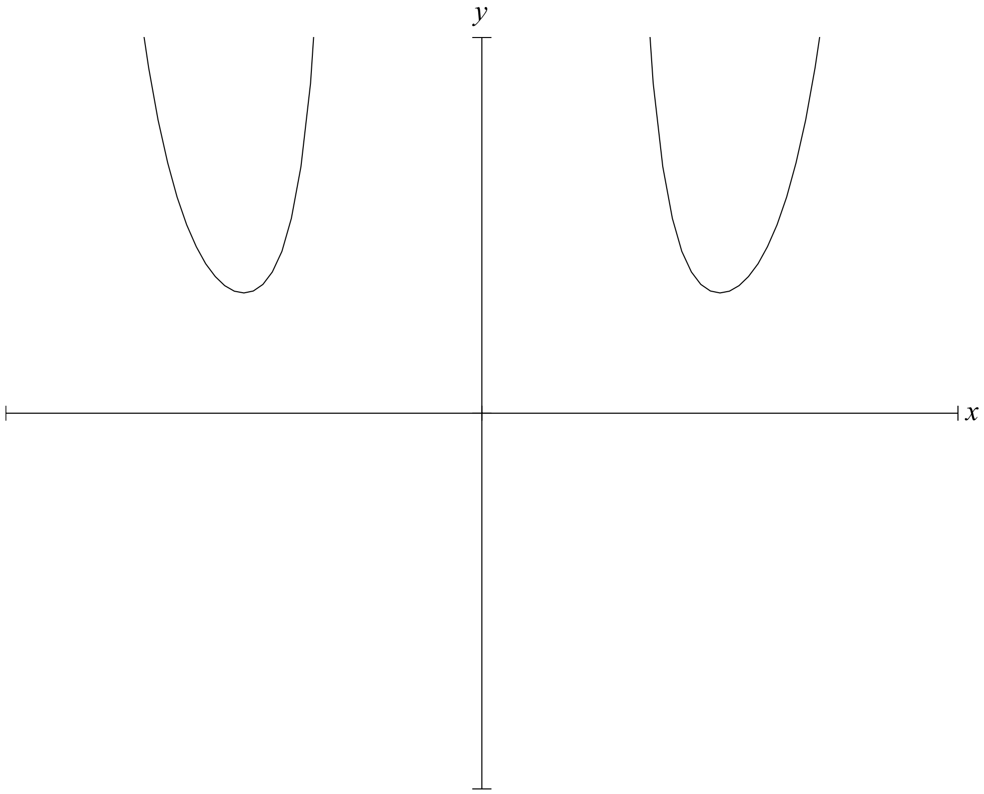
3. (a) (i) $(x^2 + {}^{1}/_{x^2})^5 = {}^5C_0(x^2)^5({}^{1}/_{x^2})^0 + {}^5C_1(x^2)^4({}^{1}/_{x^2})^1 + {}^5C_2(x^2)^3({}^{1}/_{x^2})^2$

$+ {}^5C_3(x^2)^2({}^{1}/_{x^2})^3 + {}^5C_4(x^2)^1({}^{1}/_{x^2})^4 + {}^5C_5(x^2)^0({}^{1}/_{x^2})^5$

$= x^{10} + 5x^8.{}^{1}/_{x^2} + 10x^6.{}^{1}/_{x^4} + 10x^4.{}^{1}/_{x^6} + 5x^2.{}^{1}/_{z^8} + {}^{1}/_{z^{10}}$ **See Note** 1

$= x^{10} + 5x^6 + 10x^2 + {}^{10}/_{x^2} + {}^{5}/_{x^6} + {}^{1}/_{x^{10}}$

(ii)



(b) (i) $(p - 2p^3)^6 = {}^6C_0p^6(-2p^3)^0 + {}^6C_1p^5(-2p^3)^1 + {}^6C_2p^4(-2p^3)^2$

$+ {}^6C_3p^3(-2p^3)^3 + {}^6C_4p^2(-2p^3)^4 + {}^6C_5p^1(-2p^3)^5 + {}^6C_6p^0(-2p^3)^6$

$= p^6 + 6p^5.(-2p^3) + 15p^4.(4p^6) + 20p^3.(-8p^9) + 15p^2.(16p^{12})$

$+ 6p.(-32p^{15}) + 64p^{18}$

$= p^6 - 12p^8 + 60p^{10} - 160p^{12} + 240p^{14} - 192p^{16} + 64p^{18}$

Notes

1. Use the index laws carefully.

Solutions Exercise Set 1.2 cont.

4. (a) $(x^2 + {}^1\!/_x)^4 = {}^4C_0(x^2)^4({}^1\!/_x)^0 + {}^4C_1(x^2)^3({}^1\!/_x)^1 + {}^4C_2(x^2)^2({}^1\!/_x)^2 + {}^4C_3(x^2)^1({}^1\!/_x)^3 + {}^4C_4(x^2)^0({}^1\!/_x)^4$

$$= x^8 + 4x^6.\frac{1}{x} + 6x^4.\frac{1}{x^2} + 4x^2.\frac{1}{x^3} + \frac{1}{x^4}$$

$$= x^8 + 4x^5 + 6x^2 + \frac{4}{x} + \frac{1}{x^4}$$

$\therefore$ Coefficient of x^2 is 6 and coefficient of x^{-1} is 4

(b) $({}^1\!/_x - x^2)^4 = ({}^1\!/_x)^4 + 4({}^1\!/_x)^3(-x^2)^1 + 6({}^1\!/_x)^2(-x^2)^2 + 4({}^1\!/_x)^1(-x^2)^3 + (-x^2)^4$

$$= \frac{1}{x^4} - \frac{4x^2}{x^3} + \frac{6x^4}{x^2} - \frac{4x^6}{x} + x^8$$

$$= \frac{1}{x^4} - \frac{4}{x} + 6x^2 - 4x^5 + x^8$$

$\therefore$ Coefficient of x^2 is 6 and coefficient of x^{-1} is –4

5. $(1.01)^4 = (1 + 0.01)^4 = 1^4 + {}^4C_1 1^3(0.01)^1 + {}^4C_2 1^2(0.01)^2 + {}^4C_3 1^1(0.01)^3 + (0.01)^4$

$$= 1 + 4 \times 0.01 + 6 \times 0.0001 + 4 \times 0.000001 + 0.00000001$$

$$= 1.040604$$

Checking: On the calculator $(1.01)^4 = 1.040604$ ✓

Solutions Exercise Set 1.3 page 1.16

1. $s(n) = (-1)^n(n^2 - 3)$

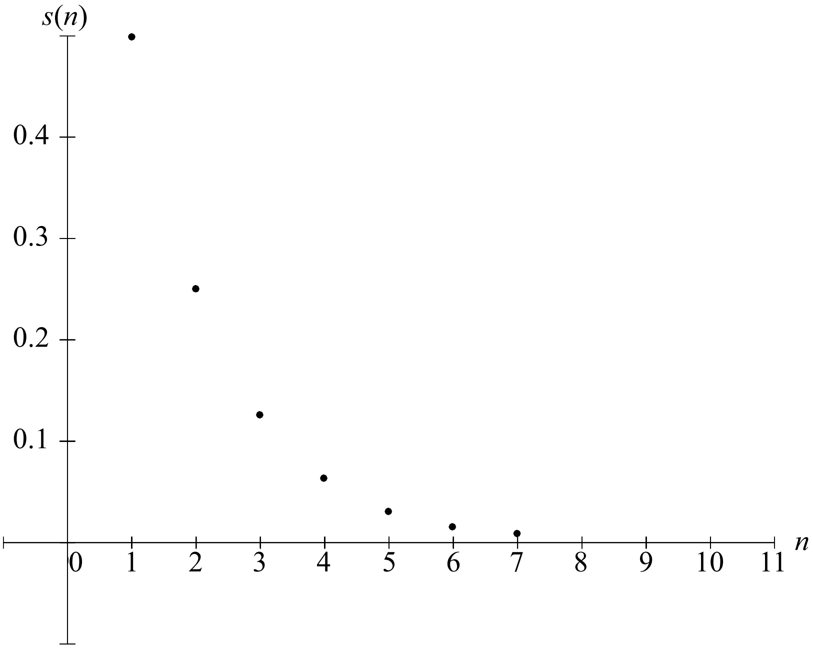
$$\begin{aligned} s(1) &= (-1)^1(1^2 - 3) &&= -1(1 - 3) &&= 2 \\ s(2) &= (-1)^2(2^2 - 3) &&= 1(4 - 3) &&= 1 \\ s(3) &= (-1)^3(3^2 - 3) &&= -1(9 - 3) &&= -6 \\ s(4) &= (-1)^4(4^2 - 3) &&= 1(16 - 3) &&= 13 \end{aligned}$$

$$s(10) = (-1)^{10}(10^2 - 3) = 1(100 - 3) = 97$$

$$s(20) = (-1)^{20}(20^2 - 3) = 1(400 - 3) = 397$$

2.

n	1	2	3	4	5	6	7
$s(n) = (1/2)^n$	$1/2$	$1/4$	$1/8$	$1/16$	$1/32$	$1/64$	$1/128$



This sequence converges. e.g. when $n = 10$, $s(n) = \frac{1}{2^{10}} \simeq 0$

when $n = 100$, $s(n) = \frac{1}{2^{100}} \simeq 0$ etc.

$\therefore$ limit of $s(n)$ as $n \to \infty = 0$

3.

n	1	2	3	4	5	6	
$s(n) = (-1)^n$	–1	+1	–1	+1	–1	+1	etc.

This sequence does not converge. It oscillates between –1 and 1.

Solutions Exercise Set 1.4 page 1.17

1. (a) (i) 2, 4, 6, 8, 10, 12, 14, 16, 18, …

(ii) 0, 3, 8, 15, 24, 35, 48, 63, …

(iii) 1, ½, ¼, ⅛, $1/16$, $1/32$, $1/64$, $1/128$, $1/256$, …

(b) (i) $u_n = 2n$

(ii) $u_n = n^2 - 1$

(iii) $u_n = \frac{1}{2^{n-1}}$ or $u_n = (1/2)^n - 1$

(c) (i) $u_{20} = 2 \times 20 = 40$; $u_{100} = 2 \times 100 = 200$; $u_{1\,000} = 2 \times 1\,000 = 2\,000$

(ii) $u_{20} = 20^2 - 1 = 399$; $u_{100} = 100^2 - 1 = 9\,999$; $u_{1\,000} = 1\,000^2 - 1 = 999$

(iii) $u_{20} = \frac{1}{2^{20-1}} = \frac{1}{2^{19}}$; $u_{100} = \frac{1}{2^{100-1}} = \frac{1}{2^{99}}$; $u_{1\,000} = \frac{1}{2^{1000\ -1}} = \frac{1}{2^{999}}$

2. (a) (i) 1, 2, 3, 4 … ⇒ $u_n = u_{n-1} + 1$ given $u_1 = 1$

(ii) ½, ¼, ⅛, $1/16$ … ⇒ $u_n = \frac{1}{2} u_{n-1}$ given $u_1 = \frac{1}{2}$

(iii) 1, –1, –3, –5, –7 … ⇒ $u_n = u_{n-1} - 2$ given $u_1 = 1$

(b) (i) Divergent

(ii) Convergent. Limit of $s(n)$ as $n \to \infty$ is 0

(iii) Divergent

Solutions Exercise Set 1.5 page 1.19

1. (a) For $s(n) = 0.1n$

$$S_4 = s(1) + s(2) + s(3) + s(4)$$

$$= 0.1^1 + 0.1^2 + 0.1^3 + 0.1^4$$

$$= 0.1111$$

(b) For $s(n) = \frac{1}{2^n}$

$$S_6 = s(1) + s(2) + s(3) + s(4) + s(5) + s(6)$$

$$= \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6}$$

$$= \frac{63}{64} = 0.984375$$

(c) For $s(n) = e^{3n}$

$$S_{10} = s(1) + s(2) + \ldots + s(9) + s(10)$$

$$= e^{3 \times 1} + e^{3 \times 2} + e^{3 \times 3} + e^{3 \times 4} + \ldots + e^{3 \times 9} + e^{3 \times 10}$$

$$= 1.124639986 \times 10^{13}$$

Make sure you correctly interpret the display of the calculator which gives the result in scientific notation form.

(d) For $s(n) = (-1)^n$

$$S_5 = s(1) + s(2) + s(3) + s(4) + s(5)$$

$$= (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 + (-1)^5$$

$$= -1 + 1 - 1 + 1 - 1$$

$$= -1$$

Solutions Exercise Set 1.5 cont.

2. (a) (i) Convergent. As $n \to \infty$, $s(n) \to 0$

$s(n) = 0.1n$ can be written as $s(n) = 0.1 \times 0.1^{n-1}$

This gives the general term $u_n = 0.1 \times 0.1^{n-1}$ and comparing with the general geometric series $u_n = ar^{n-1}$ we see a = 1 and $r = 0.1$
$\therefore$ as $-1 < r < 1$, $s(n)$ converges and its sum to infinity

$$S_\infty = \frac{a}{1-r} = \frac{0.1}{1-0.1} = \frac{0.1}{0.9} = \frac{1}{9}$$

(ii) Convergent. As $n \to \infty$, $s(n) \to 0$

$s(n) = \frac{1}{2^n}$ can be written as $s(n) = \tfrac{1}{2} \times (\tfrac{1}{2})^{n-1}$

Comparing with the general geometric series $u_n = ar^{n-1}$ gives $a = \tfrac{1}{2}$ and $r = \tfrac{1}{2}$ $\therefore$ as $-1 < r < 1$, $s(n)$ converges and its sum to infinity

$$S_\infty = \frac{a}{1-r} = \frac{\tfrac{1}{2}}{1-\tfrac{1}{2}} = 1$$

(iii) Divergent. As $n \to \infty$, $s(n) \to \infty$

(iv) Not convergent, $s(n)$ oscillates between –1 and 1.

Solutions Exercise Set 1.6 page 1.26

1. Prove $\frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots + (\frac{1}{2})^n = 1 - (\frac{1}{2})^n$

Step 1: When $n = 1$, $S_1 = (\frac{1}{2})^1 = \frac{1}{2}$ and $1 - (\frac{1}{2})n = 1 - (\frac{1}{2})^1 = \frac{1}{2}$
$\therefore$ statement is true for $n = 1$

Step 2: Assume true for any $n = k$
When $n = k$, $S_k = \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots + (\frac{1}{2})^k = 1 - (\frac{1}{2})^k$

Step 3: Using the assumption in Step 2 we need to show
$\frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots + (\frac{1}{2})^k + (\frac{1}{2})^{k+1} = 1 - (\frac{1}{2})^{k+1}$

From Step 2 we can substitute for most of LHS (for the first k terms).
$(\frac{1}{2}) + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots + (\frac{1}{2})^k + (\frac{1}{2})^{k+1} = 1 - (\frac{1}{2})^k + (\frac{1}{2})^{k+1}$

We now need to rearrange the RHS i.e. $1 - (\frac{1}{2})^k + (\frac{1}{2})^{k+1}$ into the form $1 - (\frac{1}{2})^{k+1}$ to prove the statement

$$\begin{aligned} 1 - (\tfrac{1}{2})^k + (\tfrac{1}{2})^{k+1} &= 1 - (\tfrac{1}{2})^k + (\tfrac{1}{2})(\tfrac{1}{2})^k \\ &= (\tfrac{1}{2})^k \{-1 + \tfrac{1}{2}\} + 1 \\ &= (\tfrac{1}{2})^k \{-\tfrac{1}{2}\} + 1 \\ &= 1 - (\tfrac{1}{2})^{k+1} \end{aligned}$$

See Note 1

$\therefore$ As the statement is true for $n = k + 1$, assuming it was true for any $n = k$ and it is true for $n = 1$, it is true for all n i.e. for all positive integers.

Notes

1. You may have used different algebraic manipulations to what is shown here.

Solutions Exercise Set 1.6 cont.

2. Prove $1 + x^1 + x^2 + x^3 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x}$ (if $x \neq 1$)

Step 1: When $n = 1$, $S_1 = 1 + x$ and $\frac{1 - x^{n+1}}{1 - x} = \frac{1 - x^{1+1}}{1 - x} = \frac{1 - x^2}{1 - x}$

$$= \frac{(1 - x)(1 + x)}{(1 - x)} = 1 + x$$

$\therefore$ Statement is true for $n = 1$

Step 2: Assume true for any $n = k$
When $n = k$, $S_k = 1 + x + x^2 + x^3 + \dots + x^k = \frac{1 - x^{k+1}}{1 - x}$

Step 3: Using the assumption in Step 2 we need to show

$$1 + x + x^2 + x^3 + \dots + x^k + x^{k+1} = \frac{1 - x^{(k+1)+1}}{1 - x}$$

From Step 2, we are assuming $1 + x + x^2 + x^3 + \dots + x^k = \frac{1 - x^{k+1}}{1 - x}$

$$\therefore\ 1 + x + x^2 + x^3 + \dots + x^k + x^{k+1} = \frac{1 - x^{k+1}}{1 - x} + x^{k+1}$$ {Adding x^{k+1} to each side}

We now need to rearrange the RHS into the form $\frac{1 - x^{(k+1)+1}}{1 - x}$ to prove the statement.

$$\text{RHS} = \frac{1 - x^{k+1}}{1 - x} + x^{k+1}$$

$$= \frac{1 - x^{k+1} + x^{k+1}(1 - x)}{1 - x}$$ {Putting terms on common denominator of $(1 - x)$}

$$= \frac{1 - x^{k+1} + x^{k+1} - x^{k+1+1}}{1 - x}$$

$$= \frac{1 - x^{(k+1)+1}}{1 - x}$$

Thus we have shown, by mathematical induction, that

$$1 + x + x^2 + x^3 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad (\text{if } x \neq 1)$$

Solutions Exercise Set 1.6 cont.

3. Prove $1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Step 1: When $n = 1$, $S_1 = 1$ and $\frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)(2 \times 1 + 1)}{6}$

$$= \frac{1 \times 2 \times 3}{6} = 1$$

$\therefore$ Statement is true for $n = 1$

Step 2: Assume true for any $n = k$
When $n = k$, $S_k = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$

Step 3: Using the assumption in Step 2
$S_{k+1} = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

We now need to rearrange the RHS into the form $\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$ to prove the statement

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

{Putting terms on common denominator}

$$= \frac{(k+1)\{k(2k+1) + 6(k+1)\}}{6}$$

{Taking $(k + 1)$ out as a common factor}

$$= \frac{(k+1)\{2k^2 + k + 6k + 6\}}{6}$$

$$= \frac{(k+1)\{2k^2 + 7k + 6\}}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

{Factorising $2k^2 + 7k + 6$

$$= \frac{(k+1)((k+1)+1)(2k+2+1)}{6}$$

$$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$\therefore$ Using mathematical induction the statement is proved true for any positive integer.

Solutions Exercise Set 1.6 cont.

4. Prove $(s + 1t)(s + 2t) + (s + 3t) + \ldots + (s + nt) = n\left(s + \frac{n+1}{2}t\right)$

Step 1: When $n = 1$, $S_1 = s + t$ and $n\left(s + \frac{n+1}{2}t\right) = 1\left(s + \frac{1+1}{2}t\right) = s + t$
$\therefore$ Statement is true for $n = 1$

Step 2: Assume true for any $n = k$

When $n = k$, $S_k = (s + t) + (s + 2t) + (s + 3t) + \ldots + (s + kt) =$
$k\left(s + \frac{k+1}{2}t\right)$

Step 3: Using the assumption in Step 2.
$S_{k+1} = (s + t) + (s + 2t) + (s + 3t) + \ldots + (s + kt) + (s + (k + 1)t)$
$= k\left(s + \frac{k+1}{2}t\right) + (s + (k + 1)t)$

We now need to rearrange $k\left(s + \frac{k+1}{2}t\right) + (s + (k + 1)t)$ into the form

$(k + 1)\left(s + \frac{(k+1)+1}{2}t\right)$ to prove the statement.

$$k\left(s + \frac{k+1}{2}t\right) + (s + (k + 1)t)$$

$$= ks + k\left(\frac{k+1}{2}\right)t + s + (k + 1)tl$$

$$= (k + 1)\left(\frac{kt}{2} + t\right) + s(k + 1)$$

{Taking $(k + 1)$ as a common factor out of 2 terms and s as a common factor out of the other 2 terms}

$$= (k + 1)\left\{\frac{kt + 2t}{2} + s\right\}$$

$$= (k + 1)\left(s + \frac{(k+1)+1}{2}t\right)$$

$$= (k + 1)\left(s + \frac{(k+1)+1}{2}t\right)$$

Thus we have proved, using mathematical induction, that the statement is true.