

MULTIDIMENSIONAL SOLITONS in Dispersive Complex Media: Structure and Stability. Applications

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Outline

- **Basic equations**
- **Classes of nonlinear GNLS-DNLS and GKP models:**
 - *Analytically solved cases: types of solutions (classification) and their stability*
 - *Numerical analysis. Soliton collisions*
 - *Influence of dissipation*
- **Applications of the GKP and DNLS models:**
 - *Application in a plasma (FMS and Alfvén waves)*
 - *Nonlinear waves in media with variable dispersion*



Basic Equations

Initial equations:

$$\partial_t \mathbf{v} + (\mathbf{v} \nabla) \mathbf{v} + (c^2 / \rho) \nabla \rho = 0,$$

$$\partial_t \rho + \nabla (\rho \mathbf{v}) = 0.$$

$$\partial_t \Phi + \frac{1}{2} (\nabla \Phi)^2 + \frac{c^2 (\rho - \rho_0)}{2\rho} + \frac{c^2 z}{\rho} = 0$$

$$\Delta \Phi = 0$$

$$\partial_t \eta + \partial_x \eta \partial_x \Phi + \partial_y \eta \partial_y \Phi - \partial_z \Phi = 0,$$

$$\partial_t \Phi + \frac{1}{2} (\nabla \Phi)^2 + (c^2 / \rho) \eta = 0,$$

$$z = \eta(x, y, t),$$

$$\partial_z \Phi \Big|_{z=-\rho_0} = 0$$

Functions and variables:

✓ Waves on water surface:

$$\rho \equiv H \quad c(\rho) = c_0 = \sqrt{gH}$$

✓ Ion-acoustic waves in a plasma:

$$c(\rho) = c_0 = c_s = \sqrt{T_e / m}$$

✓ FMS waves in magnetized plasma:

$$\rho \equiv B$$

$$c(\rho) = c(B) = v_A = B / \sqrt{4\pi n m}$$

Classes of nonlinear models

Modified **GNLS-DNLS-GKP** model (BK model)

$$\partial_t u + \hat{A}(t, u)u = f, \quad f = \kappa \int_{-\infty}^x \Delta_{\perp} u dx, \quad \Delta_{\perp} = \partial_y^2 + \partial_z^2.$$

Some classes of systems:

➤ **GNLS class of eqs** if $\hat{A}(t, u) = i[\gamma|u|^2 - \beta\partial_x^2] + \alpha/2$ **then**

$$\partial_t u + i\gamma|u|^2 u - i\beta\partial_x^2 u + (\alpha/2)u = \sigma \int_{-\infty}^x \Delta_{\perp} u dx$$

where $\alpha, \beta, \gamma = \varphi(t, x, y, z)$, $(\alpha/2)u$ describes dissipative effects, u is an amplitude of the envelop pulse (V.Yu. Belashov et al. Particle Phys. 2018, Chicago, USA)

➤ **DNLS class of eqs** if $\hat{A}(t, u) = 3s|p|^2 u^2 \partial_x - \partial_x^2(i\lambda + \nu)$ **and**

$$u = h = (B_y + iB_z)/2B_0 |1 - \beta|^{1/2}, \quad \mathbf{h} = \mathbf{B}_{\perp} / B_0, \quad p = (1 + ie)$$

then

$$\partial_t h + s\partial_x(|h|^2 h) - i\lambda\partial_x^2 h - \nu\partial_x^2 h = \sigma \int_{-\infty}^x \Delta_{\perp} h dx$$

Classes of nonlinear models

Modified **GNLS-DNLS-GKP** model

$$\partial_t u + \hat{A}(t, u)u = f, \quad f = \kappa \int_{-\infty}^x \Delta_{\perp} u dx, \quad \Delta_{\perp} = \partial_y^2 + \partial_z^2.$$

if

$$\hat{A}(t, u) = \alpha u \partial_x - \partial_x^2 (v - \beta \partial_x - \gamma \partial_x^3)$$

➤ **GKP** class of eqs

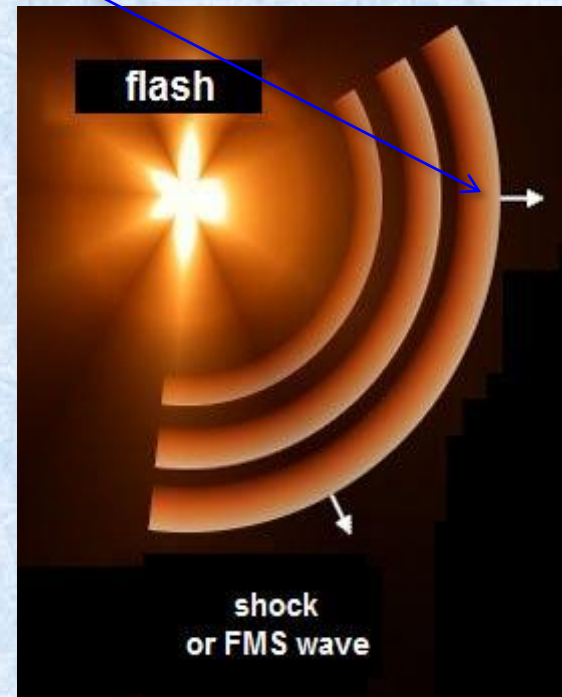
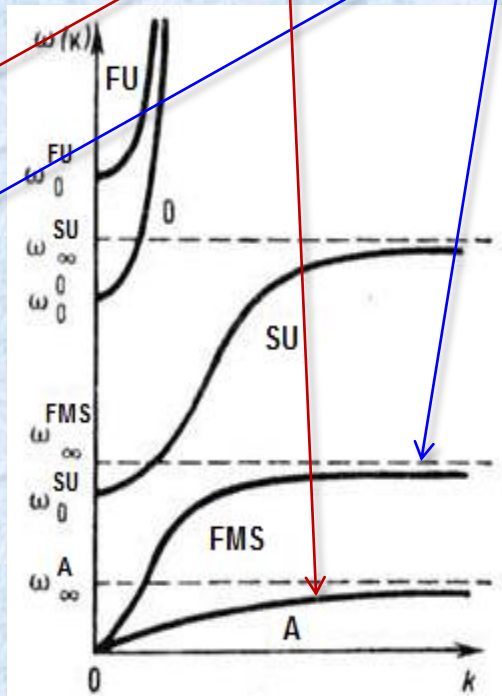
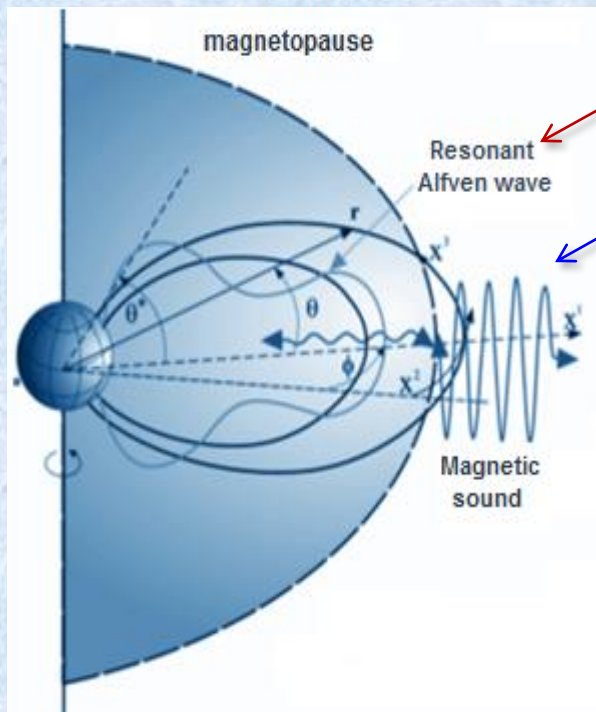


$$\partial_x \left(\partial_t u + \alpha u \partial_x u - v \partial_x^2 u + \beta \partial_x^3 u + \gamma \partial_x^5 u \right) = \kappa \Delta_{\perp} u, \quad \Delta_{\perp} = \partial_y^2 + \partial_z^2$$

which has **numerous applications** in physics of ocean, atmosphere, ionosphere and magnetosphere, and in space plasma

Some examples

- The BK model describes the evolution of **3D Alfvén and FMS waves in magnetized plasma**



FMS and Alfvén waves: magnetosphere and solar corona

GKP equation

$$\partial_x \left(\partial_t u + \alpha u \partial_x u - \nu \partial_x^2 u + \beta \partial_x^3 u + \gamma \partial_x^5 u \right) = \kappa \Delta_{\perp} u$$

$$\omega \approx c_0 k_x \left[1 + k_{\perp}^2 / 2k_x^2 - i\nu k_x / c_0 + (-\beta k_x^2 + \gamma k_x^4) / c_0 \right]$$

where, for example, for FMS waves

$$\alpha = \frac{3}{2} v_A \sin \theta, \quad \underline{v_A} = B / \sqrt{4\pi n m}, \quad m = m_e + m_i, \quad \kappa = -(v_A / 2)$$

$$\beta = v_A (c^2 / 2\omega_{0i}^2) (\cot^2 \theta - m_e / m_i) \quad \nu = (\rho_0 / 2\rho) (c_{\infty}^2 - c_0^2) \tau \int_0^{\infty} \xi \varphi(\xi) d\xi$$

$$\gamma = v_A (c^4 / 8\omega_{0i}^4) \left[3 \left(m_e / m_i - \cot^2 \theta \right)^2 - 4 \cot^4 \theta \left(1 + \cot^2 \theta \right) \right] = f[\theta, v_A]$$

If Landau damping is significant we should also include in right-hand side of eq. the term

$$-\partial_x \hat{L}[u] = -\sigma \partial_x \left[\int_{-\infty}^{\infty} \frac{dk}{2\pi} |k| \int_{-\infty}^{\infty} u(x') e^{ik(x-x')} dx' \right]$$

For waves on “shallow water” the coefficients in the GKP equation:

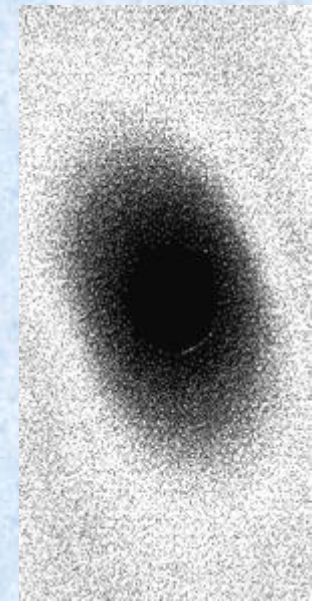
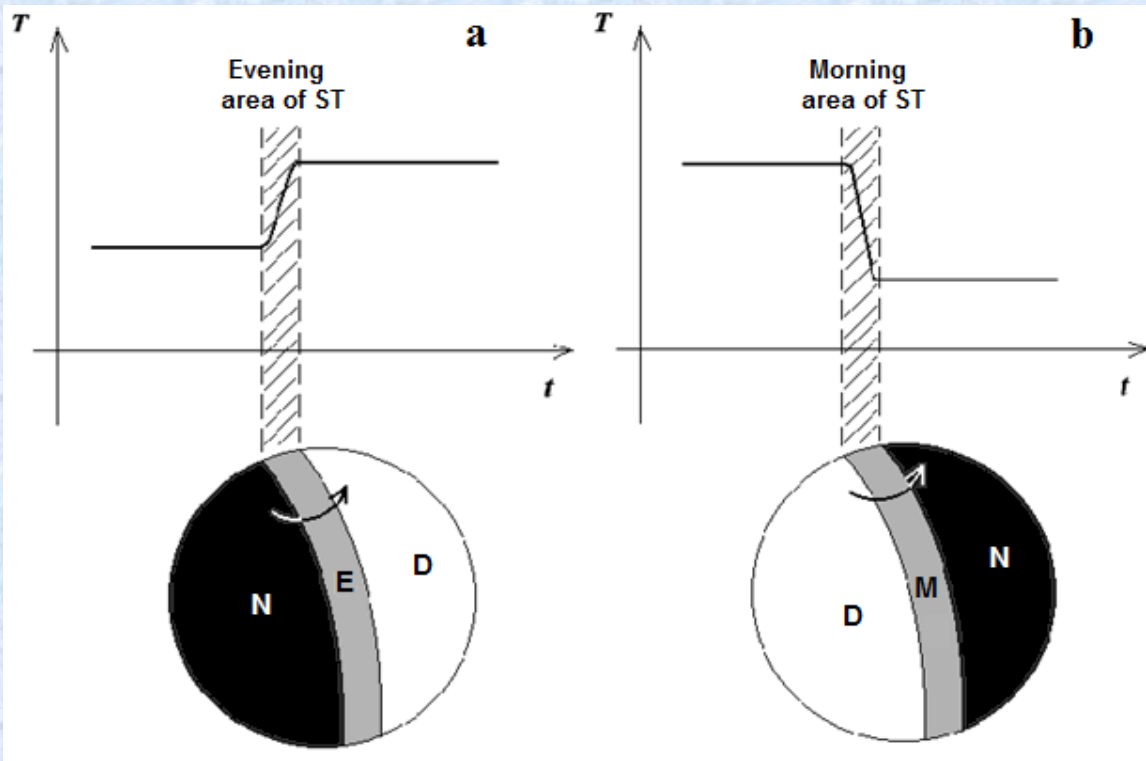
$$\alpha = 3c_0 / 2H, \quad c_0 = (\underline{gH})^{1/2}, \quad \kappa = -(c_0 / 2) \quad \beta = \frac{c_0}{6} \left(\frac{3\sigma}{\rho g} - \underline{H^2} \right)$$
$$\gamma = (c_0 / 6) \left[\underline{H^2} \left(\frac{2}{5} \underline{H^2} - \sigma / \rho g \right) - \frac{1}{12} \left(3\sigma / \rho g - \underline{H^2} \right)^2 \right]$$
$$= f [H(t, x, y)]$$

■ The example: **2D nonlinear waves on surface of “shallow water”**



Some examples

- One more characteristic example is the propagation of the **solitons-like IGW** at heights of the ionosphere's *F*-layer **in regions with sharp gradients of the ionospheric parameters**, including the regions of the **fronts of solar terminator (ST)** and **spot of the solar eclipse (SE)**



SE spot

■ Analysis of solutions' stability

The **GKP** eq. in Hamiltonian form

$$\partial_t u = \partial_x (\delta H / \delta u)$$

$$H = \int \left[-\frac{\varepsilon}{2} (\partial_x u)^2 + \frac{\lambda}{2} (\partial_x^2 u)^2 + \frac{1}{2} (\nabla_\perp \partial_x v)^2 - u^3 \right] d\mathbf{r}$$

Variational problem

$$\delta(H + v P_x) = 0, \quad P_x = \frac{1}{2} \int u^2 d\mathbf{r}$$

All finite solutions are the stationary points of Hamiltonian H at fixed momentum projection P_x

Stability problem

In dynamical system the points that correspond to *minimum or maximum of H are absolutely stable*

Deformation of H conserving momentum projection P_x

$$u(x, \mathbf{r}_\perp) \rightarrow \zeta^{-1/2} \eta^{(1-d)/2} u(x/\zeta, \mathbf{r}_\perp / \eta)$$

Hamiltonian of the GKP equation

$$H(\zeta, \eta) = a\zeta^{-2} + b\zeta^2\eta^{-2} - c\zeta^{-1/2}\eta^{(1-d)/2} + e\zeta^{-4}$$

$$a = -(\varepsilon/2) \int (\partial_x u)^2 d\mathbf{r}, \quad b = (1/2) \int (\nabla_\perp \partial_x v)^2 d\mathbf{r}, \quad c = \int u^3 d\mathbf{r},$$

$$e = (\lambda/2) \int (\partial_x^2 u)^2 d\mathbf{r}$$

✓ Necessary condition of extremum:

✓ Sufficient condition of minimum:

$$\partial_\zeta H = 0, \quad \partial_\eta H = 0 \quad \left| \begin{array}{cc} \partial_\zeta^2 H(\zeta_i, \eta_j) & \partial_{\zeta\eta}^2 H(\zeta_i, \eta_j) \\ \partial_{\eta\zeta}^2 H(\zeta_i, \eta_j) & \partial_\eta^2 H(\zeta_i, \eta_j) \end{array} \right| > 0,$$
$$\partial_\zeta^2 H(\zeta_i, \eta_j) > 0$$

2D case Changes of $H(\zeta, \eta)$ at $d=2$ for various values of the integral coefficients along lines

$$\eta = [(4b/c)^2 \zeta^5]^{1/3}$$

3D case Changes of $H(\zeta, \eta)$ at $d=3$ for various values of the integral coefficients along lines

$$\eta = (2b/c)\zeta^{5/2}$$

So, we have proved a possibility of existence of **absolutely** and **locally** stable 2D and 3D soliton solutions and obtained the conditions of soliton stability in dependence on dispersive parameters, i.e. the characteristics of medium

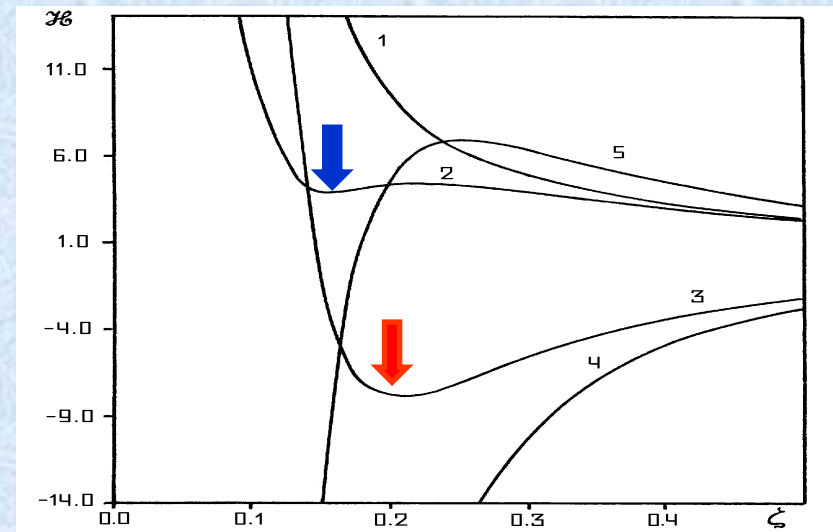
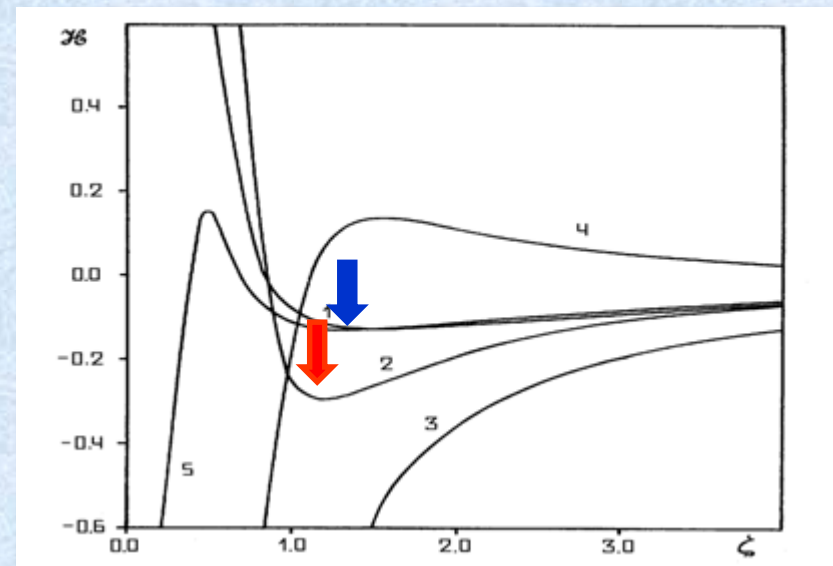
V.Yu. Belashov, Soviet Phys. Doklady, 1991

For the **3-GNLS** and **3-DNLS** eqs. $\partial_t h = \partial_x (\delta H / \delta h)$

we investigated the Hamiltonian boundness in real vector space \mathbf{C} by analogy with the case of the GKP eq.

The result (*V.Yu. Belashov, Doklady Phys., 1999*):

We have proved a possibility of existence of the **absolutely** and **locally** stable 3D solutions in the 3-GNLS and 3-DNLS models and obtained the conditions of stability of the soliton solutions (i.e. regions of values of the equation coefficients)



■ Possible types of solutions

Qualitative analysis of the GKP dynamical system enables us **to classify possible solutions** in a $(n-1) \times d$ phase space (n – order of equation, d – space dimension)

$$\partial_x \left(\partial_t u + \alpha u \partial_x u - \nu \partial_x^2 u + \beta \partial_x^3 u + \delta \partial_x^4 u + \gamma \partial_x^5 u \right) = \kappa \Delta_{\perp} u$$

2D case $\Delta_{\perp} = \partial_y^2$:

$$\begin{aligned} \partial_t u + \alpha u \partial_{\eta} u - \mu \partial_{\eta}^2 u + \beta \partial_{\eta}^3 u + \delta \partial_{\eta}^4 u + \gamma \partial_{\eta}^5 u &= 0, \\ \partial_t u + \alpha u \partial_{\zeta} u - \mu \partial_{\zeta}^2 u + \beta \partial_{\zeta}^3 u + \delta \partial_{\zeta}^4 u + \gamma \partial_{\zeta}^5 u &= 0. \end{aligned}$$

$$\eta = x + y + \kappa t, \quad \zeta = x - y + \kappa t$$

Each of eqs. is equivalent to the set:

for cases:

$$V > 0, \beta = -1 \longrightarrow$$

$$V < 0, \beta = 1 \longrightarrow$$

$$V < 0, \beta = -1 \longrightarrow$$

$$\left\{ \begin{array}{l} x_1 = \dot{w}, \quad x_2 = \dot{x}_1, \quad x_3 = \dot{x}_2, \\ \begin{pmatrix} + \\ - \\ - \end{pmatrix} \gamma \dot{x}_3 = \begin{pmatrix} - \\ + \\ + \end{pmatrix} \delta C x_3 \begin{pmatrix} + \\ + \\ - \end{pmatrix} C^2 x_2 \begin{pmatrix} + \\ - \\ - \end{pmatrix} \mu C^3 x_1 + w - \alpha w^2 / 2, \end{array} \right.$$

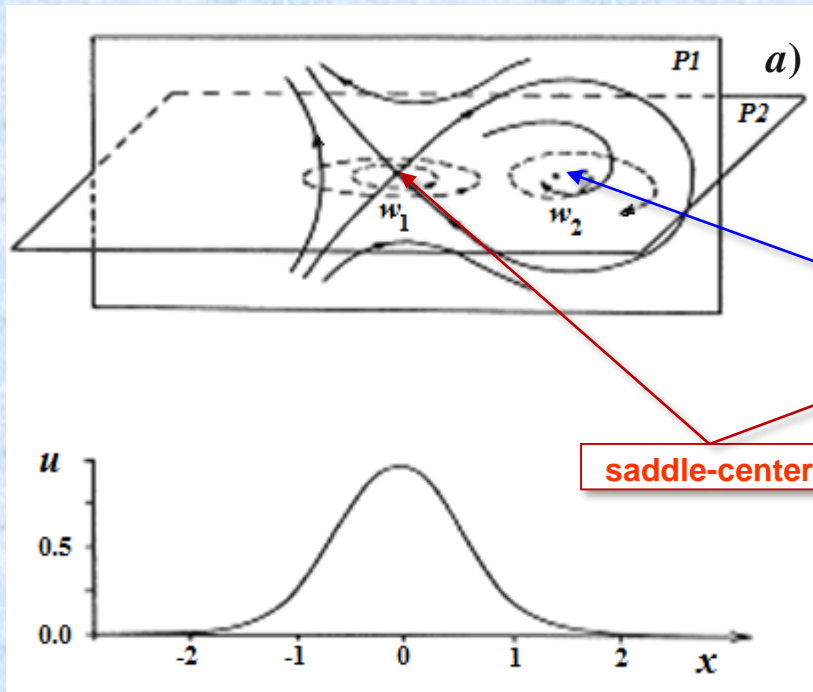
where $w = u(\eta, \zeta, t) / V$; $C = |V|^{-1/4}$

Phase space in 1D case is (w, x_1, x_2, x_3)

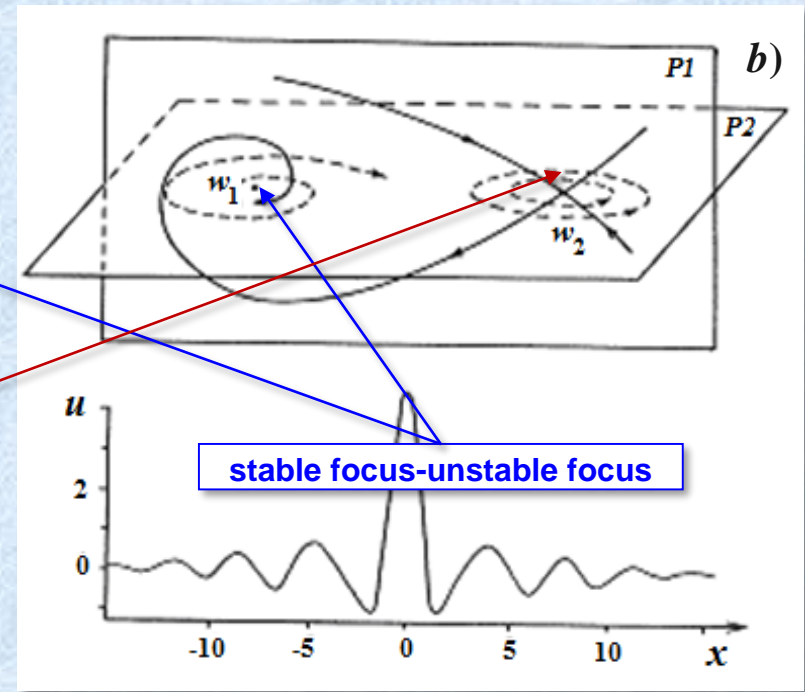


Phase portraits and solutions profiles at $d=1$: **dispersive** cases **without** dissipation and instability

$$\partial_x \left(\underbrace{\partial_t u}_{=0} + \alpha u \underbrace{\partial_x u}_{=0} - \underbrace{\nu \partial_x^2 u}_{=0} + \beta \partial_x^3 u + \delta \partial_x^4 u + \gamma \partial_x^5 u \right) = \kappa \Delta_{\perp} u$$



a) $\nu = \delta = 0, \gamma = 1, \beta = -1$

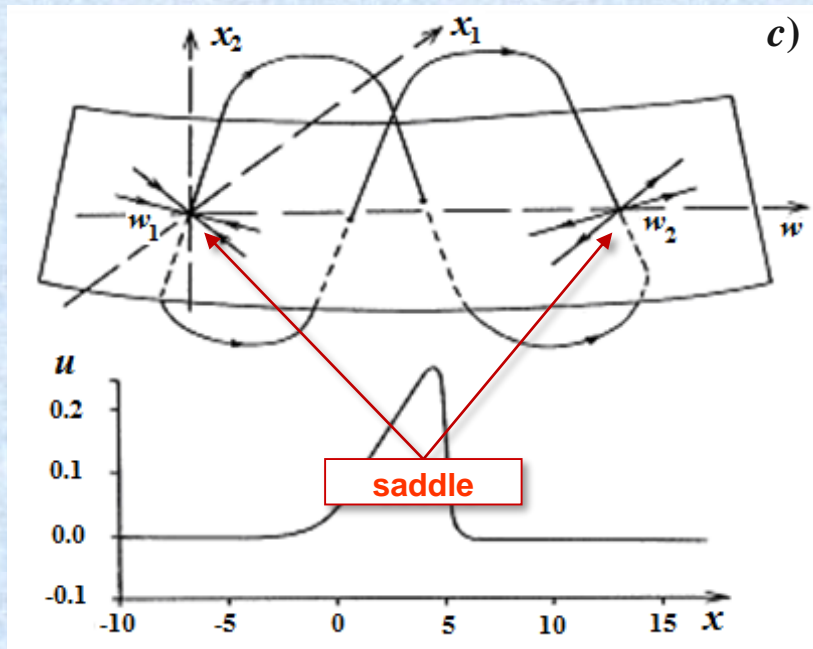


b) $\nu = \delta = 0, \gamma = 1, \beta = 3.16$

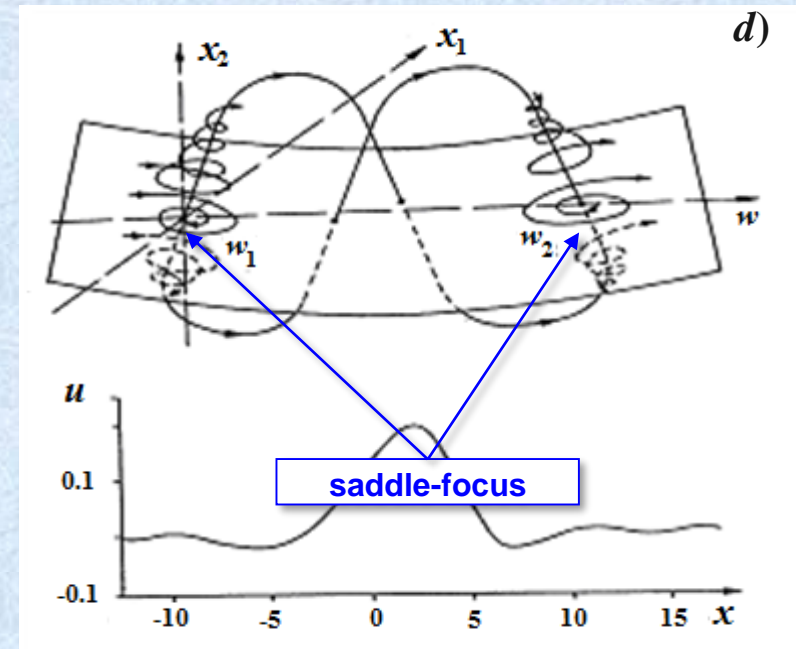
V.Yu. Belashov, E.S. Belashova, J. Phys. & Astron., 2017

Phase portraits and solutions profiles at $d=1$: *non-dispersive* cases *with* dissipation and instability

$$\partial_x \left(\underbrace{\partial_t u}_{=0} + \underbrace{\alpha u \partial_x u}_{=0} - \underbrace{\nu \partial_x^2 u}_{=0} + \beta \partial_x^3 u + \delta \partial_x^4 u + \gamma \partial_x^5 u \right) = \kappa \Delta_{\perp} u$$



c) $\beta = \gamma = 0$, $\nu = 0.1$, $\delta = 1 \times 10^{-6}$



d) $\beta = \gamma = 0$, $\nu = 0.01$, $\delta = 1$

V.Yu. Belashov, E.S. Belashova, J. Phys. & Astron., 2017

Asymptotes of 2D solutions $w = u(\eta, \zeta, t)/V$; $\eta = x + y + \kappa t$, $\zeta = x - y + \kappa t$

a) conservative equations ($\mu = \delta = 0$)

- For cases $V > 0$, $\gamma = -1$ and $V < 0$, $\gamma = -1$:

$$w = A_1 \exp \left\{ (2\gamma)^{-1/2} \left[C^2 + \sqrt{C^4 \pm 4\gamma} \right]^{1/2} \chi \right\}$$

monotonous asymptotes

- For case $V < 0$, $\gamma = 1$:

$$w = A_2 \exp \left\{ \left(2C^{-1}\gamma^{-1/2} \right)^{-1} \left(2C^{-2}\gamma^{1/2} - 1 \right)^{1/2} \chi \right\} \times \cos \left\{ \left(2C^{-1}\gamma^{-1/2} \right)^{-1} \left(2C^{-2}\gamma^{1/2} + 1 \right) \chi + \Theta \right\}$$

oscillating asymptotes

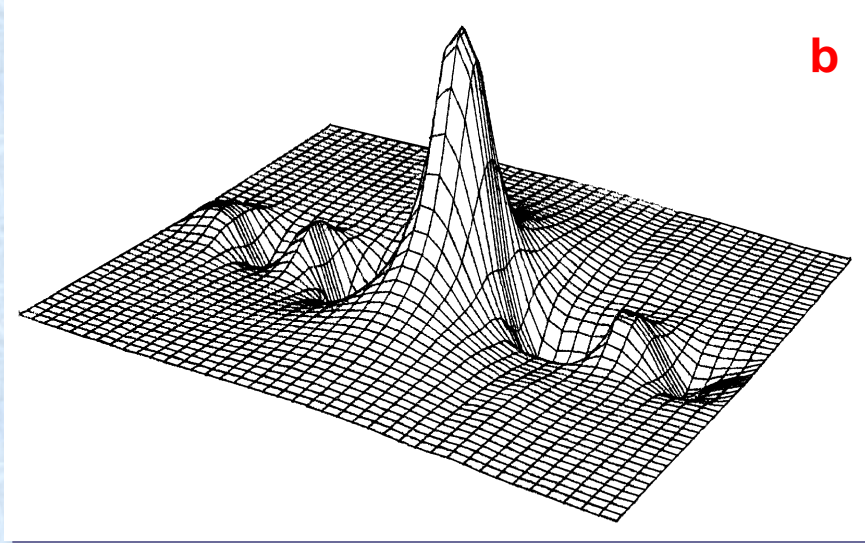
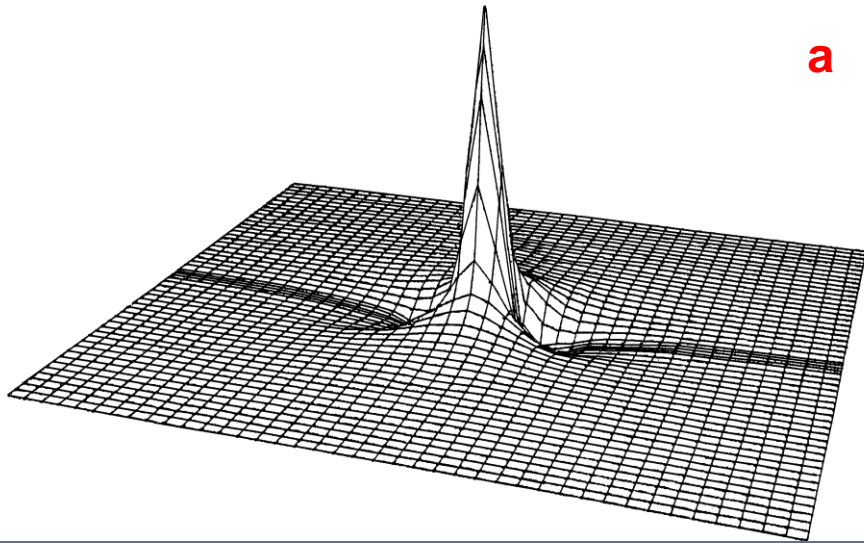
where A_1, A_2 and Θ are arbitrary constants, and

$$C = |V|^{-1/4} \quad \chi = (\eta \pm \zeta + (\kappa - V)t)$$

V.Yu. Belashov, E.S. Belashova,
J. Phys. & Astron., 2017



As an example, in 2D case the GKP equation can have **stable soliton solutions** with **monotonous** and **oscillating** asymptotes in dependence on signs of V and β



General view of 2D solutions of the GKP equation:

(a) $\gamma=1, \beta = -0.8$ ($t = 0.2$);

(b) $\gamma=1, \beta = 3.16$ ($t = 0.5$)

This kind of solution for the first time has been obtained in **L.A. Abramyan, Yu.A. Stepanyants, ZhETF, 1985**

Asymptotes of 2D solutions $w = u(\eta, \zeta, t) / V$; $\eta = x + y + \kappa t$, $\zeta = x - y + \kappa t$

b) dissipative equations with instability ($\beta = \gamma = 0$)

- For case $\delta > (4/27) \mu^3 C^8$:

$$w = A_1 \exp \left[(2\delta C)^{-1/3} Q_1^+ \chi \right] + \exp \left[-(16\delta C)^{-1/3} Q_1^+ \chi \right] \times \left\{ A_2 \cos \left[\sqrt{3} (16\delta C)^{-1/3} Q_1^- \chi + \Theta_1 \right] + A_3 \sin \left[\sqrt{3} (16\delta C)^{-1/3} Q_1^- \chi + \Theta_2 \right] \right\}$$

oscillating asymptotes

- For case $\delta = (4/27) \mu^3 C^8$:

$$w = A_1 \exp \left[(\delta C / 4)^{-1/3} \chi \right] + A_2 (1 + A_3 \chi) \exp \left[-(2\delta C)^{-1/3} \chi \right]$$

- For case $\delta < (4/27) \mu^3 C^8$:

$$w = A_1 \exp \left[(\delta C / 4)^{-1/3} \operatorname{Re}(Q^\pm) \chi \right] + A_2 \exp \left[(2\delta C)^{-1/3} \chi \left[\operatorname{Re}(Q^\pm) - \sqrt{3} |\operatorname{Im}(Q^\pm)| \right] \right] + A_3 \exp \left\{ -(2\delta C)^{-1/3} \chi \left[\operatorname{Re}(Q^\pm) + \sqrt{3} |\operatorname{Im}(Q^\pm)| \right] \right\}$$

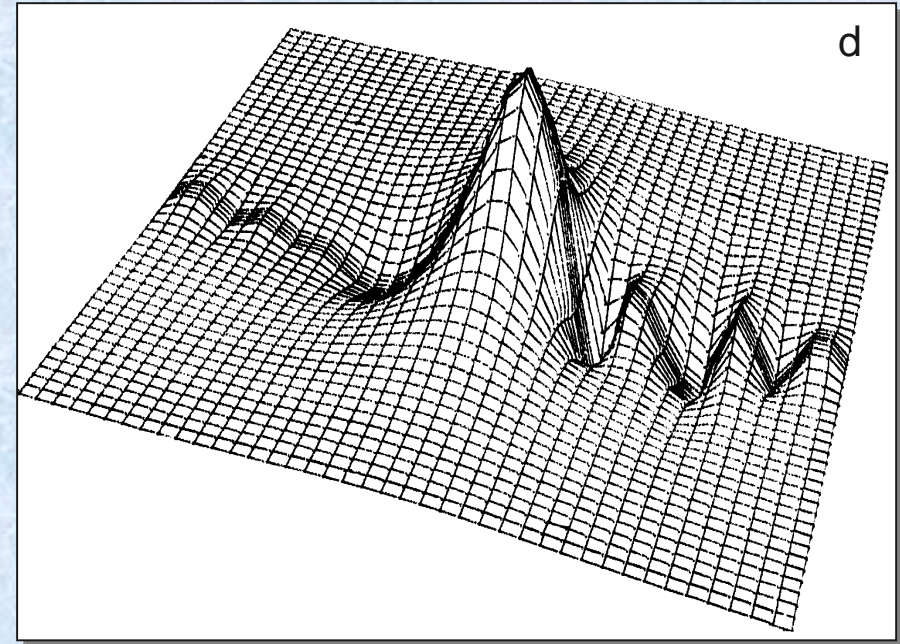
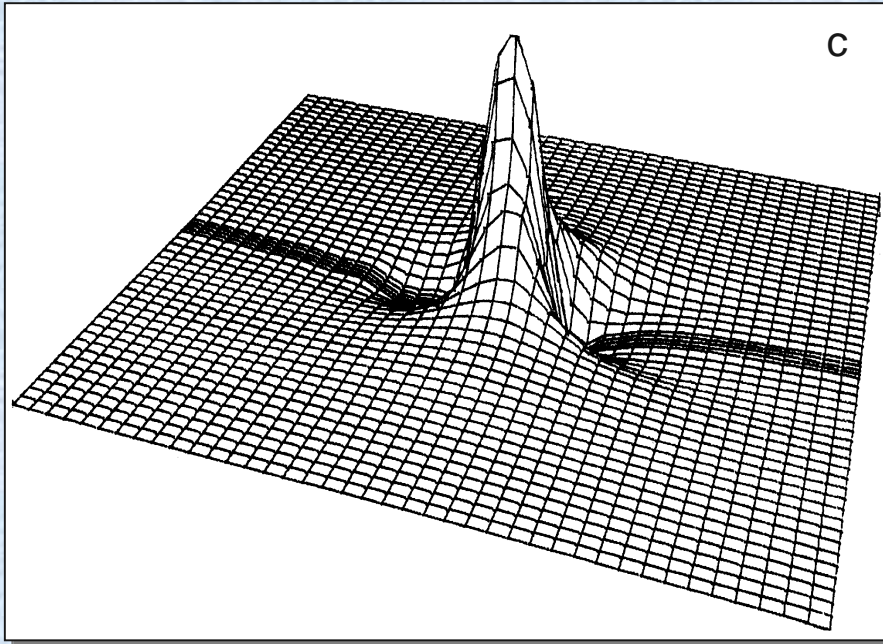
monotonous asymptotes

where A_1, A_2, A_3 and Θ_1, Θ_2 are arbitrary constants, and

$$Q_1^\pm = Q^+ \pm Q^- \quad Q^\pm = \left[1 \pm \sqrt{1 - 40^3 C^8 / 27\delta} \right]^{1/3} \quad C = |V|^{-1/4} \quad \chi = (\eta \pm \zeta + (\kappa - V)t)$$



In case $\beta = \gamma = 0$ the 2D GKP equation can have **unstable solutions** with **monotonous** and **oscillating** asymptotes in dependence on value of δ



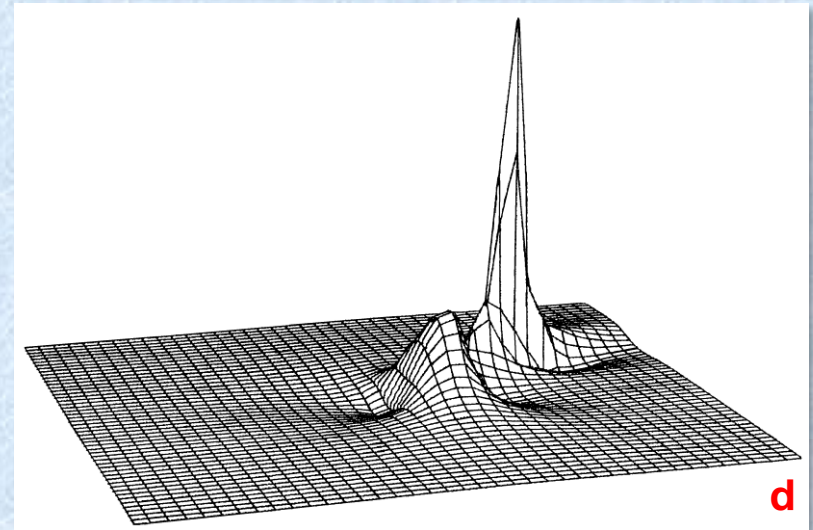
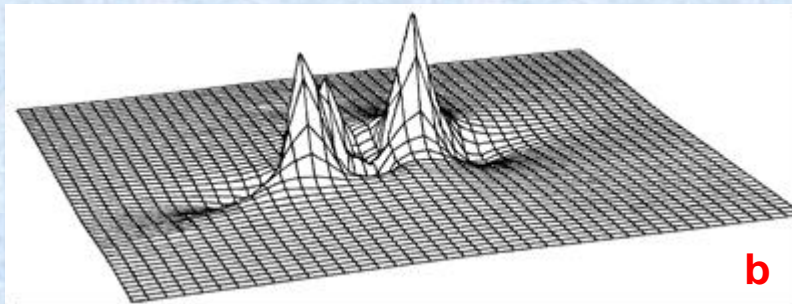
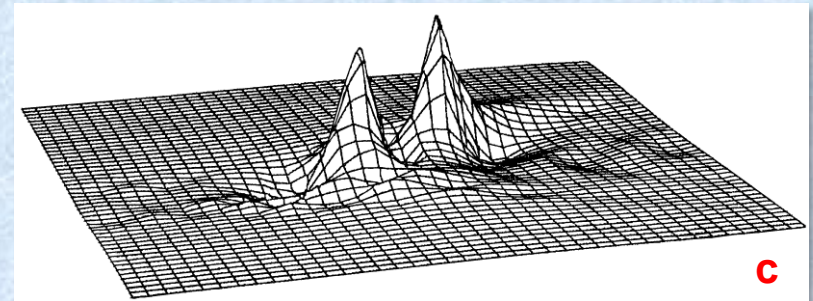
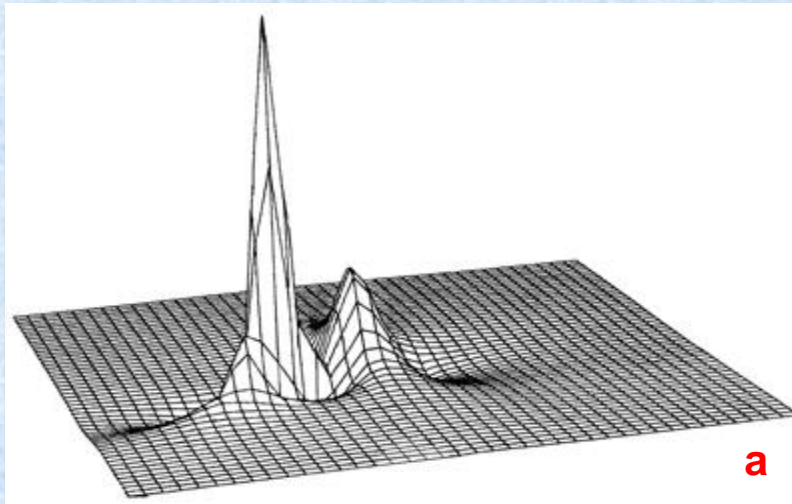
General view of 2D solutions of the GKP equation with $\Delta_{\perp} = \partial_y^2$ and $\beta = \gamma = 0, V > 0$:

(c) $\mu = 1, \delta = 1 \times 10^{-6}, \delta \leq (4/27) \mu^3 C^8; t = 0.2$;

(d) $\mu = 1, \delta = 1, \delta > (4/27) \mu^3 C^8; t = 0.5$

■ Interaction of 2D solitons of the GKP equation (*trivial case*)

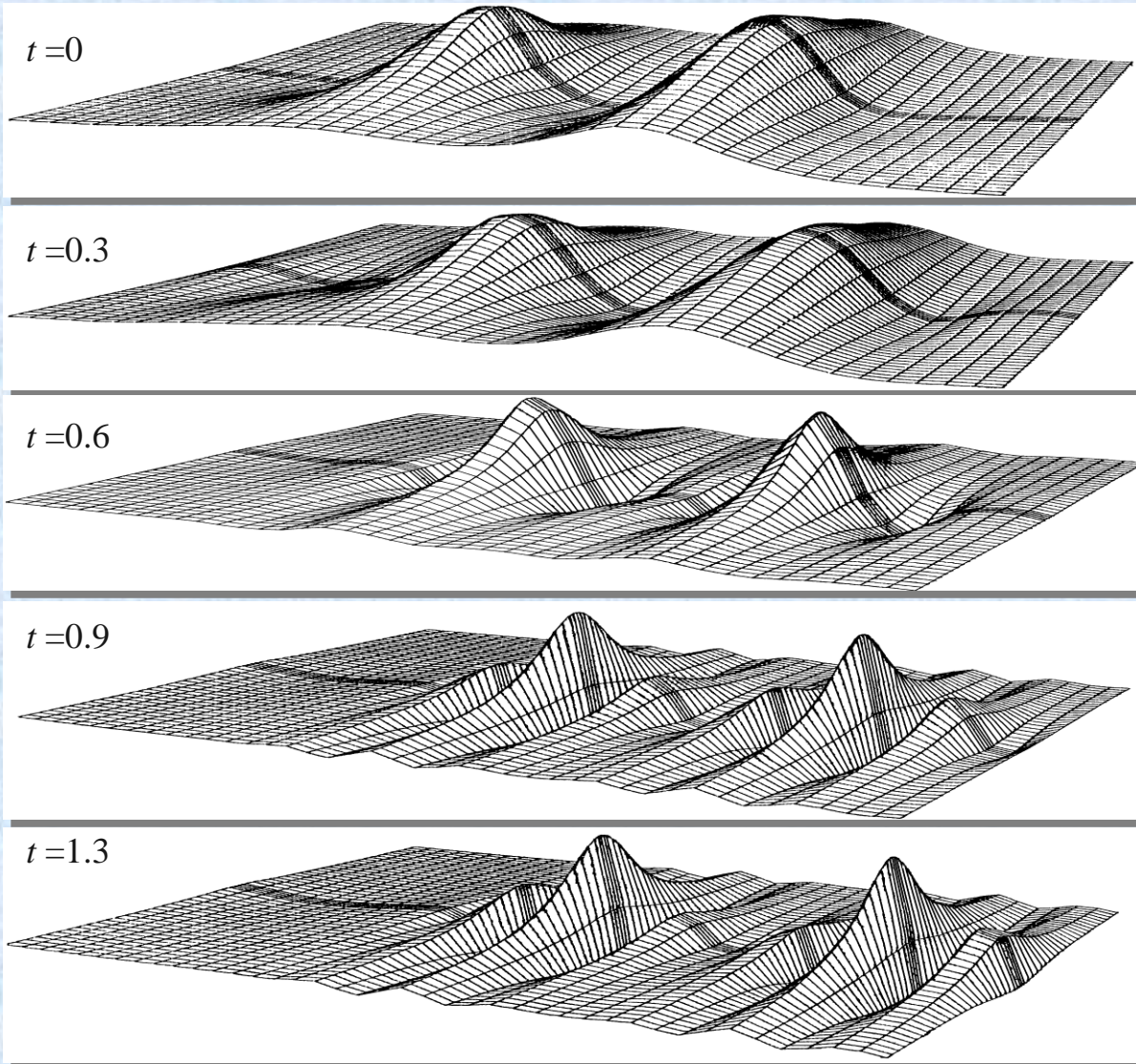
V.Yu. Belashov, V.I. Karpman, Phys. Lett. A, 1991



Slanting collision of 2D solitons with algebraic asymptotics and essentially differing amplitudes

N. Singh, Y. Stepanyants, Wave Motion, 2016

Bound states formation – 2D bi-solitons of GKP eq.



V.Yu. Belashov, V.I. Karpman, Phys. Lett. A, 1991

for the first time:

L.A. Abramyan, Yu.A. Stepanyants, ZhETF, 1985

K.A. Gorshkov, D.E. Pelinovskii, Yu.A. Stepanyants, ZhETF, 1993

V.Yu. Belashov, S.V. Vladimirov, Solitary Waves in Dispersive Complex Media. Theory, Simulation, Applications. Springer Verlag, 2005.

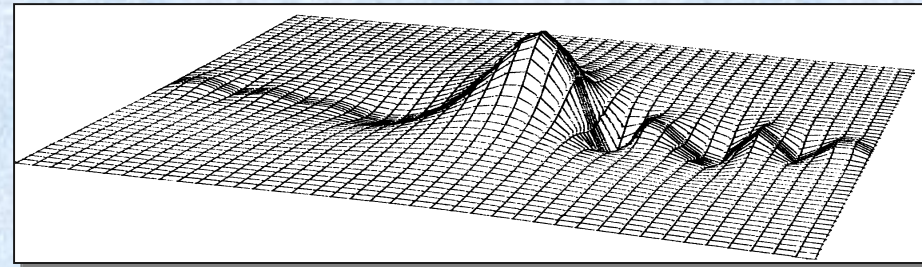
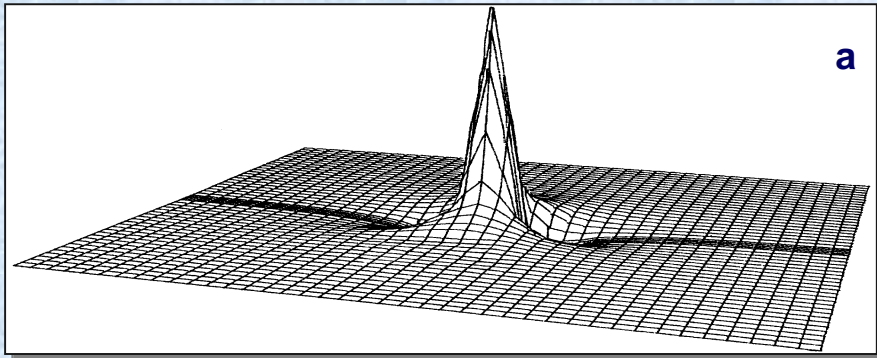
Formation of 2D bi-soliton at $u_1(0)=1.35$, $u_2(0)=1.3$, $\Delta x(0)=6$

■ Influence of dissipation on the soliton structure

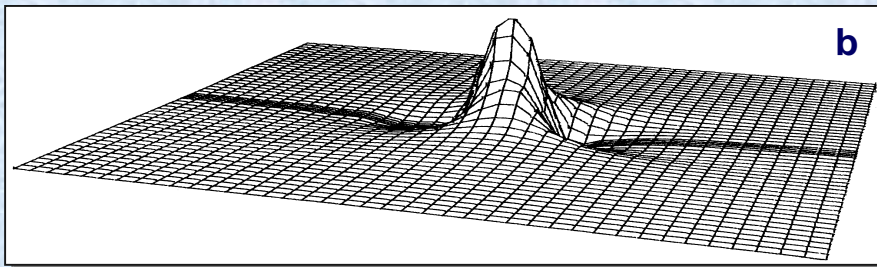
GKP equation

$$\partial_x \left(\partial_t u + \alpha u \partial_x u - \nu \partial_x^2 u + \beta \partial_x^3 u + \gamma \partial_x^5 u \right) = \kappa \Delta_{\perp} u$$

Numerical results



Evolution of 2D soliton: $\nu=1, \beta<0, \gamma>0$; $t=0.2$



Evolution of 2D soliton: $\nu=1, \beta, \gamma>0$:

a - $t=0$; **b** - $t=0.1$

V. Yu. Belashov, S. V. Vladimirov, Solitary Waves in Dispersive Complex Media. Theory, Simulation, Applications. Springer Verlag, 2005

Dissipation in a system, on a level with general damping of the wave field amplitude, directly influences on the structure of 2D solitons. In all cases we observe:

- **effect of elongation** of the soliton “tail”;
- **decreasing of frequency** of oscillations and **damping of oscillations** behind of the main maximum;
- **asymmetrical changes of integrals** P and H in front and back “cavities” (where $u < 0$)

Applications of the GKP and DNLS models

■ Propagation of the FMS wave beam in magnetized plasma

$$4\pi nT / B^2 \ll 1, \quad \omega < \omega_B = eB / m_i c$$

$$k\lambda_D \ll 1, \quad k_x^2 \gg k_\perp^2, \quad v_x \ll v_A = B^2 / 4\pi n m_i$$

Dispersion relation: $\omega \approx v_A k_x \left(1 + k_\perp^2 / k_x^2 + \chi(\theta) \lambda_D^2 k_x^2 \right)$

If $4\pi nT / B^2 < m_e / m_i$ structure of FMS waves depends on sign of dispersion coefficient

$$\beta = -v_A \chi(\theta) \lambda_D^2 = v_A \frac{c^2}{2\omega_{0i}^2} \left(\frac{m_e}{m_i} - \cot^2 \theta \right)$$

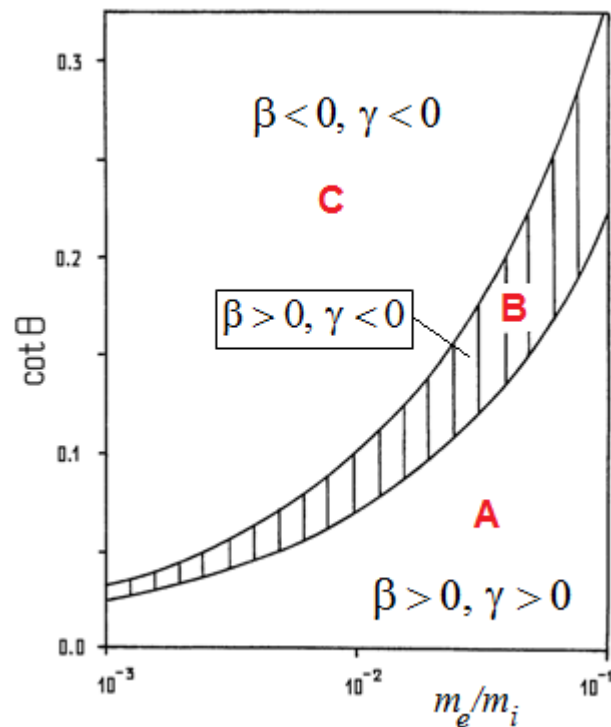
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and near the cone of angles where dispersion changes its sign (where $\beta \rightarrow 0$), depends also on sign of

$$\gamma = v_A \frac{c^4}{8\omega_{0i}^4} \left[3 \left(\frac{m_e}{m_i} - \cot^2 \theta \right)^2 - 4 \cot^4 \theta (1 + \cot^2 \theta) \right]$$

GKP equation

$$\partial_x \left(\partial_t h + \alpha h \partial_x h + \beta \partial_x^3 h + \gamma \partial_x^5 h \right) = - (v_A / 2) \Delta_\perp h, \quad h = B_\sim / B_0$$



B – case of *negative* dispersion

A and C – cases of “*mixed*” dispersion

Regions **B** and **C** – “magnetic sound” scattering

Region **A** – sub-focusing – nonlinear saturation – defocusing and formation of the stationary FMS beam

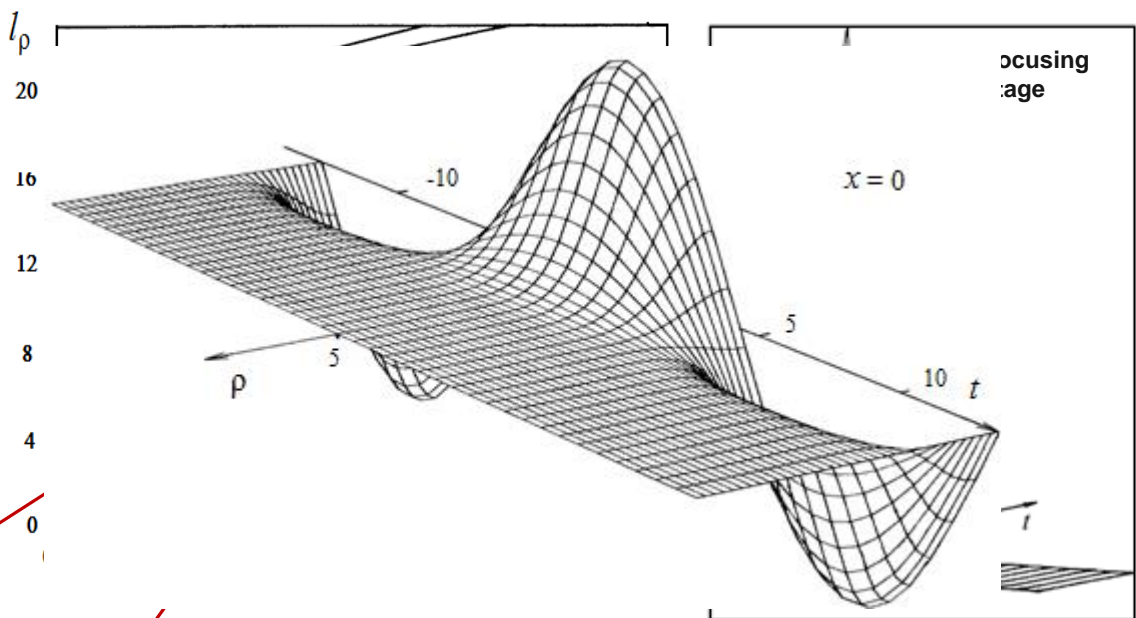
Problem: *there is 3D stationary FMS wave beam propagating in plasma at angle to magnetic field near the cone*

$$\theta = \arctan(m_i / m_e)^{1/2}$$

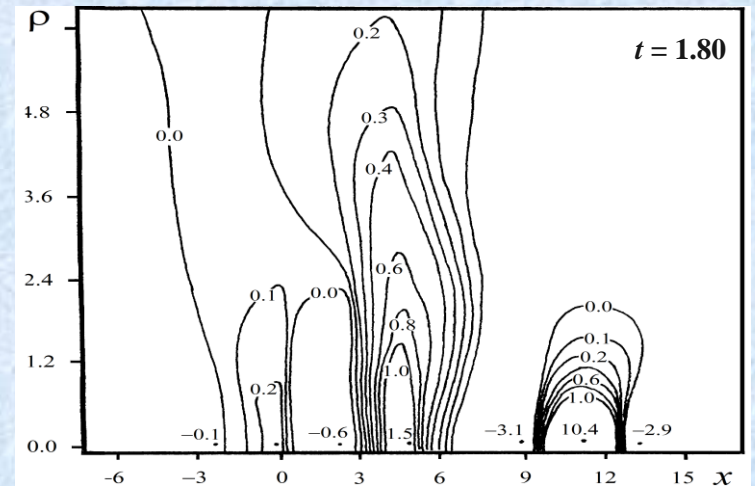
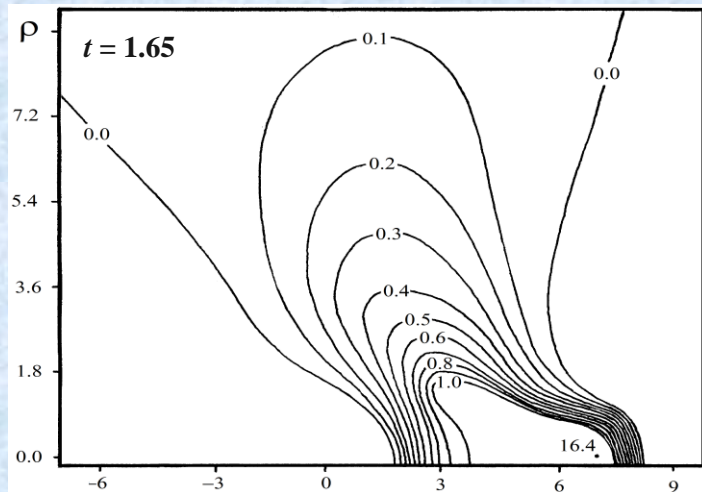
Scale transforms – transition to *boundary problem*:

$$\partial_t \left(\partial_x h + 6h \partial_t h - \varepsilon \partial_t^3 h - \lambda \partial_t^5 h \right) = \Delta_{\perp} h, \quad \Delta_{\perp} = \partial_{\rho}^2 + (1/\rho) \partial_{\rho}$$

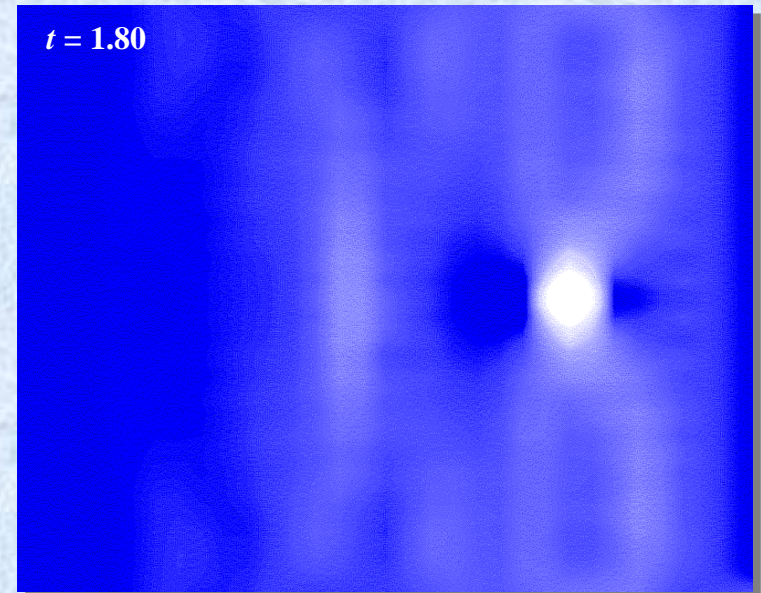
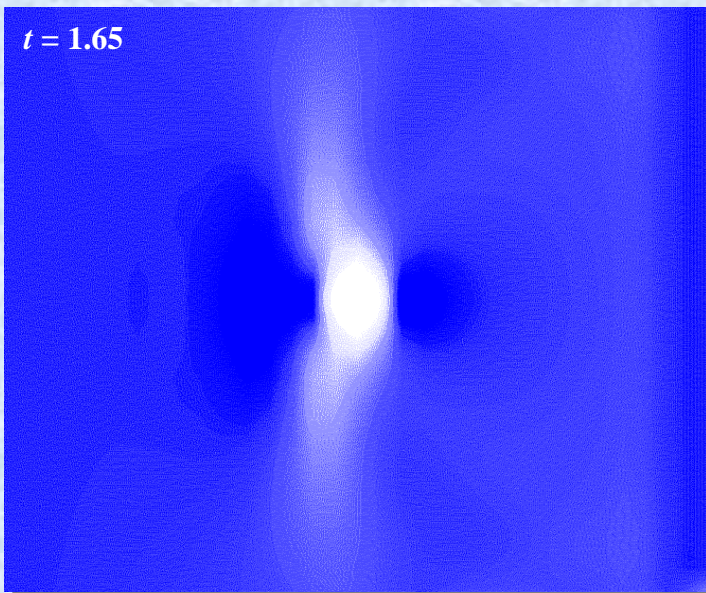
$$h_0 = h(t, 0, \rho) = \cos(mt) \exp(-\rho^2)$$



Evolution with formation of stable FMS wave beam

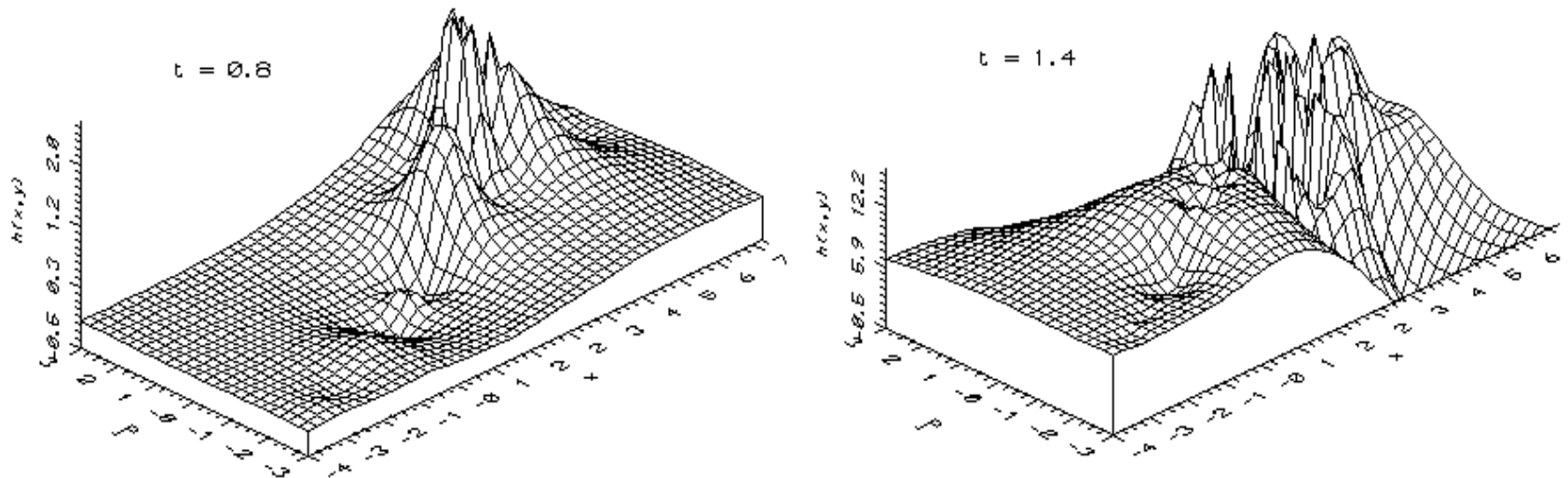


Solution in (x, ρ) -plane corresponding the stage of maximum of amplitude of beam



Evolution of 3D FMS wave in a plasma with stochastic fluctuations of wave field

$$\partial_t \left(\partial_x h + 6h \partial_t h - \varepsilon \partial_t^3 h - \lambda \partial_t^5 h + \eta(t) \right) = \Delta_{\perp} h$$

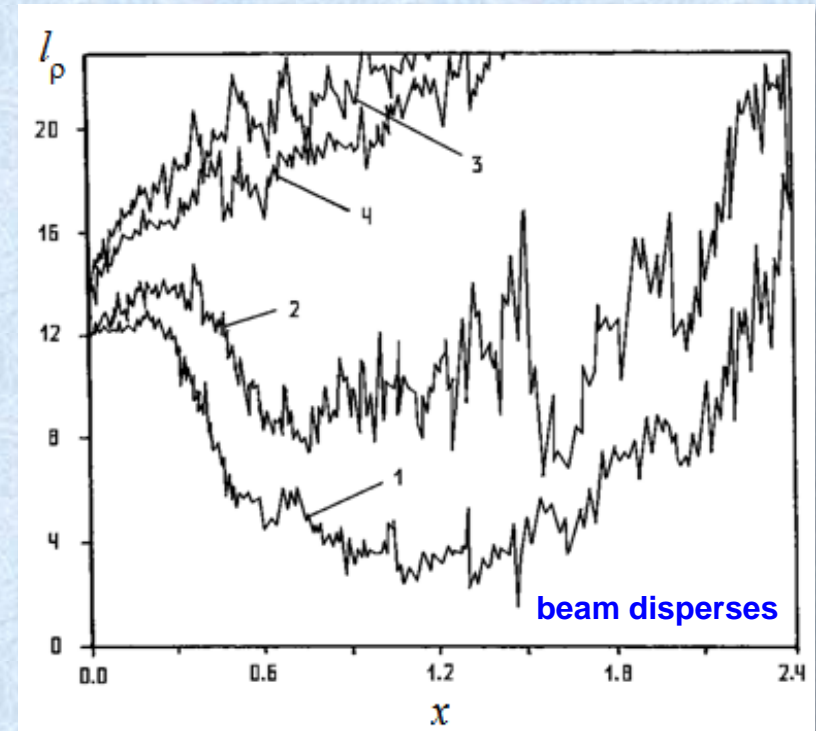
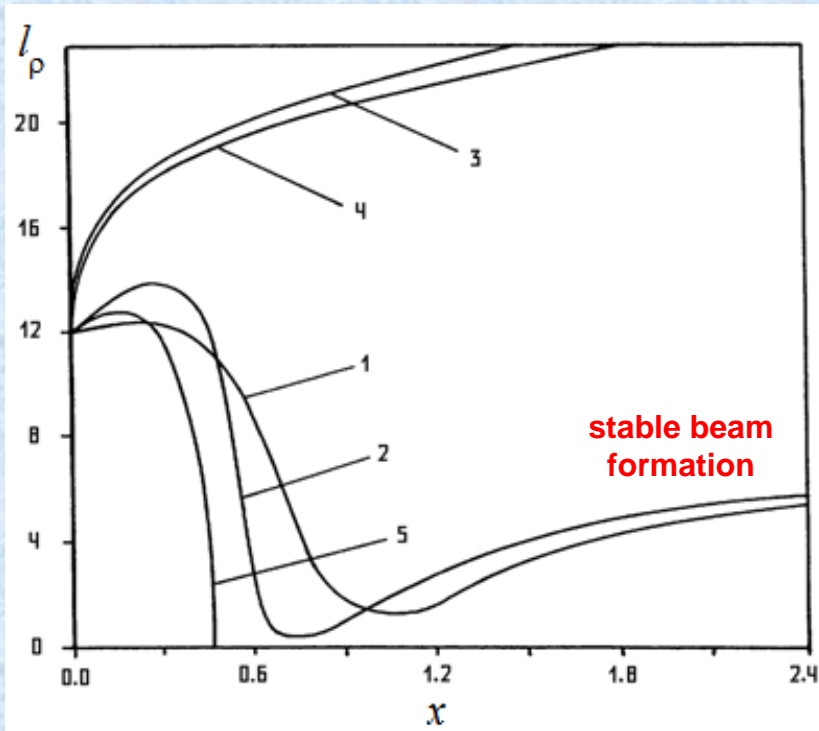


Evolution of the 3D FMS waves in a plasma with Gaussian noise $\eta = \eta(t)$
at standard deviation $\sigma = 0.02$ ($\lambda=1, \varepsilon=2.24$)

We observe *wave destruction with formation of the high frequency stochastic wave field*

Evolution in a plasma with stochastic fluctuations of the wave field

$$\partial_t \left(\partial_x h + 6h \partial_t h - \varepsilon \partial_t^3 h - \lambda \partial_t^5 h + \eta(t, x, \rho) \right) = \Delta_{\perp} h$$



Change of cross-section size of the wave beam at its propagation along the x -axis in a plasma $\eta=0$ (left) and $\eta=\eta(t, x, \rho)$ (right) at standard deviation of noise $\sigma = 0.04$ for different λ and ε

V.Yu. Belashov, Phys. Lett. A, 1995 and V.Yu. Belashov, E.S. Belashova, Geomag. and Aeronomy, 2016

■ Nonlinear effects for 3D Alfvén solitons in a plasma

3-DNLS equation:

$$\partial_t h + s \partial_x \left(|h|^2 h \right) - i \lambda \partial_x^2 h - v \partial_x^2 h = \sigma \int_{-\infty}^x \Delta_{\perp} h dx$$

$$h = (B_y + i B_z) / 2 B_0 |1 - \beta|^{1/2}, \quad \mathbf{h} = \mathbf{B}_{\perp} / B_0, \quad p = (1 + ie)$$

Statement of the problem:

problem geometry: $\Delta_{\perp} = \partial_{\rho}^2 + (1/\rho) \partial_{\rho}, \quad \rho^2 = y^2 + z^2$

Two kinds of initial conditions:

1) soliton-like axially symmetric pulse:

$$h(x, \rho, 0) = h_0(x) \exp \left[i \varphi(x) - \rho^2 / l_{\rho}^2 \right]$$

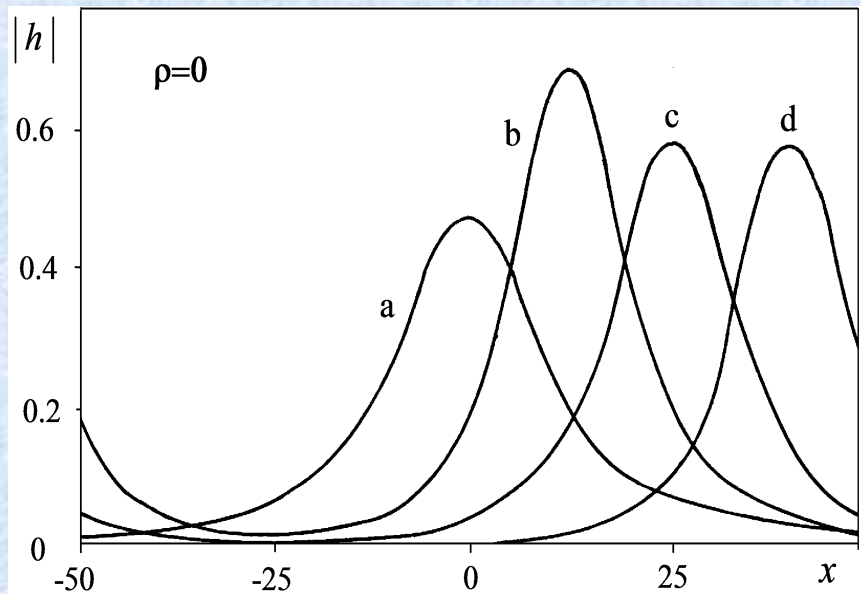
where $h_0(x) = 2\sqrt{2}\delta \sin \vartheta \left[\cosh(4\delta^2 \sin \vartheta x) + \cos \vartheta \right]^{-1/2}, \quad \varphi(x) = -2s\delta^2 \cos \vartheta x - (3s/4) \int_{-\infty}^x h_0^2(x) dx$
 $0 < \vartheta < \pi$

2) modulated plane wave:

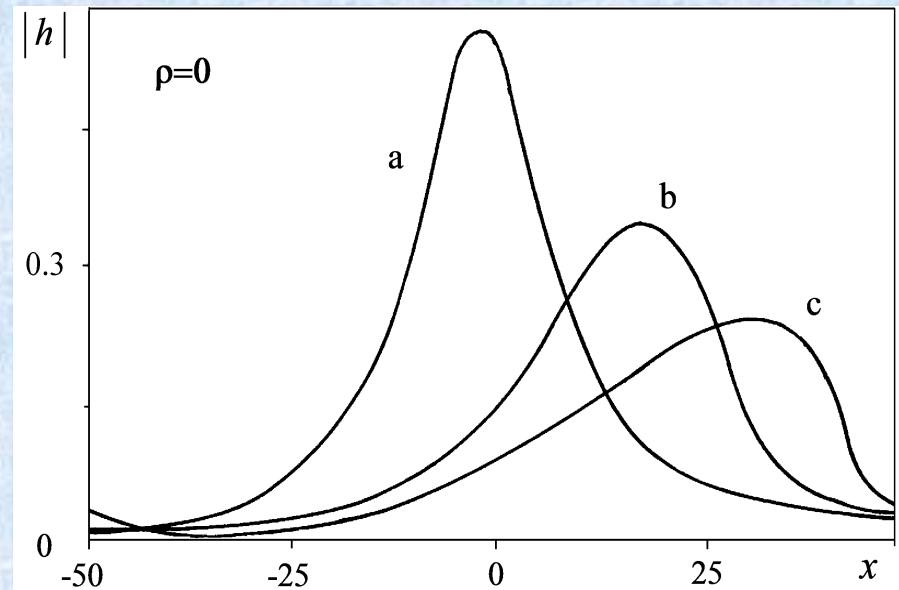
$$h(x, \rho, 0) = H_0 \exp \left(2\pi i x / \lambda - x^2 / l_x^2 - \rho^2 / l_{\rho}^2 \right)$$

where λ is the **wavelength**, H_0 is the **amplitude**, and l_x and l_{ρ} are the **characteristic scales of the Gaussian envelop modulation in the x and ρ -directions**

Numerical results



$t = 1.65$
Evolution of a **3D right circularly polarized axially symmetric nonlinear pulse** for $\lambda=1$, $s=-1$, $\kappa=1$;
 $\mathbf{H} > -3bd/(1+2d^2) > 0$: a) $t=0$, b) $t=25$, c) $t=50$, d) $t=75$



$t = 1.80$
Evolution of a **3D right circularly polarized modulated plane wave pulse** for $\lambda=-1$, $s=1$, $\kappa=1$; $\mathbf{H} < 0$:
a) $t=0$, b) $t=50$, c) $t=100$

The initial pulse weakly limited in the transverse ρ -direction when the **stability condition**

$$\mathbf{H} > -3bd / (1 + 2d^2), \quad b = \lambda s \int h h^* \partial_x \varphi \, d\mathbf{r},$$

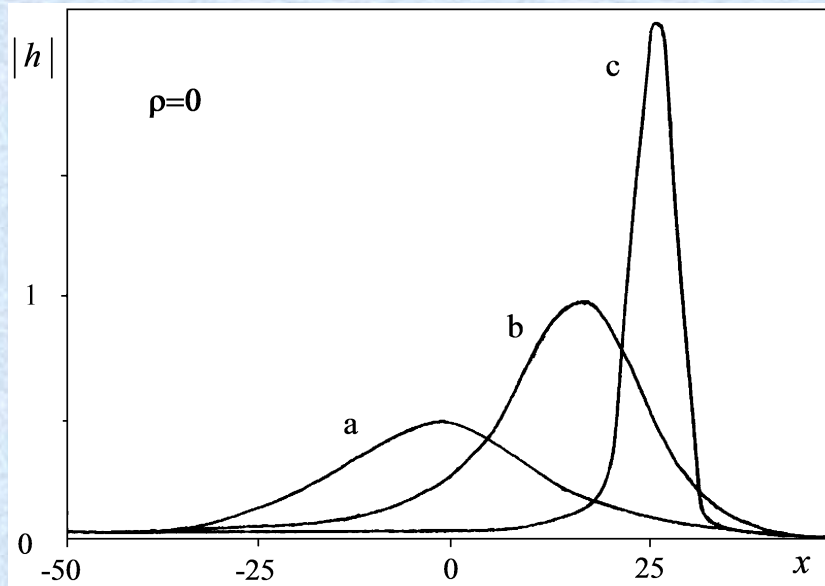
is satisfied, the evolution results in formation of the **stable 3D (axially-symmetric) solution**

At the opposite signs of λ and s , that is equivalent to change

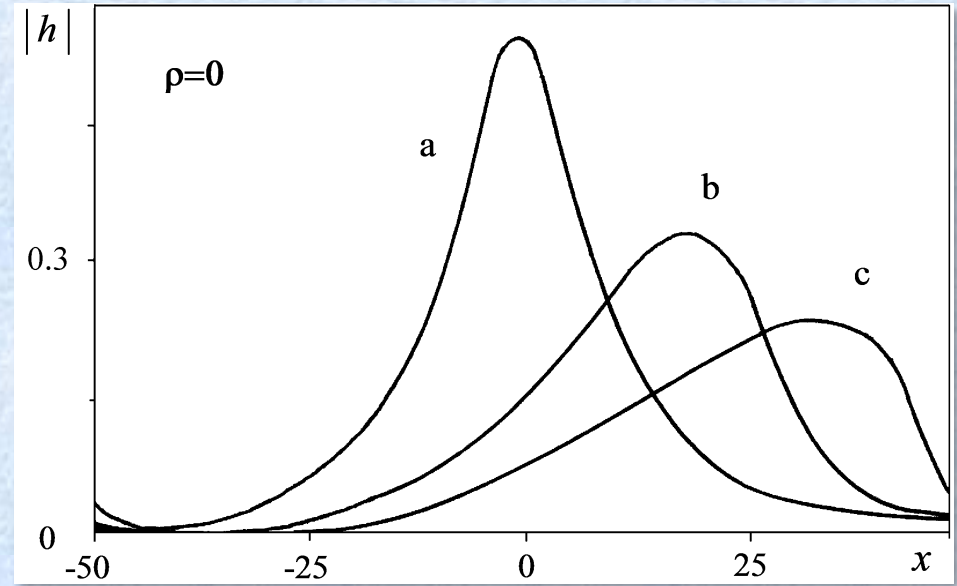
$$t \rightarrow -t, \quad \kappa \rightarrow -\kappa$$

the Hamiltonian of the 3-DNLS equation becomes negative, and a **3D Alfvén wave spreads with evolution**





Evolution of a **3D right circularly polarized axially symmetric nonlinear pulse** for $\lambda=1$, $s=-1$, $\kappa=0.2$; $0 < H < -3bd/(1+2d^2)$: a) $t=0$, b) $t=25$, c) $t=30$



Evolution of a **3D nonlinear left circularly polarized modulated plane wave pulse** for $\lambda=s=\kappa=1$; $H > 0$: a) $t=0$, b) $t=50$, c) $t=100$

Here the condition of the existence of the local minimum of H

$$H > -3bd / (1 + 2d^2), \quad b = \lambda s \int h h^* \partial_x \varphi \, dr,$$

is not satisfy, and one can observe **development of the 3D collapsing solutions of the 3-DNLS eq.**

For 3D nonlinear **left circularly polarized modulated plane wave pulse** for all above mentioned cases one can observe **a mirror opposite picture**

■ Nonlinear waves in media with variable dispersion

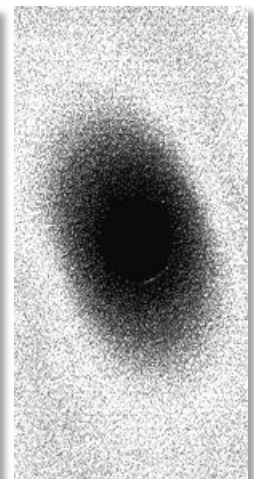
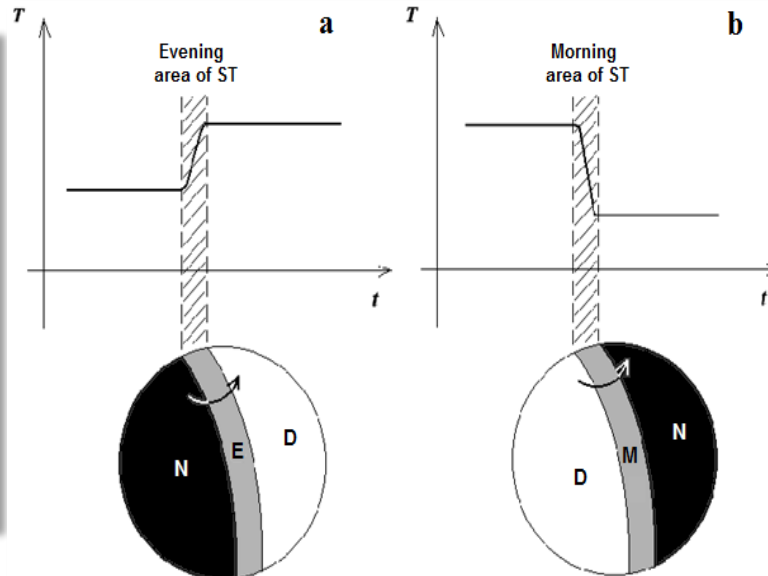
- For example, that takes place on studying of the evolution of **3D FMS waves in magnetized plasma**, when in the GKP equation

$$\beta = \underline{v_A} (c^2 / 2\omega_{0i}^2) (\cot^2 \underline{\theta} - m_e / m_i), \quad v_A = f[B(t, \mathbf{r}), n(t, \mathbf{r})], \quad \theta = (\mathbf{k} \wedge \mathbf{B})$$

- Next example: the **2D nonlinear waves on surface of “shallow water”**, when

$$\beta = (\sqrt{gH} / 6)(\underline{H}^2 - 3\sigma / \rho g) \text{ is a function of depth.}$$

- One more characteristic example is the propagation of the **solitons-like IGW** at heights of the ionosphere's F-layer **in regions with sharp gradients of the ionospheric parameters**, including the regions of the **fronts of solar terminator (ST) and spot of the solar eclipse (SE)**



SE spot

Consider an example of **2D nonlinear waves on surface of “shallow water”**, when in the GKP equation

$$\partial_x \left(\partial_t u + \alpha u \partial_x u - \nu \partial_x^2 u + \beta \partial_x^3 u + \gamma \partial_x^5 u \right) = \kappa \Delta_{\perp} u$$

β and γ are the functions of depth:

$$\beta = \frac{c_0}{6} \left(\frac{3\sigma}{\rho g} - H^2 \right), \quad \gamma = (c_0/6) \left[H^2 \left(\frac{2}{5} H^2 - \sigma/\rho g \right) - \frac{1}{12} \left(3\sigma/\rho g - H^2 \right)^2 \right] = f[H(t, x, y)]$$

Initial condition: $u(0, x, y) = u_0 \exp[-(x/l_x + y/l_y)^2]$

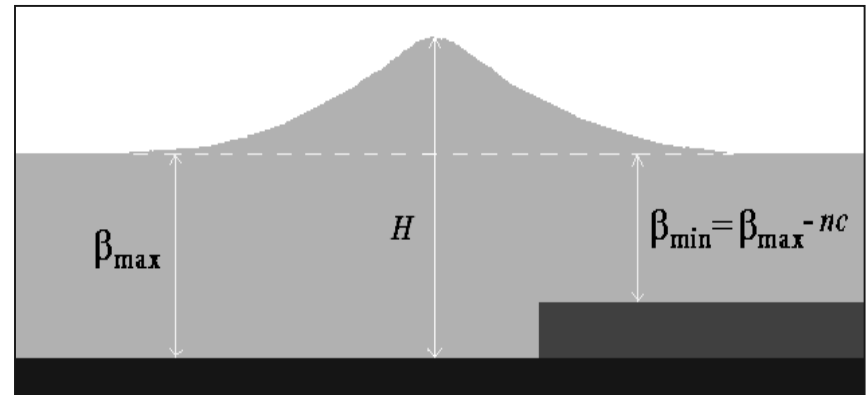
Boundary conditions were periodic

Model of depth: “step” – sharp and gradual «break of a bottom»

$$\beta(r, t) = \begin{cases} \beta_0, & r \leq a; \\ \beta_0 - nc, & r > a. \end{cases}$$

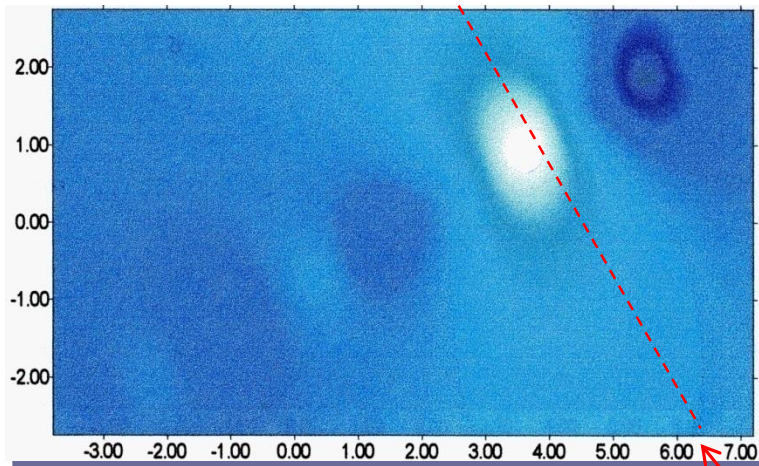
$n(t) > 0$ – **positive** step;

$n(t) < 0$ – **negative** step



Dependence $\beta(r, t)$ of type of “step”

Numerical results (2D case – GKP equation)

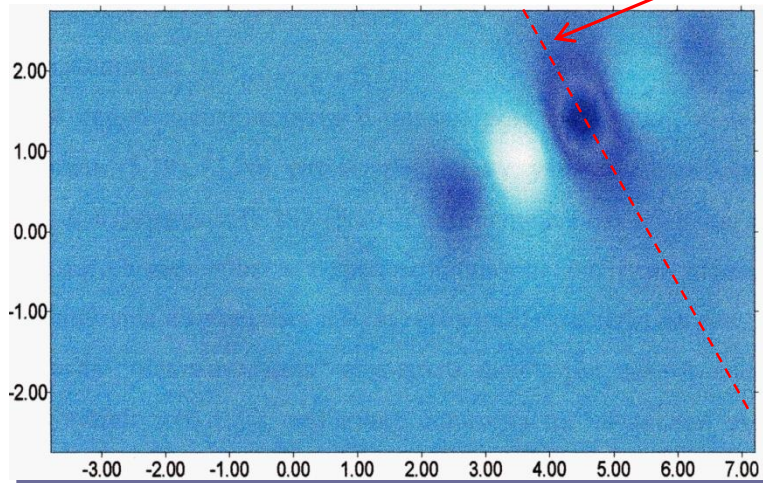


Evolution of the 2D GKP soliton for dispersion low of type “sharp break of a bottom” at $t = 0.6$

negative «step» is *directly under initial disturbance*:

- perturbation caused by sharp jump of the dispersive parameter β has a *local character*, i.e. *Is not propagate* together with leaving forward soliton;
- the *asymptotes* of separated soliton *become oscillating*;
- on a background of long-wave fluctuations of a soliton tail you can see also the *wavy oscillations*

«step»



Evolution of the 2D GKP soliton for dispersion low of type “gradual change of a height of a site of bottom” at $t = 0.8$

positive «step» is *far ahead of initial disturbance*:

- *soliton evolution at an initial stage* practically does not differ qualitatively from its evolution at $\beta = \text{const}$,
- however in further its character is defined by presence of step – *because of intensive generation on forward front of harmonics with great k_x , there is an appreciable changing of the soliton structure which can lead to overturning of a wave*

Conclusion

*In conclusion, we have discussed some problems of the multidimensional solitons dynamics in complex dispersive media being described by the BK model (GKP and 3-DNLS eqs.), taking into account the **high order dispersive corrections and influence of dissipation of viscous type**, namely:*

- *stability of 2D and 3D solitons;*
- *possible classes of stable and unstable 2D and 3D solutions and*
- *asymptotic behavior of solutions;*

and also have discussed some of possible **applications of the theory**, such as:

- *propagation of the FMS wave beam in magnetized plasma;*
- *nonlinear effects for 3D Alfvén solitons in a plasma, and*
- *evolution of 2D nonlinear waves on surface of fluids with variable dispersion.*

There are also many other applications in different physical media.



2005

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Solitary Waves in Dispersive Complex Media

Theory · Simulation
Applications

 Springer

2016

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Е.С. БЕЛАШОВА*

СОЛИТОНЫ

ТЕОРИЯ,
МОДЕЛИРОВАНИЕ,
ПРИЛОЖЕНИЯ

THANK YOU!

THANKS TO ORGANIZERS OF THE WORKSHOP!

