The forced coupled KdV equations as a model for internal waves in the atmosphere

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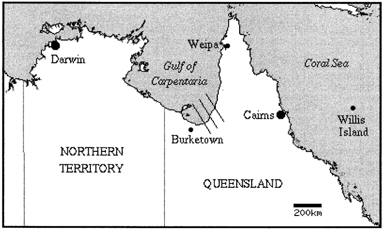
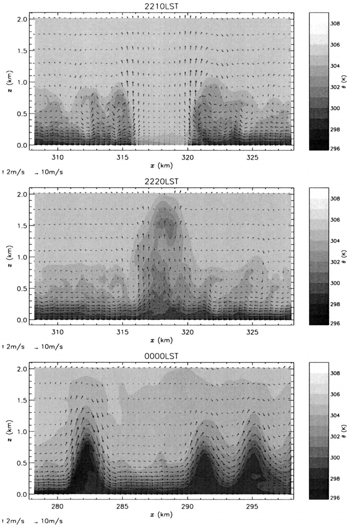
In honour of Roger Grimshaw

Morning Glory Wave

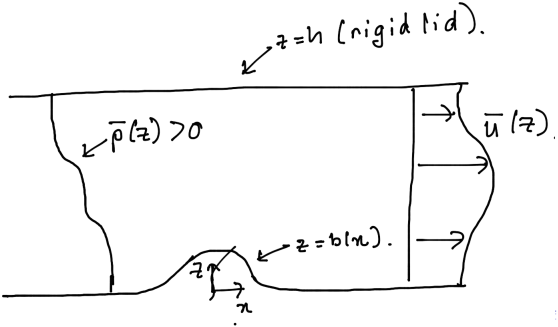


- Modelling of Goler & Reeder (2004).



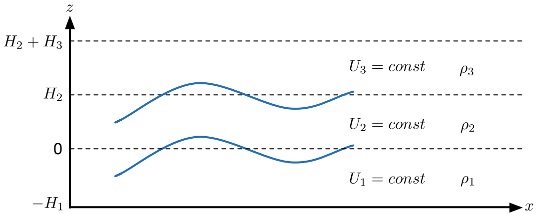
Resonant interactions of long waves

- Goler & Reeder (2004) suggest dynamics of internal waves effected by environmental shear flow, Cape York topography.
- Consider long, internal wave modes:
  - $kh \ll 1$;
  - Boussinesq approximation;
  - Isopycnal displacement ζ, s.t. $\bar{\rho} = \bar{\rho}(z - \zeta)$;
  - Buoyancy (or Brunt–Väisälä) frequency: $N^2 = -\dfrac{g\bar{\rho}_z}{\rho}$;
  - Mean environmental shear $\bar{u}(z)$.

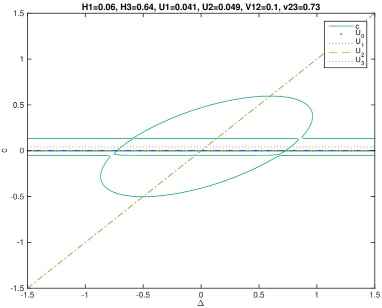


Resonant interactions of three-layer shears

- For a piecewise constant three-layer shear

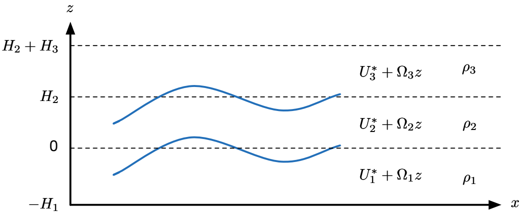


obtain regular and singular resonances:

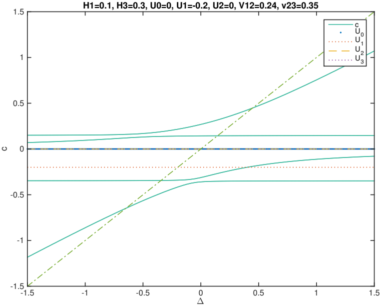


Resonant interactions of three-layer shears

- For a piecewise continuous three-layer linear shear



obtain regular resonances:



Long, weakly-nonlinear internal waves, $b \equiv 0$

- Assume nonlinearity $a/h = O(\epsilon)$, and dispersion $kh = O(\mu)$, for quasi-steady waves.
- Long, weakly nonlinear approximation $\epsilon = \mu^2 \ll 1$, Boussinesq approximation $\bar{N}h \ll (gh)^{\frac{1}{2}}$
- Introduce:

$$z = h\hat{z}, \quad x = \mu^{-1}h\hat{x}, \quad t = \mu^{-1}\bar{N}^{-1}\hat{t}, \quad N = \bar{N}\hat{N},$$
$$\zeta = \epsilon h\hat{\zeta}, \quad u = \bar{N}h(\bar{u}(\hat{z}) + \epsilon u(\hat{x}, \hat{z}, \hat{t})).$$

Resonant interaction with topography

- Assume particular wave mode approximately stationary, such that

$$\bar{u} = \bar{u}_0 + \epsilon\Delta,$$

where

$$(\bar{u}_0^2\phi_z)_z + N^2\phi = 0,$$

$$\phi = 0 \text{ on } z = 0, 1.$$

- Assume small-amplitude topography, and slow variation

$$b = \epsilon^2\hat{b}(x), \quad \tau = \epsilon t,$$

and

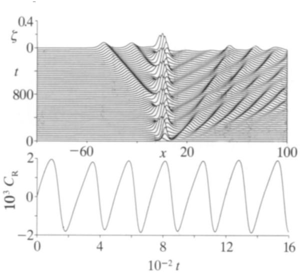
$$\zeta = A(x,\tau)\phi(z) + \epsilon\zeta^{(2)} + \ldots$$

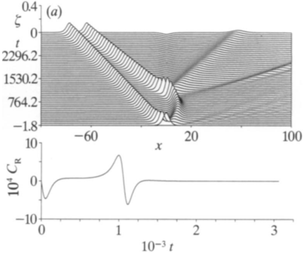
- Amplitude now satisfies forced KdV (fKdV) equation:

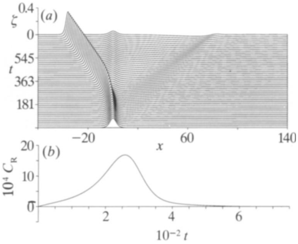
$$A_\tau + \Delta A_x + A_x + rAA_x + sA_{xxx} = -\gamma f_x.$$

Resonant interaction with topography, solitary waves

Camassa & Wu (1991) considered analytical and numerical stability of solitary wave solutions of fKdV equation:







Resonant interactions between long waves, $b \equiv 0$

- Consider case of two resonant modes, such that have $\phi_{n,m}$ where

$$\left[(\bar{u} - c_0 \mp \epsilon c)^2 \left(\phi_{n,m}\right)_z\right]_z + N^2 \phi_{n,m} = 0,$$

$$\phi_{n,m} = 0 \text{ on } z = 0, 1.$$

- Now assume

$$\theta = x - c_0 t, \quad \tau = \epsilon t, \quad \zeta^{(0} = A(x,\tau)\phi_n(z) + B(x,\tau)\phi_m(z).$$

- Since ϕ_n and ϕ_m are orthogonal, can now obtain the coupled KdV equations (Grimshaw, 2013):

$$A_\tau + \left(cA_\theta + \lambda_1 A_{\theta\theta} + \sigma B_{\theta\theta} + \mu_1 A^2 + \nu_1 B^2 + 2\nu_2 AB\right)_\theta = 0,$$
$$B_\tau + \left(-cB_\theta + \lambda_2 B_{\theta\theta} + \sigma A_{\theta\theta} + \mu_2 B^2 + \nu_2 A^2 + 2\nu_1 AB\right)_\theta = 0,$$

where coefficients are dependent on integrals of N^2, $(\bar{u} - c_0)$, ϕ_n and ϕ_m.

Resonant interactions between long waves and topography

- To leading order, assume:

$$\tau = \epsilon t, \quad b = \epsilon \hat{b}(x), \quad \bar{u} = \bar{u}_0 + \epsilon \Delta,$$
$$\zeta^{(0)} = A(x,\tau)\phi_n(z) + B(x,\tau)\phi_m(z),$$

  such that:

$$\left[(\bar{u}_0 \mp \epsilon c)\,(\phi_{n,m})_z\right]_z + N^2 \phi_{n,m} = 0, \quad \phi_{n,m} = 0 \quad \text{on} \quad z = 0, 1.$$

- Hence can show that compatibility condition at $O(\epsilon^2)$ is that amplitudes satisfy coupled fKdV equations:

$$A_\tau + \big((\Delta + c)A + \lambda_1 A_{xx} + \sigma B_{xx} + \mu_1 A^2 + \nu_1 B^2 + 2\nu_2 AB\big)_x = -\gamma_1 b_x,$$

$$B_\tau + \big((\Delta - c)B + \sigma A_{xx} + \lambda_2 B_{xx} + \nu_2 A^2 + \mu_2 B^2 + 2\nu_1 AB\big)_x = -\gamma_2 b_x.$$

Resonant interactions between long waves and topography

- Rescaling, can assume $c = 1$ and canonical form is:

$$A_\tau + \big((\Delta+1)A + \lambda_1 A_{xx} + \sigma B_{xx} + \mu_1 A^2 + \nu_1 B^2 + 2\nu_2 AB\big)_x = -\gamma_1 b_x,$$

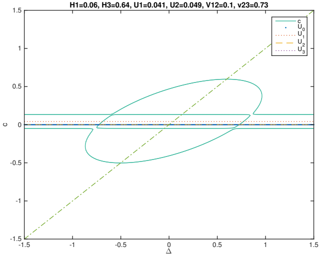
$$B_\tau + \big((\Delta-1)B + \sigma A_{xx} + \lambda_2 B_{xx} + \nu_2 A^2 + \mu_2 B^2 + 2\nu_1 AB\big)_x = -\gamma_2 b_x.$$

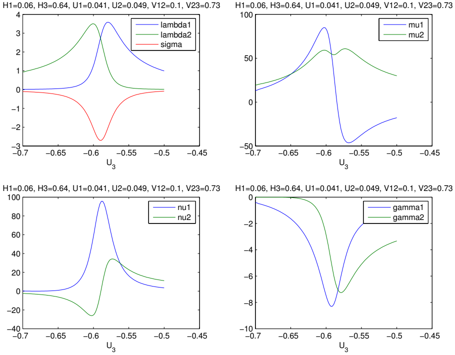
- System has Hamiltonian and momentum:

$$\mathcal{H}(A,B) = \int \Big[-\frac{1}{2}(\Delta+1)A^2 - \frac{1}{2}(\Delta-1)B^2 - \gamma_1 Ab - \gamma_2 Bb + \frac{1}{2}\lambda_1 A_x^2 + \sigma A_x B_x + \frac{1}{2}\lambda_2 B_x^2 - \frac{1}{3}\mu_1 A^3 - \nu_2 A^2 B - \nu_1 AB^2 - \frac{1}{3}\mu_2 B^3\Big]\, dx,$$

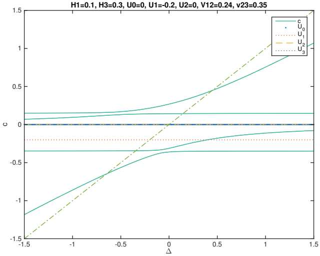
$$\mathcal{P}(A,B) = \int \big[A^2 + B^2\big]\, dx.$$

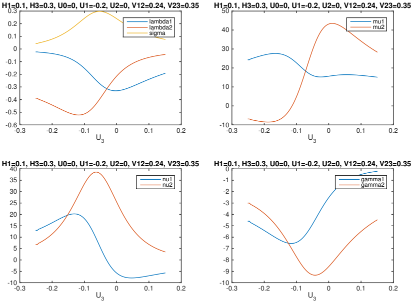
Resonant interactions between long waves and topography





Resonant interactions between long waves and topography





AITEM for coupled solitary waves (Yang, 2011)

- Let
$$\Lambda = [A\ B]^T.$$
Can write the coupled equations as
$$\Lambda_t = \partial_x \mathcal{D}[\Lambda] + \partial_x N(\Lambda),$$
where $\mathcal{D}$ represents a diffusive operator and N represents the nonlinear and forcing terms.
- Look for steady solution of the form $\Lambda(x + Vt)$ and integrating once and assuming $|\Lambda| \to 0$ and $|x| \to \infty$, then
$$\mathcal{D}\{\Lambda\} + N(\Lambda) - V\Lambda = 0.$$
- Treat this as a nonlinear diffusion equation
$$\Lambda_t = \mathcal{D}\{\Lambda\} + N(\Lambda) - V\Lambda. \tag{1}$$
Use implicit–explicit differencing for arbitrary h:
$$\Lambda^* = \Lambda^n + h[\mathcal{D}\{\Lambda\} - c^*\Lambda]^* + h[N(\Lambda) - (V - c^*)\Lambda]^n.$$

AITEM for coupled solitary waves (Yang, 2011)

- Hence

$$\Lambda^* = \Lambda^n + \left(\frac{1}{h} I - \mathcal{D}\right)^{-1} [\mathcal{D}\{\Lambda\} + N(\Lambda) - V\Lambda]^n, . \qquad (2)$$

  Let

$$\Lambda^{n+1} = \sqrt{\frac{P^n}{P^*}}\Lambda^*, \quad P^* = \langle \Lambda^*, \Lambda^* \rangle.$$

- To calculate V multiply (2) by Λ^n, sum over domain and assume $\Lambda^* = \Lambda^n$, then

$$V = \frac{\langle M^{-1}[\mathcal{D}\{\Lambda\} + N(\Lambda)]^n, \Lambda^n \rangle}{\langle M^{-1}\Lambda^n, \Lambda^n \rangle}.$$

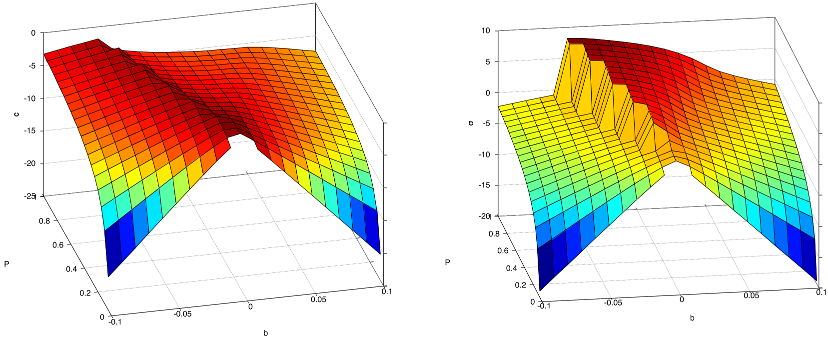
  This is equivalent to assuming

$$\frac{dP}{dt} = 0.$$

- Accelerated convergence then occurs for optimal h.

AITEM for coupled solitary waves (Yang, 2011)

Allows calculation of wave-speed and growth rate:



Unsteady simulations of solitary waves

- Can write unsteady equations in form:

$$A_\tau + \big((\Delta+1)A + \lambda_1 A_{xx} \\ + \sigma_1 B_{xx} + \mu_1 A^2 + \nu_{11} B^2 + 2\nu_{21} AB\big)_x = \gamma_1 b_x,$$

$$B_\tau + \big((\Delta-1)B + \sigma_2 A_{xx} + \lambda_2 B_{xx} \\ + \nu_{22} A^2 + \mu_2 B^2 + 2\nu_{12} AB\big)_x = \gamma_2 b_x,$$

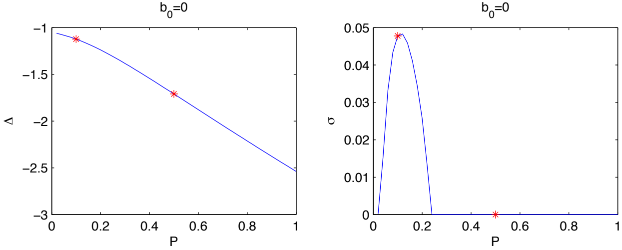
  and can assume $\mu_{1,2}$ or $\gamma_{1,2}$ are arbitrary.
- Unsteady integrations performed using Linearly Implicit RK4 (Calvo et al., 2001).

Case 1: Unforced solitary waves

$$\lambda_1 = 0.955, \lambda_2 = 0.767, \sigma_1 = -0.005, \sigma_2 = -1.326,$$

$$\mu_1 = \mu_2 = 3, \nu_{11} = 0.119, \nu_{12} = 31.97, \nu_{21} = 0.131, \nu_{22} = 35,$$

$$\gamma_1 = -0.0237, \gamma_2 = -0.8632.$$



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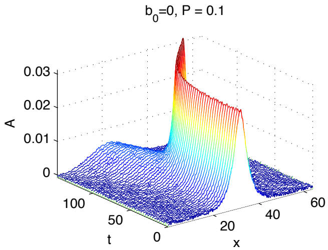

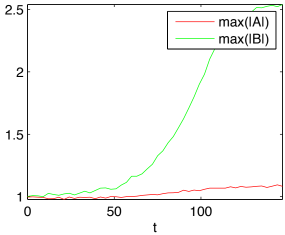

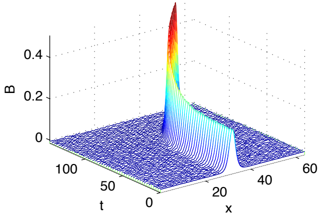

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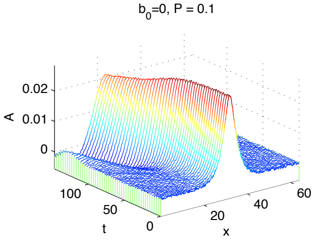
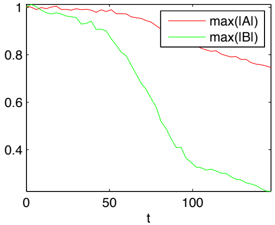
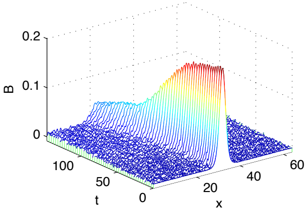


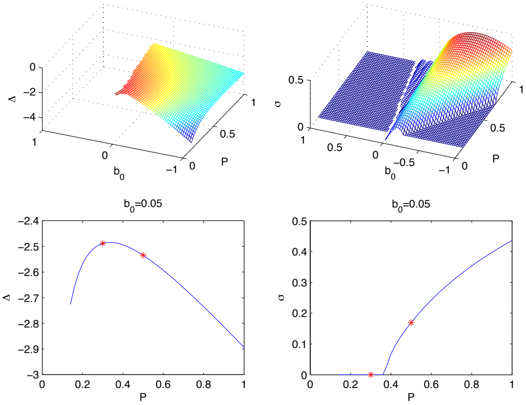




Case 2: Forced solitary waves, $b_0 = -0.5$

$$\lambda_1 = 1, \lambda_2 = 1, \sigma_1 = .1, \sigma_2 = .1, \gamma_1 = 1, \gamma_2 = -1$$

$$\mu_1 = \mu_2 = 3, \nu_{11} = 1, \nu_{12} = 1, \nu_{21} = 1, \nu_{22} = 1,$$



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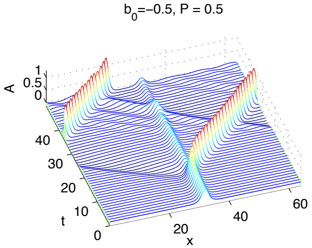
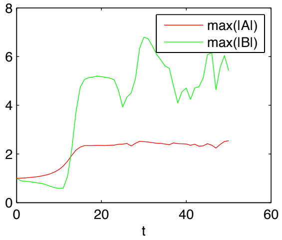
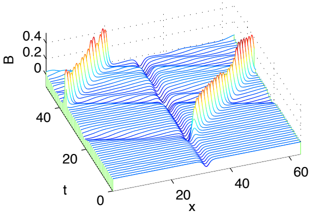


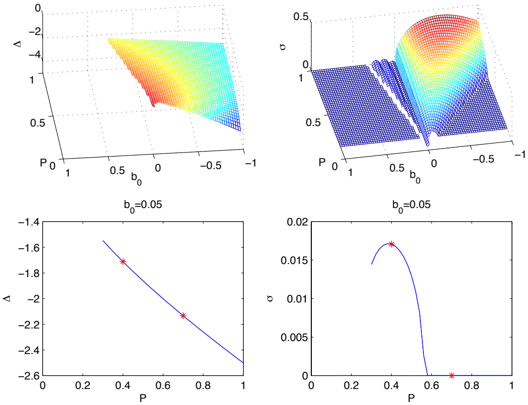




Case 2: Forced solitary waves, $b_0 = 0.05$

$$\lambda_1 = 1, \lambda_2 = 1, \sigma_1 = .1, \sigma_2 = .1, \gamma_1 = 1, \gamma_2 = -1$$

$$\mu_1 = \mu_2 = 3, \nu_{11} = 1, \nu_{12} = 1, \nu_{21} = 1, \nu_{22} = 1,$$



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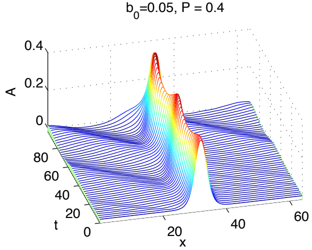

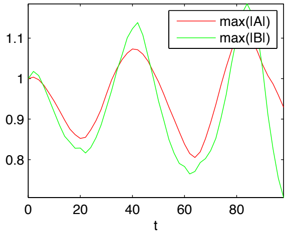

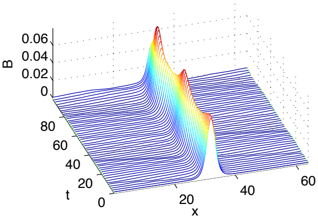

Conclusions

- Resonant interactions between topography and wave modes produce large-amplitude travelling waves.
- Are these significantly different from forced uncoupled modes? Yet to determine.
- What forms of asymmetric solutions are possible, and where is three-way coupling maximised?
- What is the effect of critical layers?
- Is there a simple criteria for stability of forced solitary waves?