UNIVERSITY OF SOUTHERN QUEENSLAND

Soliton interaction with external forcing within the Korteweg–de Vries equation

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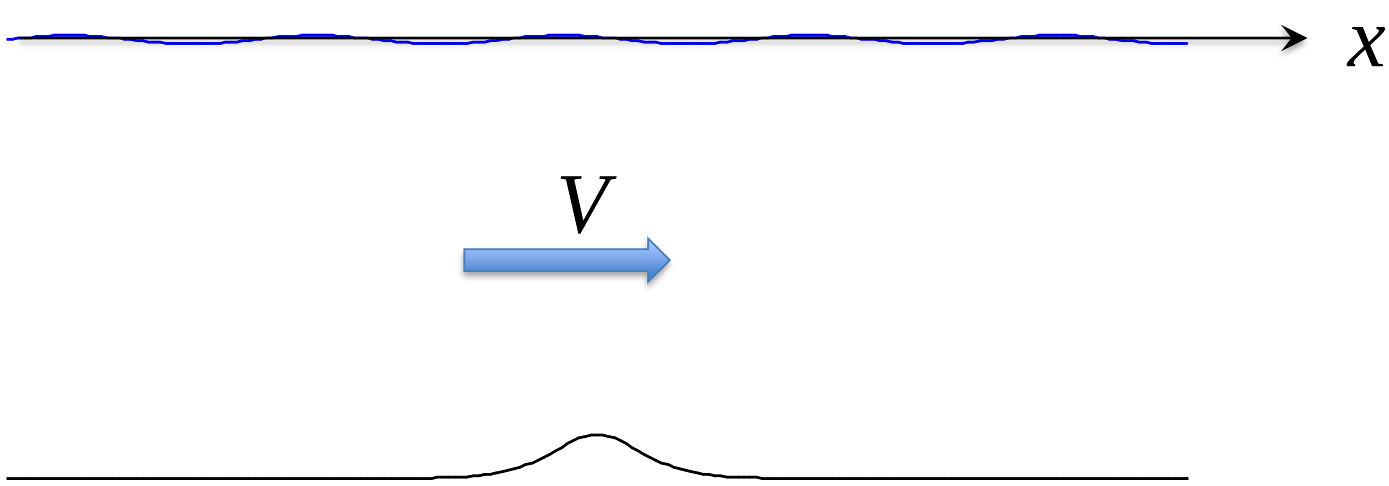
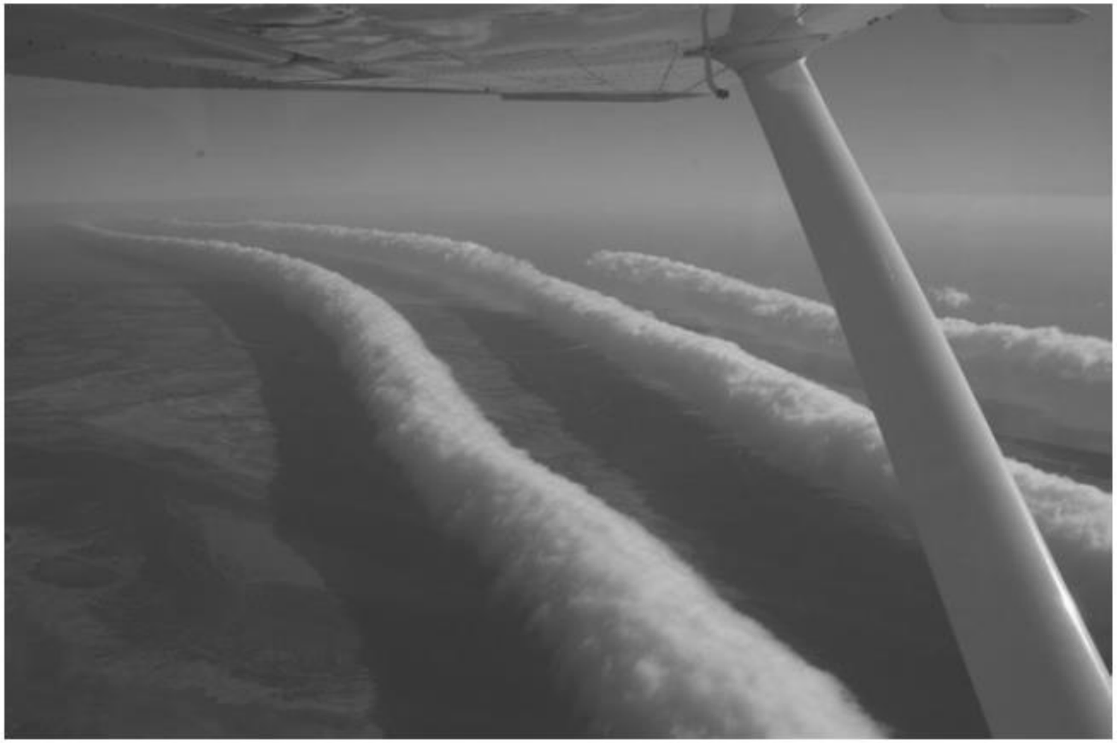
The forced Korteweg–de Vries equation

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}+\alpha u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=\varepsilon\frac{\partial f\left(x,t\right)}{\partial x}$$

Flow around an underwater obstacle



Air flow around a mounting



The Morning Glory clouds
(Mick Petroff, http://creativecommons.org/licenses/by-sa/3.0)

The forced Korteweg–de Vries equation

In the moving coordinate frame the fKdV equation reads:

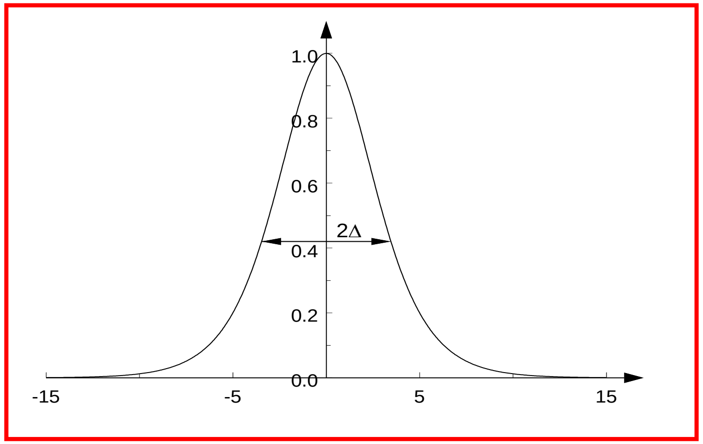
$$\frac{\partial u}{\partial t} + (c - V)\frac{\partial u}{\partial x} + \alpha u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = \varepsilon \frac{\partial f(x)}{\partial x}$$

If $\varepsilon = 0$, then there is a stationary solution – a KdV soliton

$$u = A_0 \operatorname{sech}^2 \gamma_0 \Phi$$

$$\upsilon_0 = c + \frac{\alpha A_0}{3}, \quad \gamma_0 = \sqrt{\frac{\alpha A_0}{12\beta}},$$

$$\Phi = x - x_0 - \upsilon_0 t$$



If $\varepsilon \neq 0$, but $\varepsilon << 1$, then a solitary wave is quasi-stationary and close to a KdV soliton with the slowly variable parameters ($T = \varepsilon t$):

$$\upsilon(T) = c - V + \frac{\alpha A(T)}{3}, \quad \gamma(T) = \sqrt{\frac{\alpha A(T)}{12\beta}},$$

$$\Phi = x - \Psi(T), \quad \Psi(T) = x_0 + \frac{1}{\varepsilon}\int_0^T \upsilon(\tau)\,d\tau$$

To determine soliton parameters $A(T)$, $\gamma(T)$, $\upsilon(T)$ we use the asymptotic theory developed in the papers by Grimshaw, Pelinovsky (1994) et al.

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots$$

$$\upsilon = \upsilon_0 + \varepsilon \upsilon_1 + \varepsilon^2 \upsilon_2 + \ldots$$

Equation for the soliton amplitude can be obtained from the energy balance equation:

$$\frac{\partial u}{\partial t}+(c-V)\frac{\partial u}{\partial x}+\alpha u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=\varepsilon\frac{\partial f(x)}{\partial x}\Bigg|\cdot u(x,t)$$

$$\frac{d}{dT}\frac{1}{2}\int_{-\infty}^{+\infty}u_0^2(\Phi)\,d\Phi=\int_{-\infty}^{+\infty}u_0(\Phi)\frac{df(\Phi)}{d\Phi}d\Phi$$

Substituting here soliton solution $u_0 = A\,\mathrm{sech}^2\gamma\Phi$, we obtain:

$$\boxed{\frac{dA}{dT}=\gamma\int_{-\infty}^{+\infty}\mathrm{sech}^2(\gamma\Phi)\frac{df(\Phi+\Psi)}{d\Phi}d\Phi}$$

Then we need one more equation for $\Psi(T)$.

In the first approximation equation for $\Psi(T)$ is very simple:

$$\frac{d\Psi}{dT} = \Delta V + \frac{\alpha A(T)}{3}, \quad \Delta V = (c - V)$$

In the next approximation after some manipulations we obtain:

$$\boxed{\frac{d\Psi}{dT} = \Delta V + \frac{\alpha A(T)}{3} + \frac{\varepsilon\alpha}{48\beta\gamma^2}\int_{-\infty}^{+\infty}\frac{\sinh(2\gamma\Phi) + 2(\gamma\Phi - 1)}{\cosh^2(\gamma\Phi)}\frac{df(\Phi+\Psi)}{d\Phi}d\Phi}$$

Thus, we have a complete set of equations for soliton dynamics under the action of arbitrary external force $f(x)$ of a small amplitude ε.

$$\boxed{\frac{dA}{dT} = \gamma\int_{-\infty}^{+\infty}\operatorname{sech}^2(\gamma\Phi)\frac{df(\Phi+\Psi)}{d\Phi}d\Phi}$$

1. The KdV-type forcing

Consider the particular forcing with

$$f(x) = \operatorname{sech}^2(x/\Delta_f).$$

In this case our basic equation

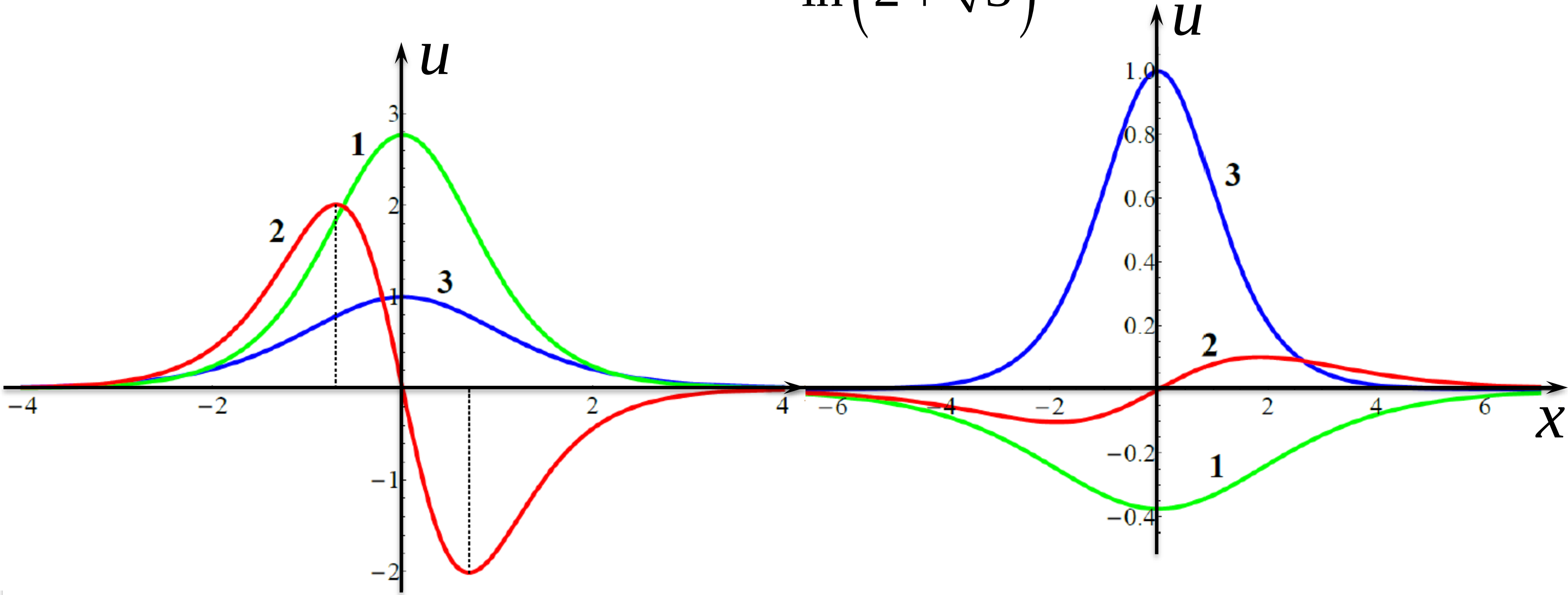
$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \alpha u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = \varepsilon\frac{\partial f(x,t)}{\partial x}$$

has exact solution (Lin, Zeng, Ma, 2001) for the <u>arbitrary</u> parameters ε and Δ_f:

$$u = A_s \operatorname{sech}^2 \frac{x - x_0 - Vt}{\Delta_f}$$

$$V = c + \frac{4\beta}{\Delta_f^2} - \frac{\varepsilon\alpha\Delta_f^2}{12\beta}, \quad A_s = \frac{12\beta}{\alpha\Delta_f^2}$$

$$K = \gamma_0 \Delta_f = \frac{2\gamma_0 D_f}{\ln\left(2+\sqrt{3}\right)}$$



The shape and polarity of the forcing $f(x)$ (green lines) for $K = 0.75$ (left) and $K = 2$ (right), red lines represent the derivatives $f_x{}'$ and blue lines show the initial KdV soliton of a unit amplitude.

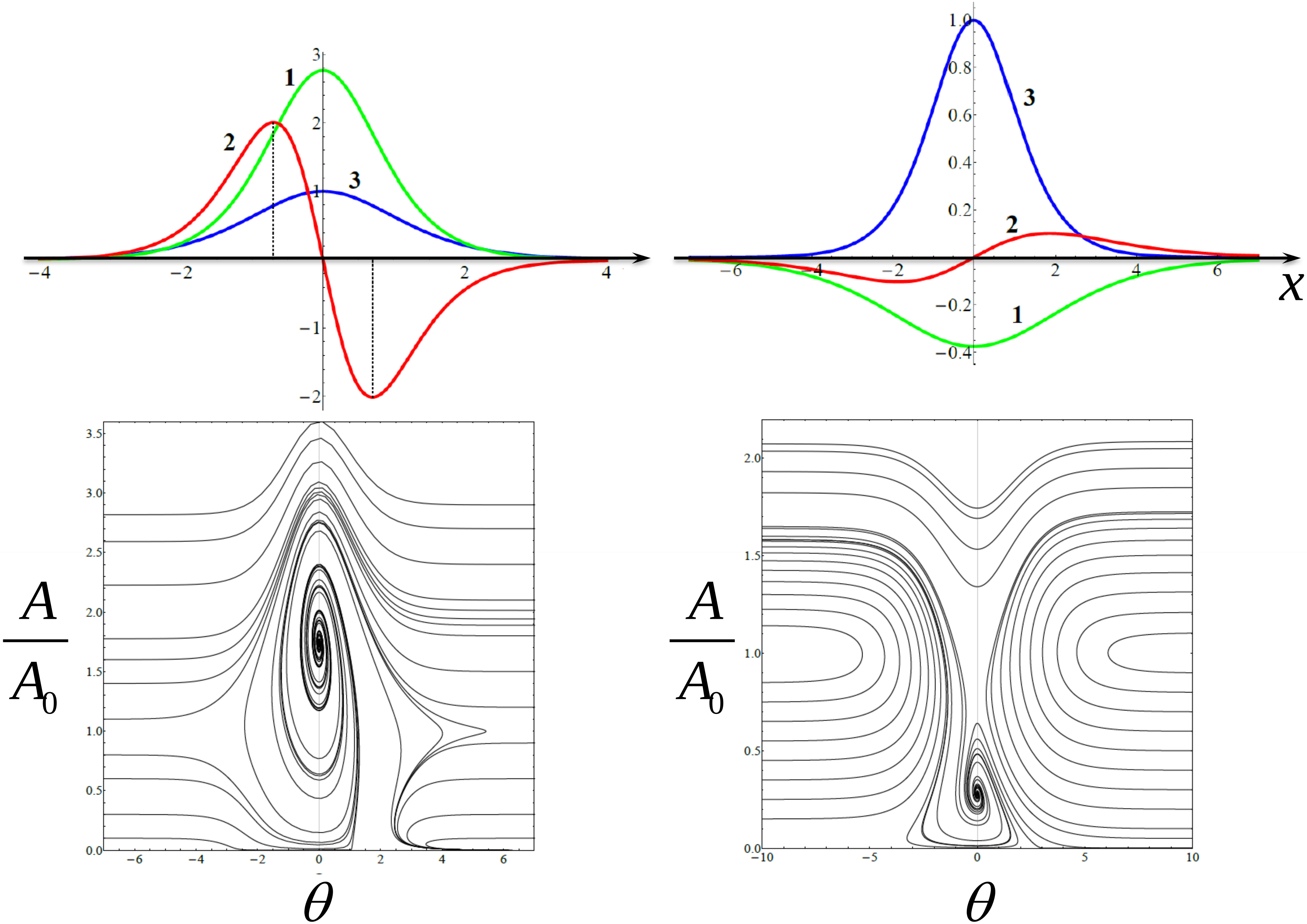
For a small-amplitude forcing, $\varepsilon << 1$, and initially unperturbed KdV soliton the asymptotic theory provides:

$$\boxed{\frac{d\gamma}{dT} = -\frac{2\varepsilon\,\alpha e^{2\theta}}{3\beta}\int_0^{+\infty}\frac{q^K}{\left(e^{2\theta}+q^K\right)^2}\frac{q-1}{\left(q+1\right)^3}dq,}\qquad \gamma\left(T\right) = \sqrt{\frac{\alpha A\left(T\right)}{12\beta}},$$

$$\boxed{\begin{aligned}\frac{d\theta}{dT} &= \Delta V\gamma + 4\beta\gamma^3 \\ &- \frac{\left(K^2-1\right)\Delta V^2 e^{2\theta}}{K^4\beta\gamma}\int_0^{+\infty}\frac{e^{2\theta}+q^K\left(3+2\theta-K\ln q\right)}{\left(e^{2\theta}+q^K\right)^2}\frac{q-1}{\left(q+1\right)^3}dq,\end{aligned}}$$

$$\theta = \gamma\Psi,\quad \Psi\left(T\right) = x_0 + \frac{1}{\varepsilon}\int_0^T \upsilon\left(\tau\right)d\tau,$$

$$\Phi = x - \Psi\left(T\right),\quad q = e^{2\Phi/\Delta_f},\quad \varepsilon = \frac{\alpha A_0^2}{3K^4}\left(1-K^2\right)$$



Phase planes of the dynamical system.

2. The KdVB-type forcing

Consider now another forcing with

$$f(x) = [\pm 1 - \tanh(x/\Delta_f)]\,\mathrm{sech}^2(x/\Delta_f).$$

In this case our basic equation

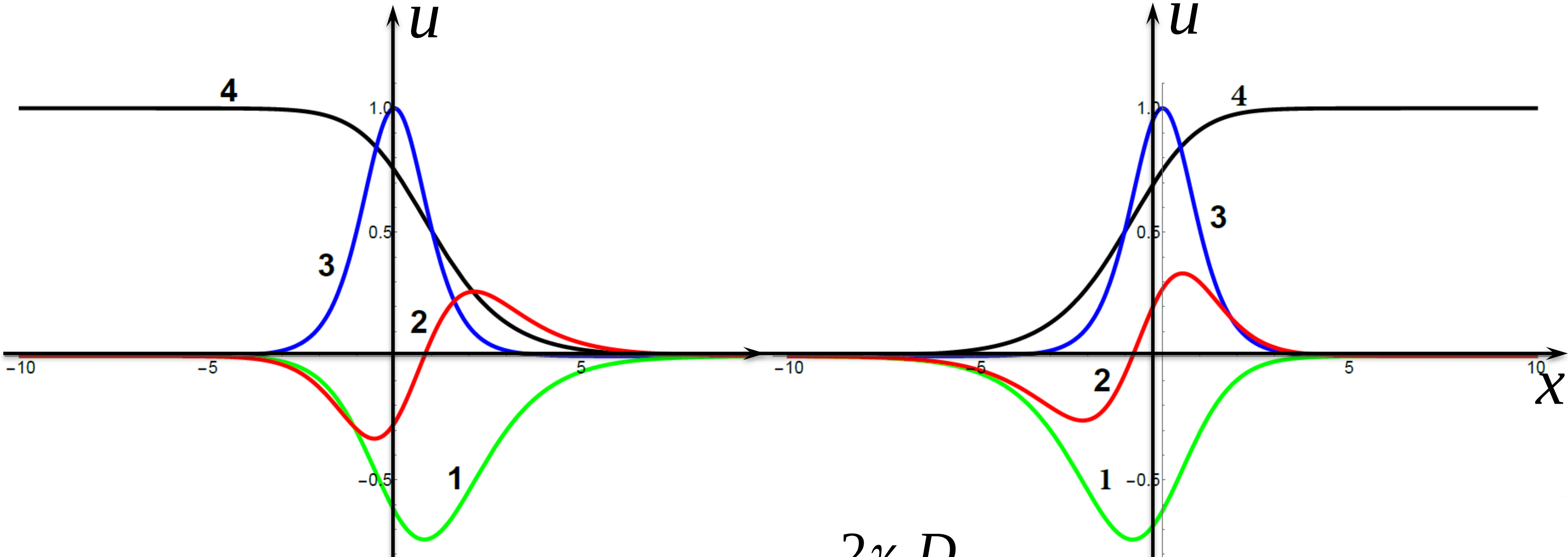
$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \alpha u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = \varepsilon\frac{\partial f\left(x,t\right)}{\partial x}$$

also has exact solution (Wang, 1996) for the arbitrary parameter Δ_f:

$$u = \varepsilon\Delta_f\left(1 \pm \tanh\frac{x - x_0 - Vt}{\Delta_f} + \frac{1}{2}\mathrm{sech}^2\frac{x - x_0 - Vt}{\Delta_f}\right)$$

$$V = c + \frac{24\beta^2}{\Delta_f^2}, \quad \varepsilon = \frac{24\beta}{\alpha\Delta_f^3}$$

$$f(x) = [\pm 1 - \tanh(x/\Delta_f)]\,\mathrm{sech}^2(x/\Delta_f)$$



$$K = \gamma_0 \Delta_f = \frac{2\gamma_0 D_f}{\ln\left(1.75 + \sqrt{2.0625}\right)}$$

The shape of the forcing $f(x)$ (green lines) for $K = 2$.
Left frame pertains to sign plus, and right frame – to sign minus.
Red lines represent the derivatives f_x' and
blue lines show the initial KdV soliton of a unit amplitude,
black lines are exact solutions of KdVB equation.

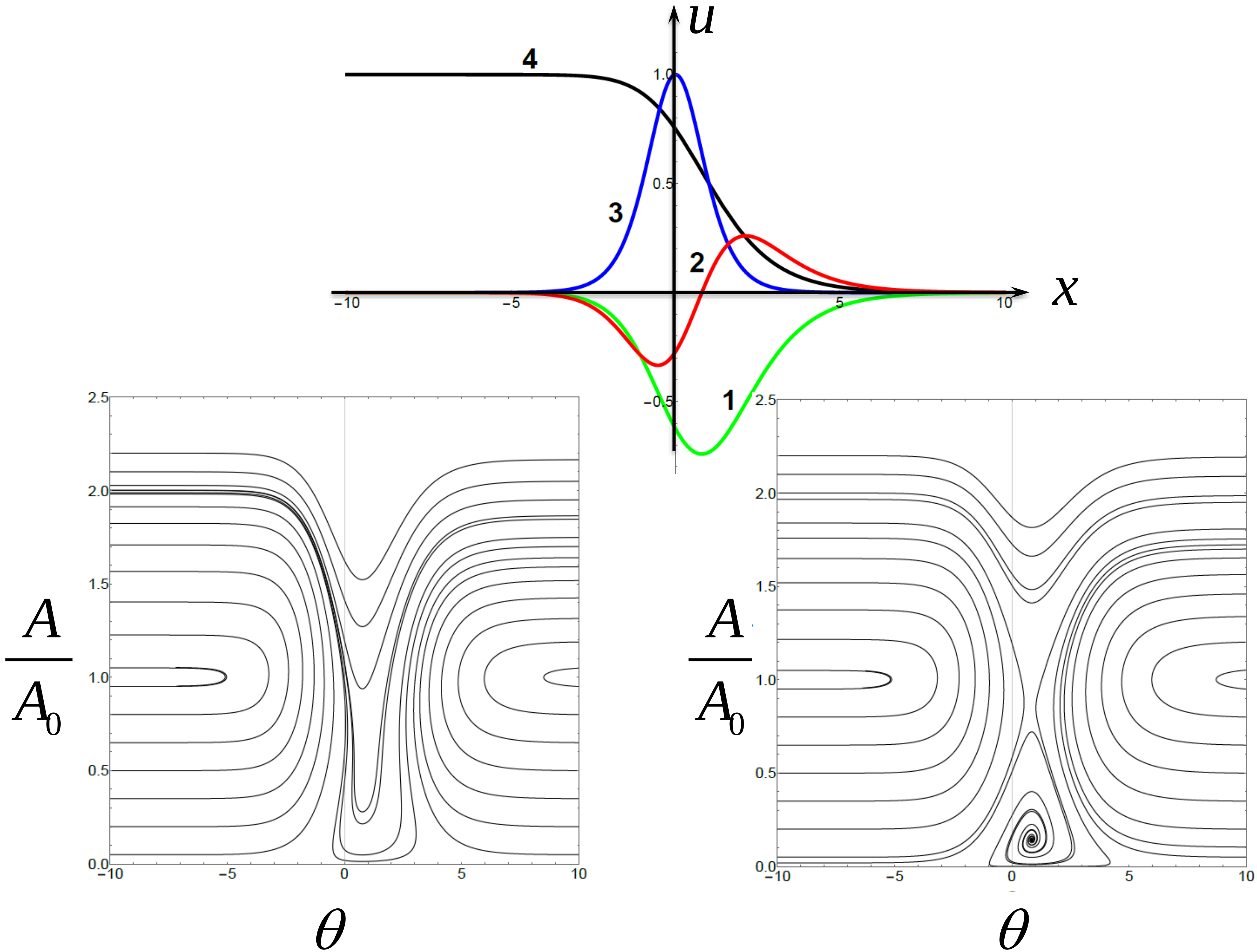
For a small-amplitude forcing, $\varepsilon << 1$, and initially unperturbed KdV soliton the asymptotic theory provides:

$$\boxed{\frac{d\gamma}{dT} = \mp \frac{320\beta}{\Delta_f^4} e^{2\theta} \int_0^{+\infty} \frac{q^{K+1}}{\left(e^{2\theta} + q^K\right)^2} \frac{q^{\pm 1} - 2}{\left(q+1\right)^4} dq,} \quad \gamma\left(T\right) = \sqrt{\frac{\alpha A\left(T\right)}{12\beta}},$$

$$\boxed{\begin{aligned}\frac{d\theta}{dT} &= \Delta V \gamma + 4\beta\gamma^3 \\ &\pm \frac{10\Delta V^2 e^{2\theta}}{27K^4\beta\gamma} \int_0^{+\infty} \frac{e^{2\theta} + q^K\left(3 + 2\theta - K\ln q\right)}{\left(e^{2\theta} + q^K\right)^2} \frac{q^{\pm 1} - 2}{\left(q+1\right)^4} q\,dq,\end{aligned}}$$

$$\theta = \gamma\Psi, \quad \Psi\left(T\right) = x_0 + \frac{1}{\varepsilon}\int_0^T \upsilon\left(\tau\right) d\tau,$$

$$\Phi = x - \Psi\left(T\right), \quad q = e^{2\Phi/\Delta_f}, \quad \varepsilon = \frac{24\beta}{\alpha\Delta_f^3}$$



Phase planes of the dynamical system for $K = 2$ (left) and 3.5 (right).

3. The Gardner-type forcing

Consider now one more forcing with

$$f(x) = [1 + B\cosh(x/\Delta_f)]^{-1}.$$

In this case our basic equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \alpha u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = \varepsilon\frac{\partial f\left(x,t\right)}{\partial x}$$

has a variety of exact solutions for the arbitrary parameters ε and Δ_f:

$$u = \frac{A_f}{1 + B\cosh\dfrac{x - x_0 - Vt}{\Delta_f}}$$

$$A_f = \frac{6\beta}{\alpha\Delta_f^2}, \quad V = c + \frac{\beta}{\Delta_f^2}, \quad \varepsilon = \frac{12\beta^2\left(1-B\right)}{\alpha\Delta_f^4\left(1+B\right)^2}$$

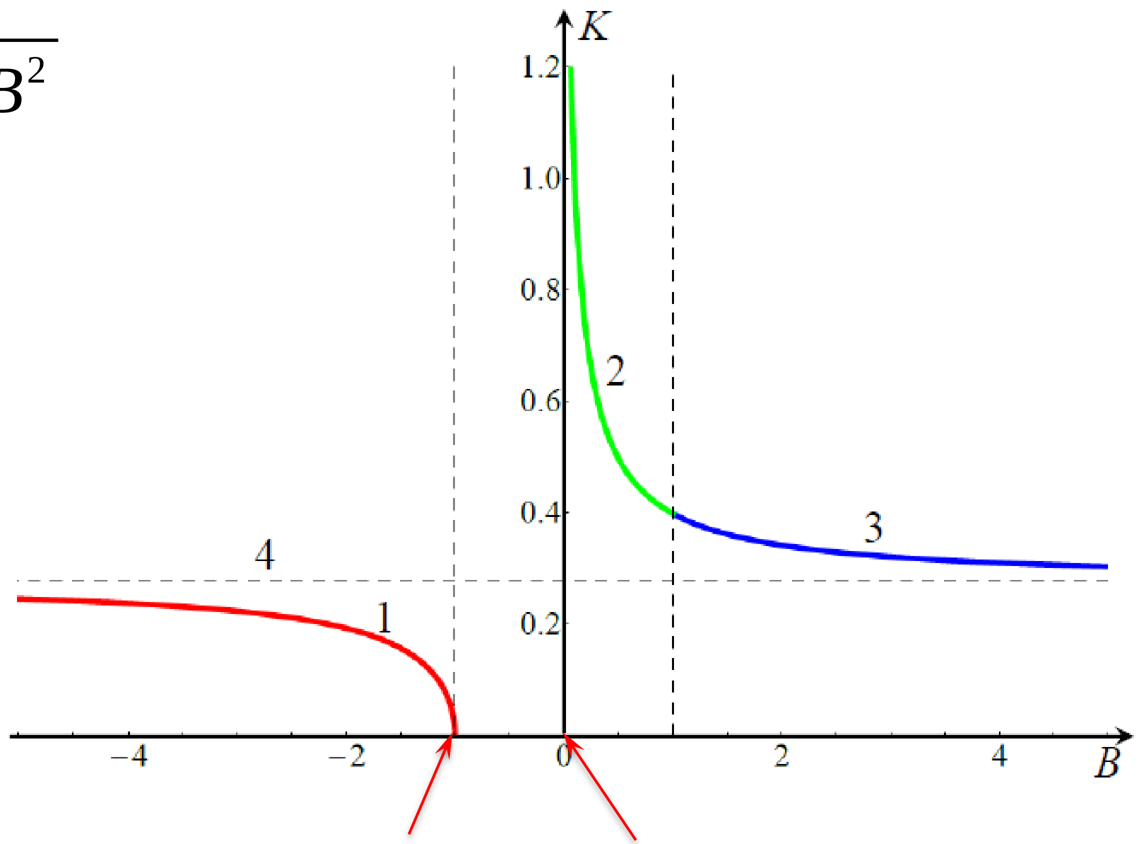
The parameter B varies in the ranges:

$-\infty < B < -1$, and $0 < B < \infty$.

There are no solutions for the range $-1 < B < 0$.

$$K = \gamma_0 \Delta_f = \frac{\gamma_0 D_f}{\ln\left[1 \pm R \pm \sqrt{2\left(R^2 + R - 42B^2\right)}\right] - \ln(6B)}$$

$$R = \sqrt{1 + 48B^2}$$



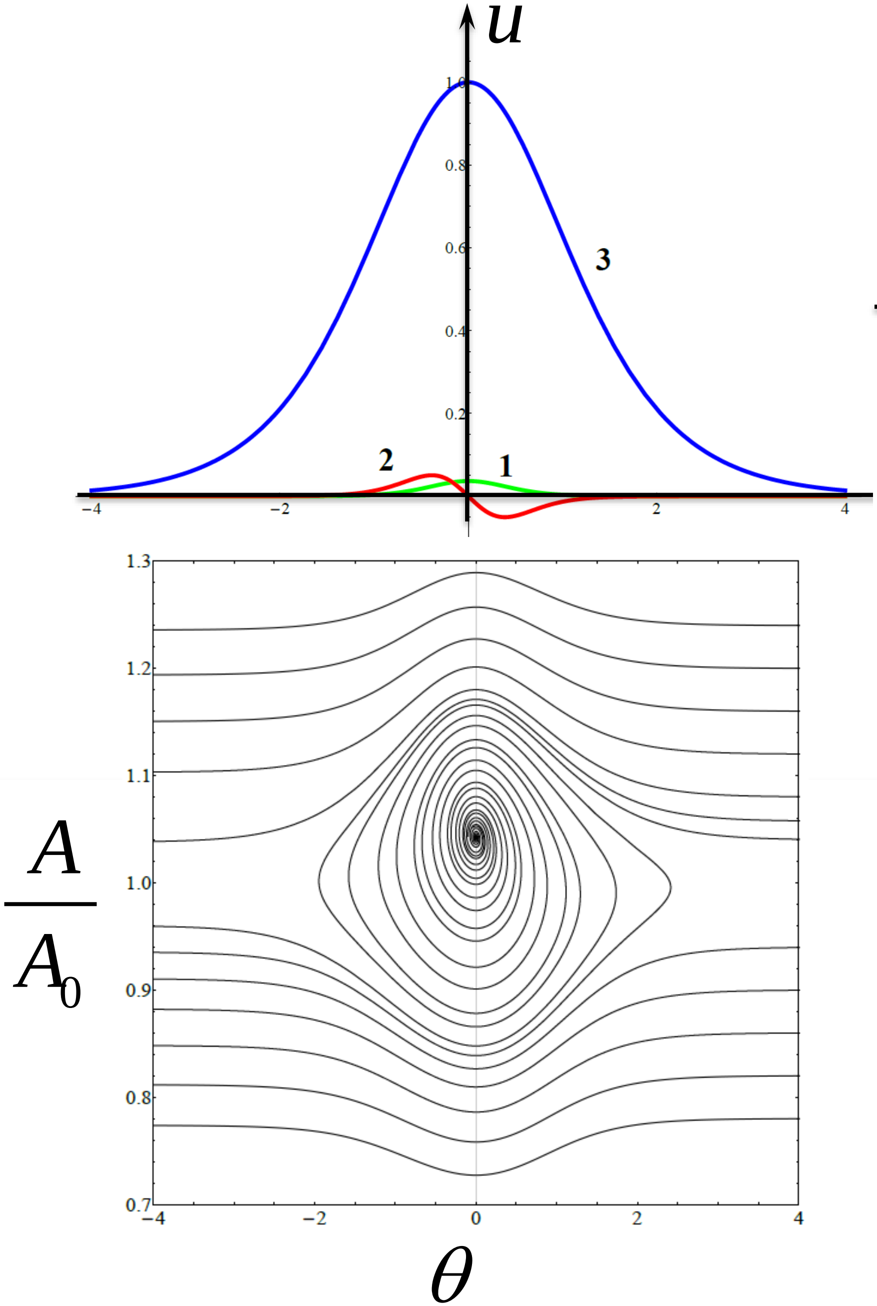
For a small-amplitude forcing, $\varepsilon << 1$, and initially unperturbed KdV soliton the asymptotic theory provides:

$$\boxed{\frac{d\gamma}{dT} = -\frac{48\left(1-B^2\right)\Delta V^2}{\beta B^3} e^{2\theta} \int_0^{+\infty} \frac{q^3\left(q^2-1\right)}{\left(e^{2\theta}+q\right)^2\left(q^2+2q/B+1\right)^4} dq,}$$

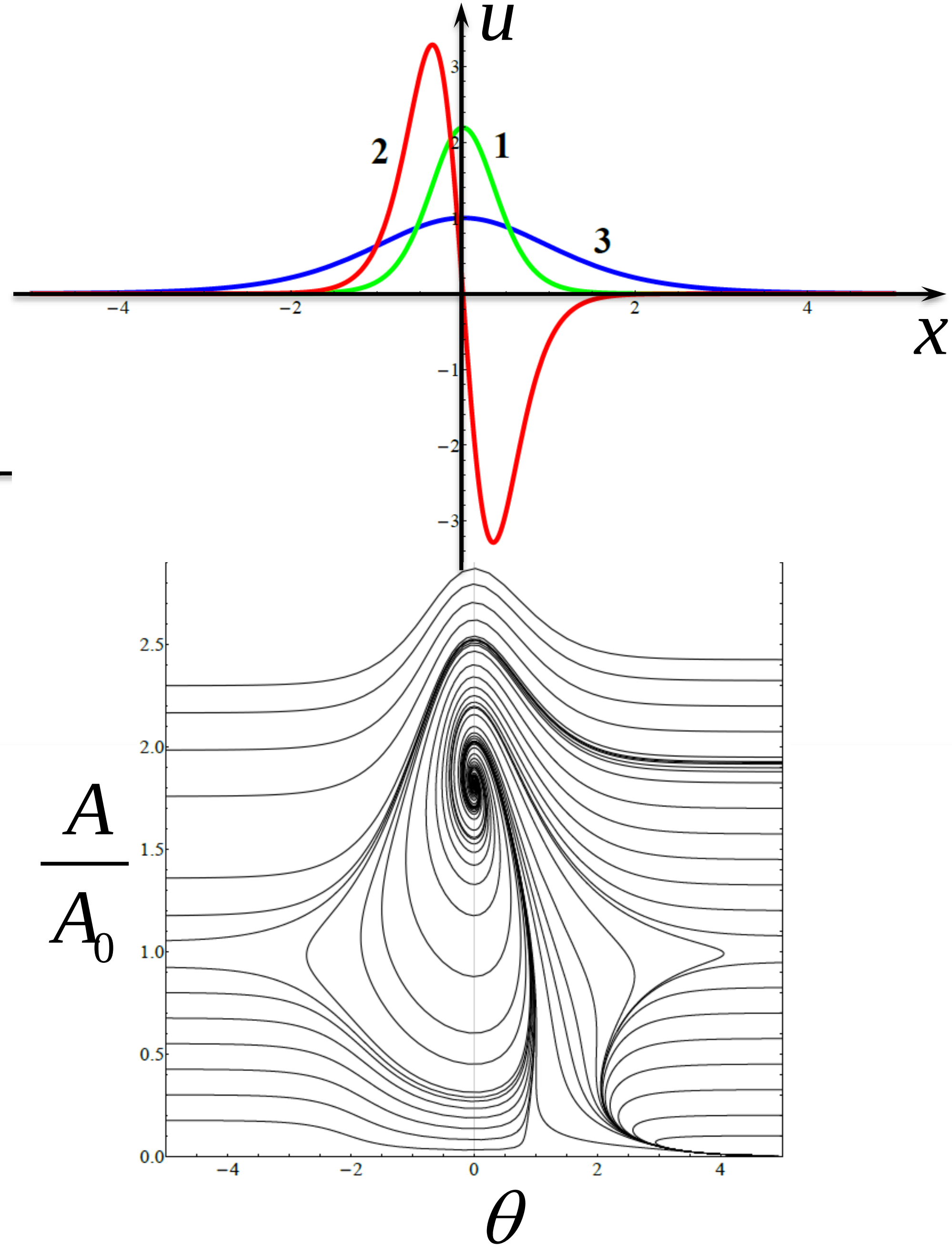
$$\boxed{\begin{aligned}&\frac{d\theta}{dT} = \Delta V\gamma + 4\beta\gamma^3\\&+\frac{12\Delta V^2\left(1-B^2\right)}{\beta\gamma B^3}\int_0^{+\infty}\frac{e^{4\theta}-q^2+2qe^{2\theta}\left(2+2\theta-\ln q\right)}{\left(e^{2\theta}+q\right)^2}\frac{q^2\left(q^2-1\right)}{\left(q^2+2q/B+1\right)^4}dq,\end{aligned}}$$

$$\gamma(T) = \sqrt{\frac{\alpha A(T)}{12\beta}}, \quad \theta = \gamma\Psi, \quad \Psi(T) = x_0 + \frac{1}{\varepsilon}\int_0^T \upsilon(\tau)\,d\tau,$$

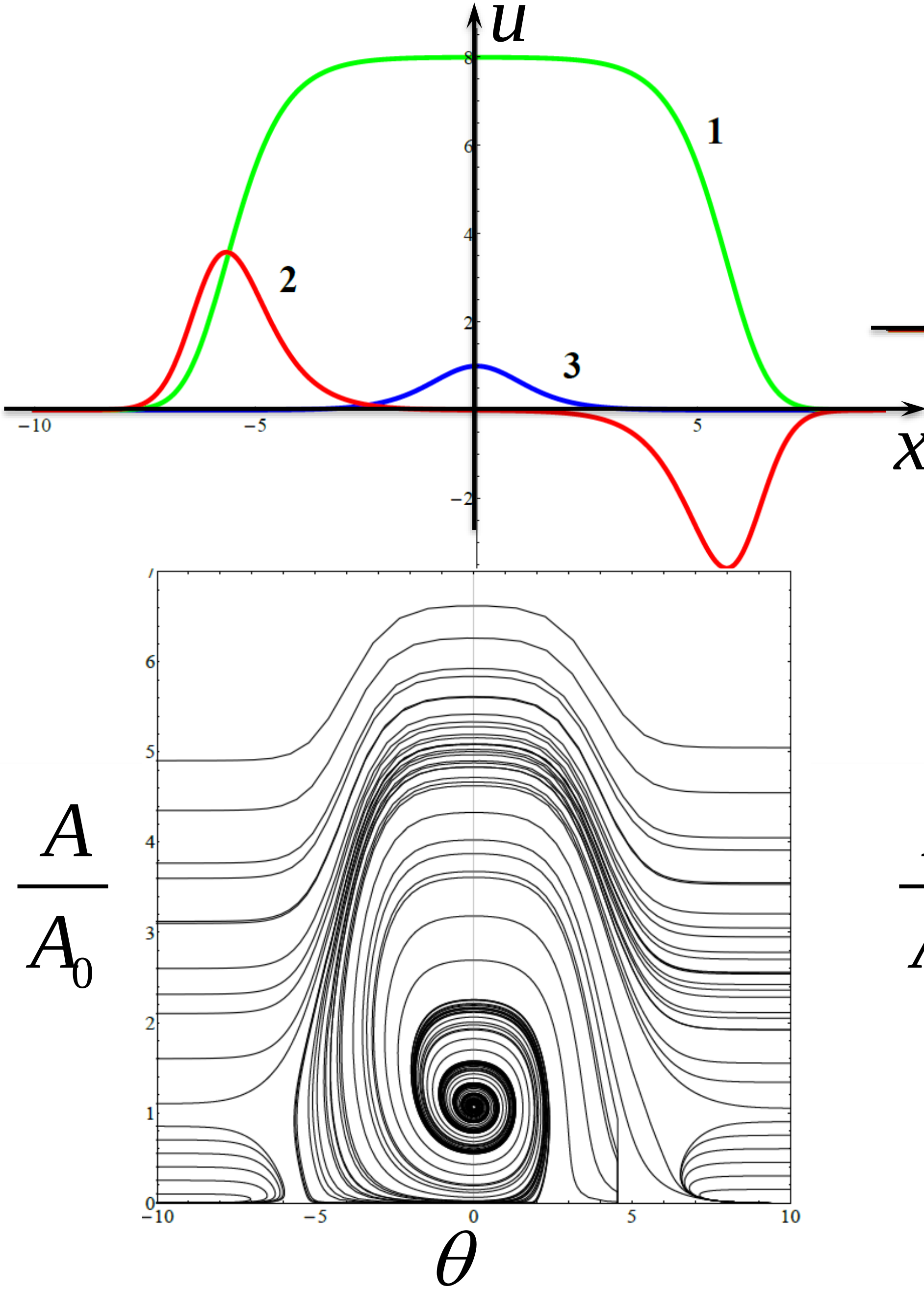
$$\Phi = x - \Psi(T), \quad q = e^{\Phi/\Delta_f}, \quad \varepsilon = \frac{12\beta^2\left(1-B\right)}{\alpha\Delta_f^4\left(1+B\right)^2}$$



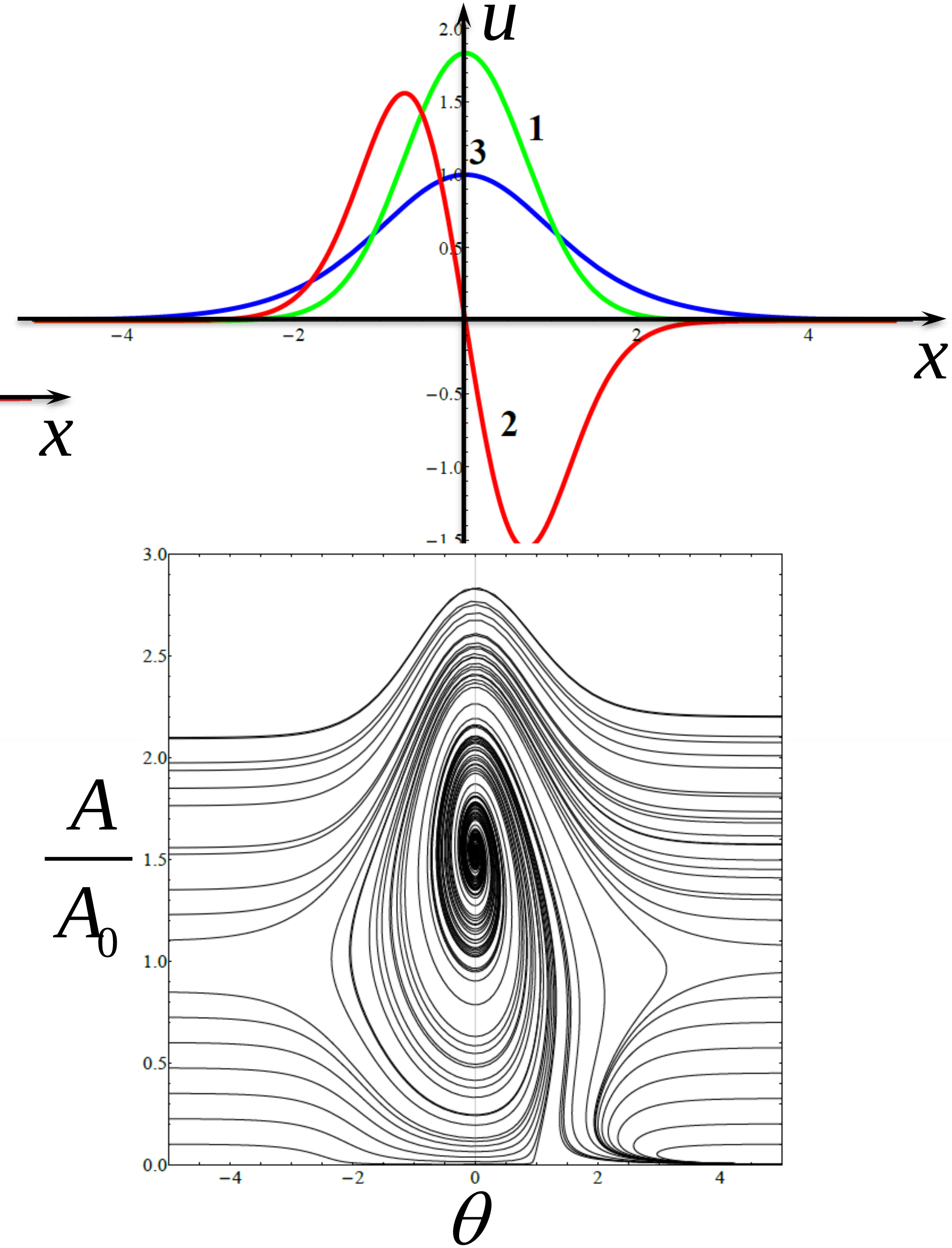
$K = 0.274$ $(B = -221.23)$;



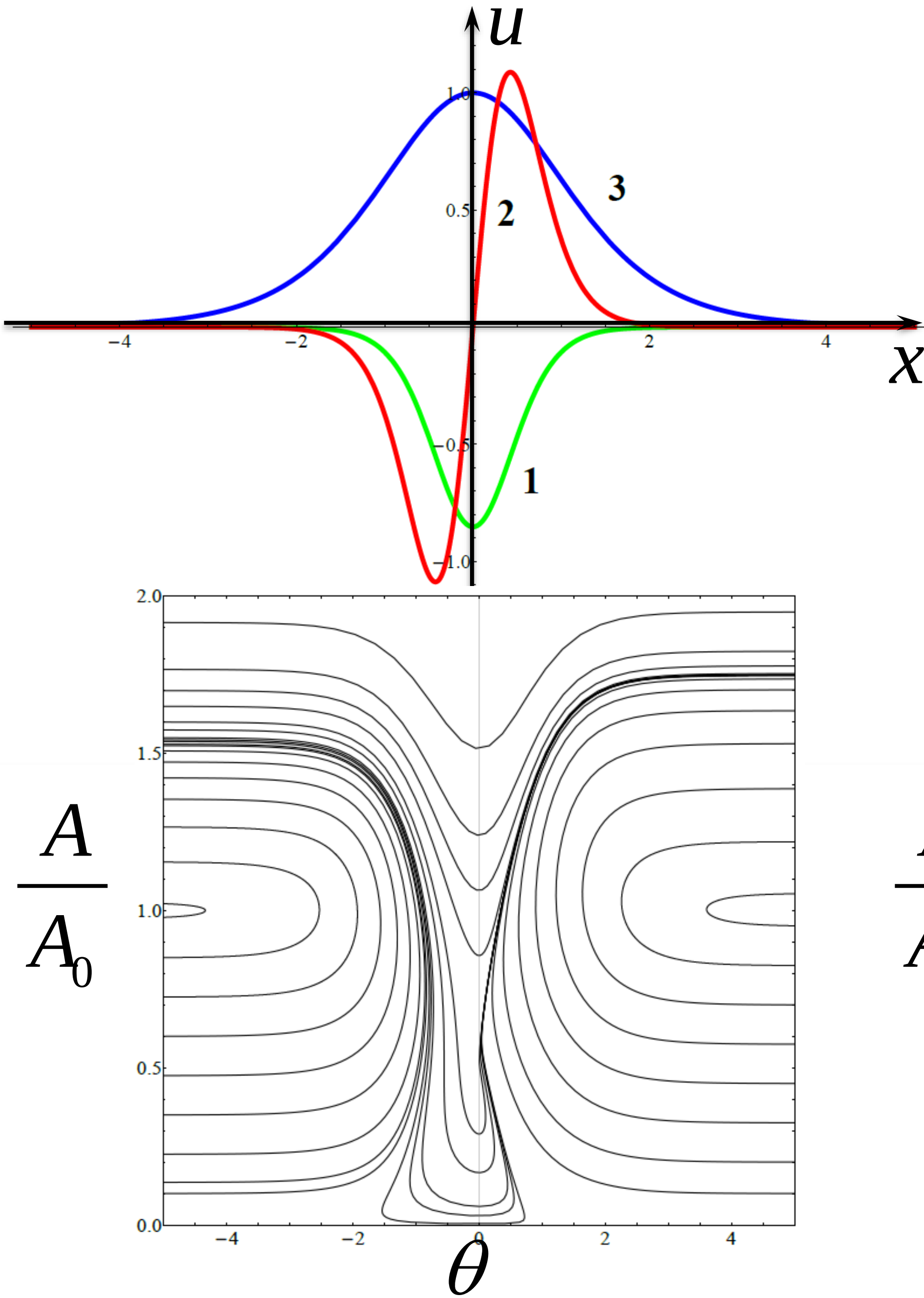
$K = 0.25$ $(B = -6.08)$



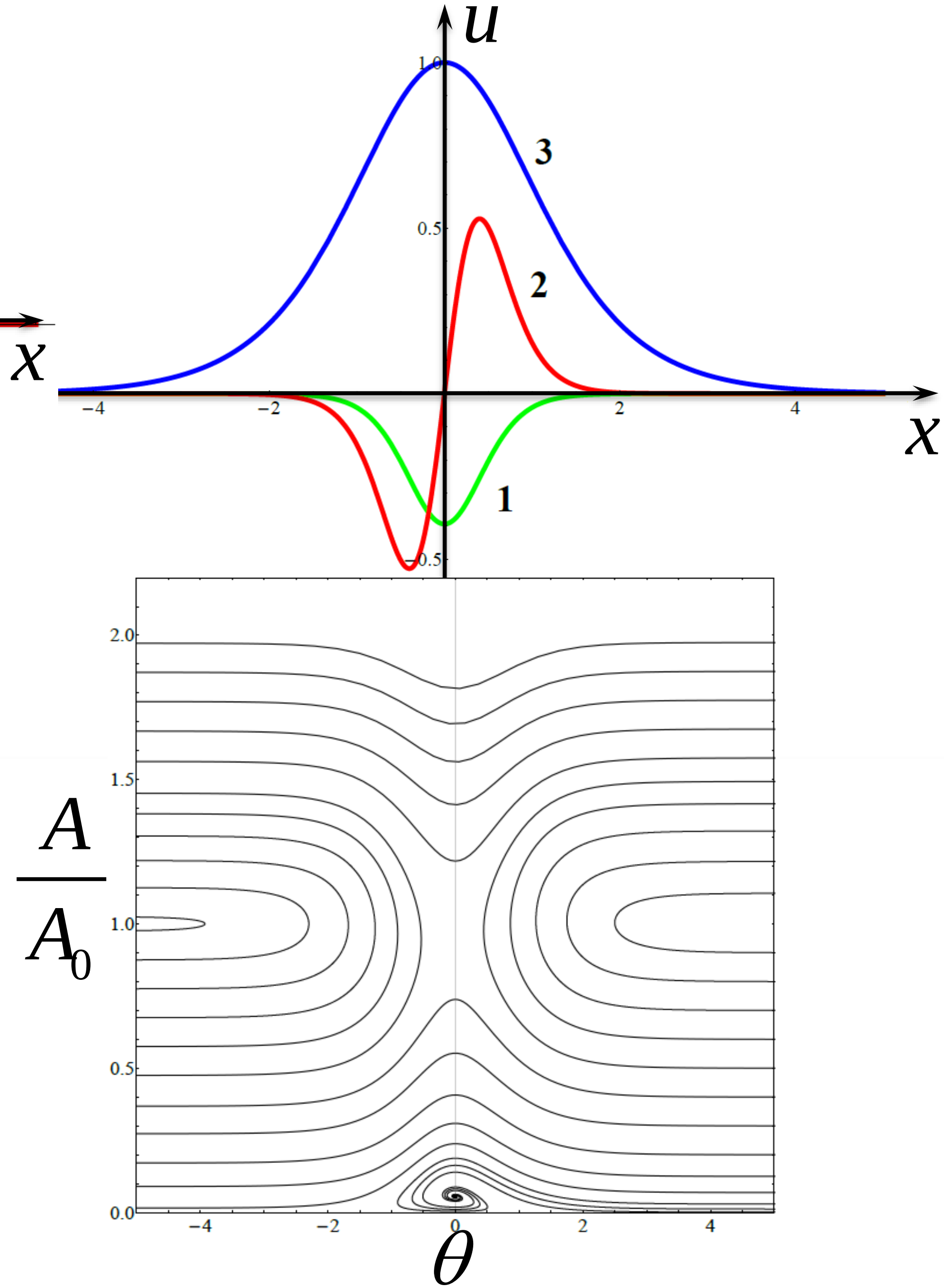
$K = 2 \ \ (B = 0.012);$



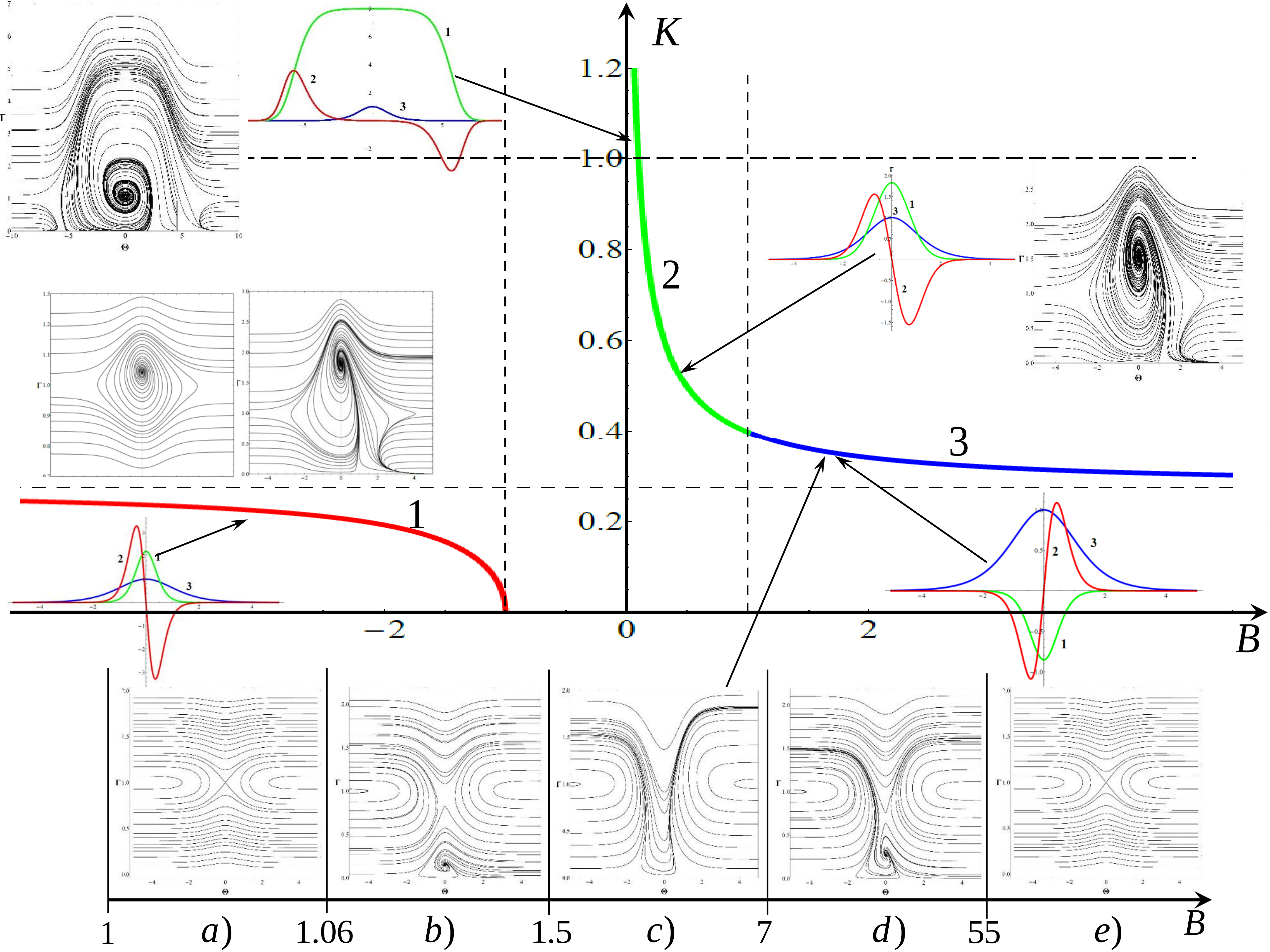
$K = 0.5 \ \ (B = 0.49)$



$K = 0.32$ $(B = 3)$;



$K = 0.288$ $(B = 10)$

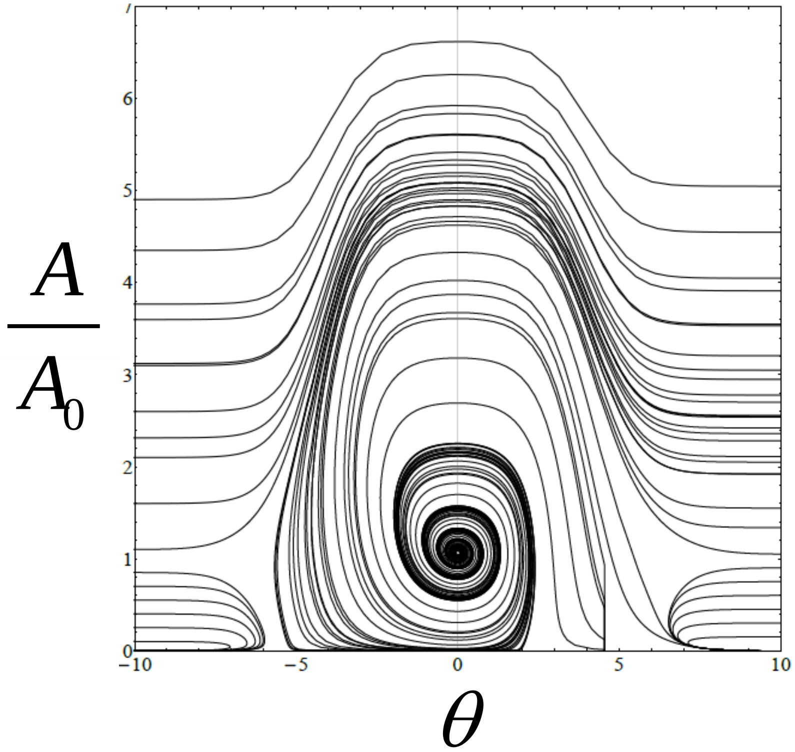


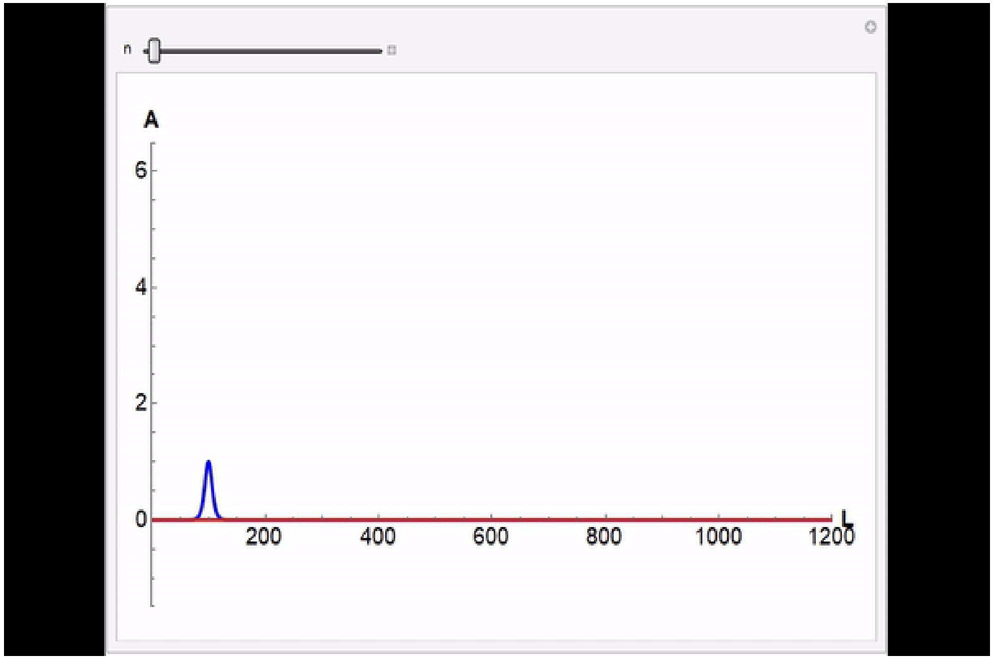
K
1.2
1.0
0.8
0.6
0.4
0.2
−2
0
2
B
1
2
3
1
1.06
1.5
7
55
B
a)
b)
c)
d)
e)
Γ
Θ

4. Results of direct numerical modelling

https://kdvforcing.wordpress.com/

$$K = 2 \quad (B = 0.012);$$

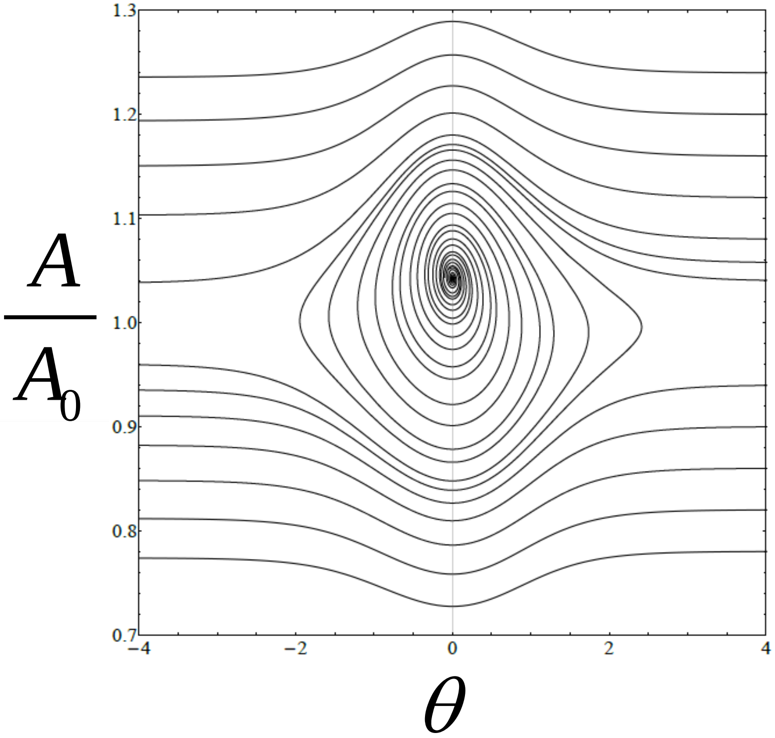


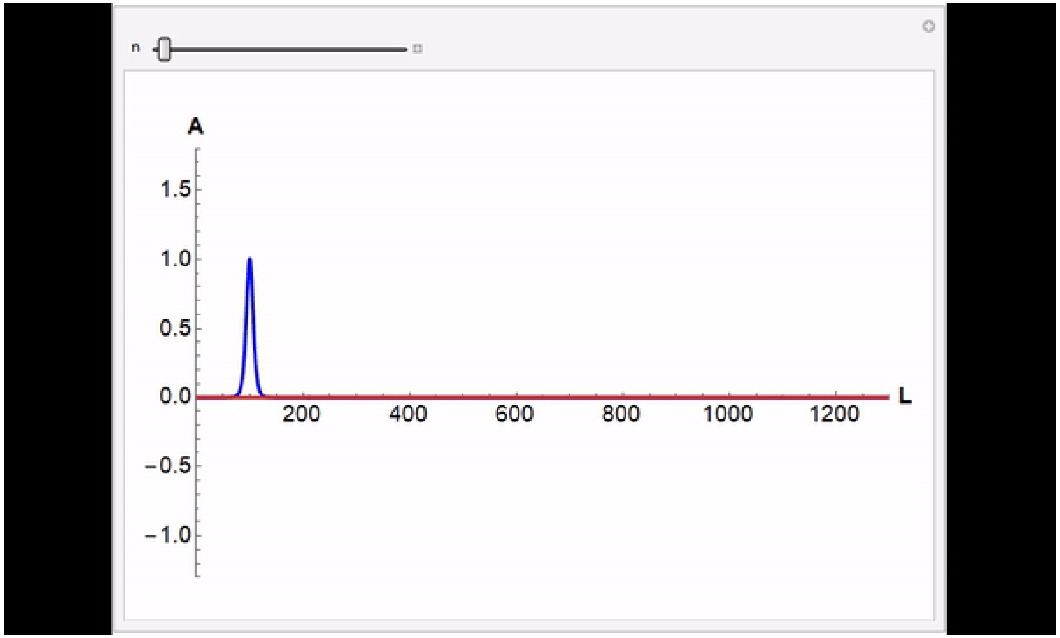


4. Results of direct numerical modelling

https://kdvforcing.wordpress.com/

$$B = 0.85$$





4. Results of direct numerical modelling

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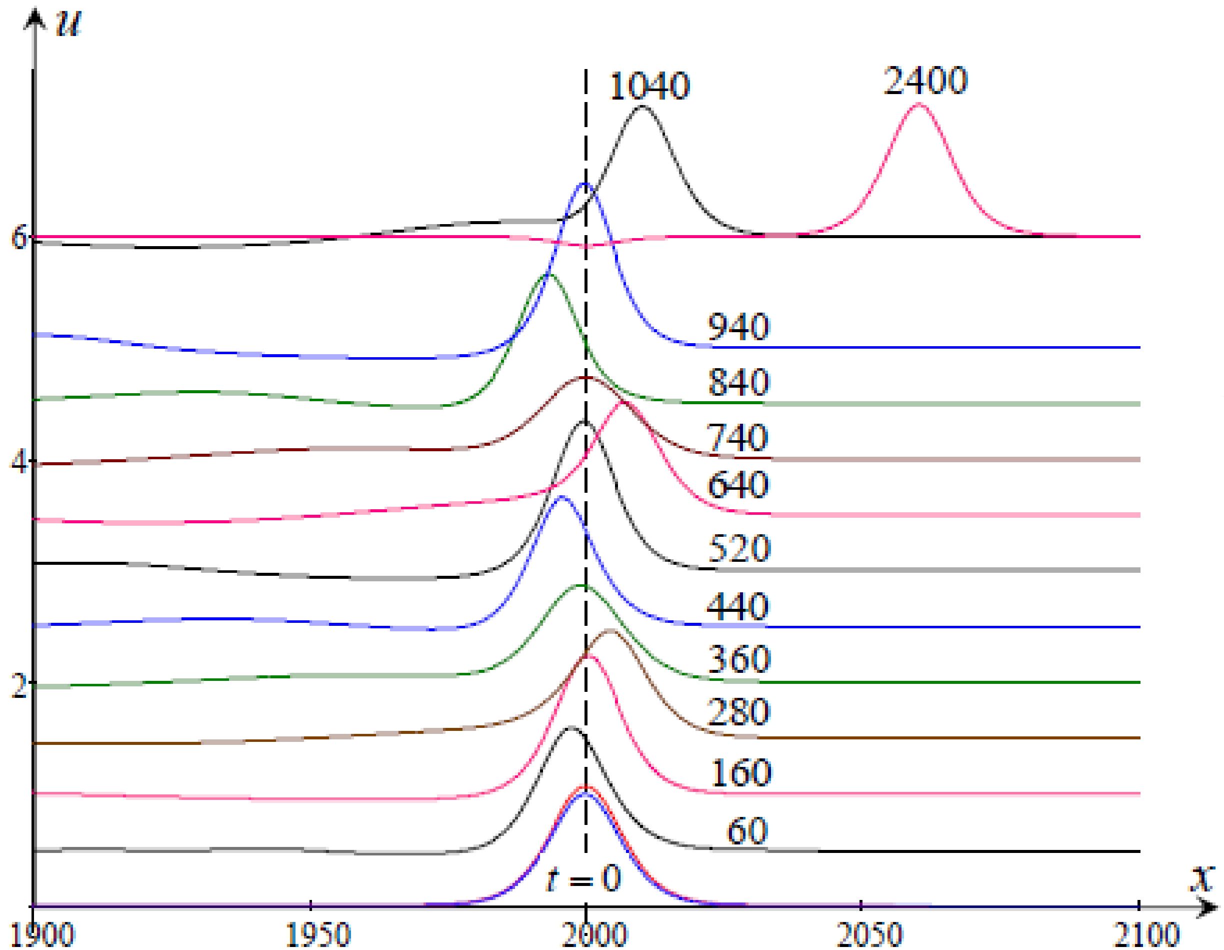
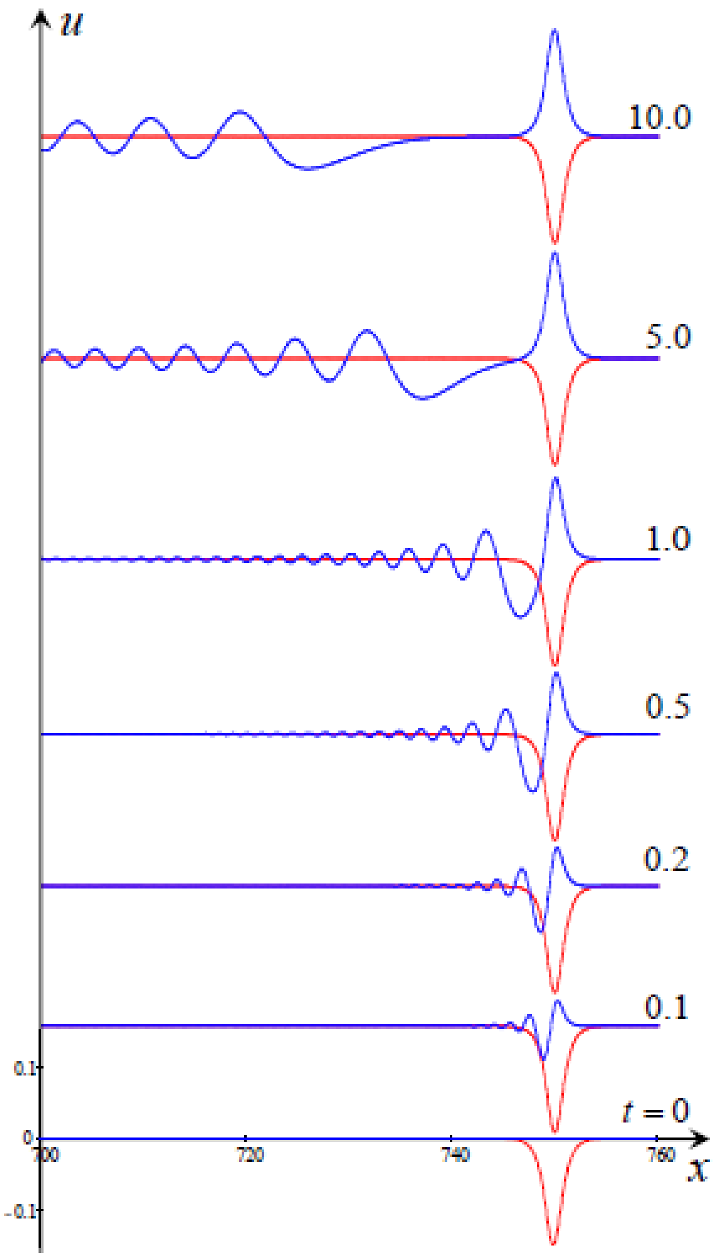
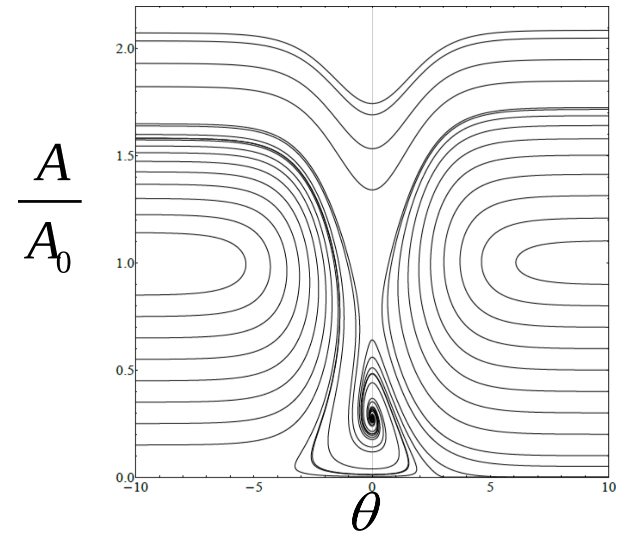
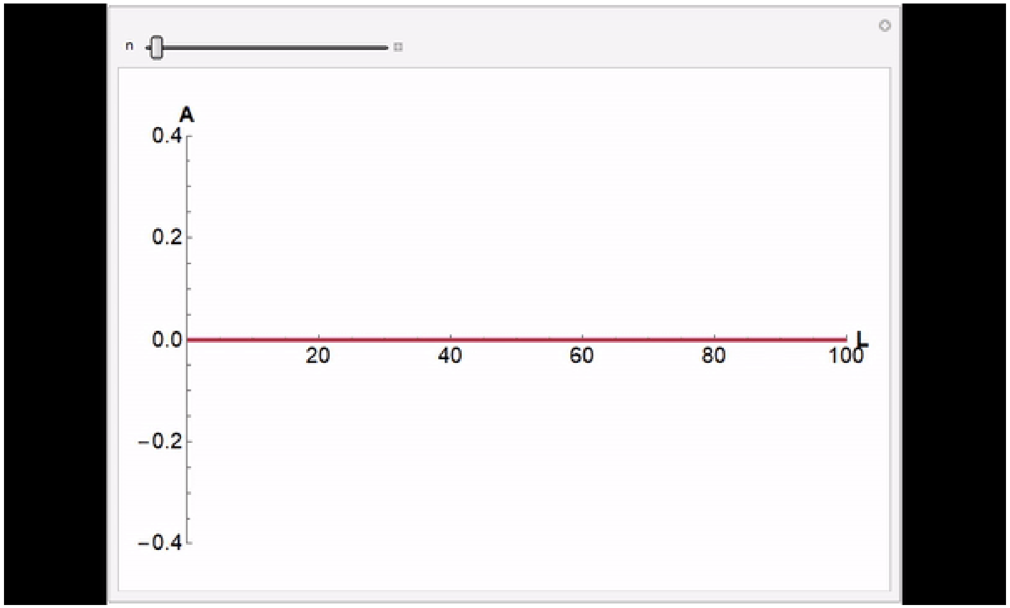
$B = 0.85$



All other results were obtained for $B = 12.5$



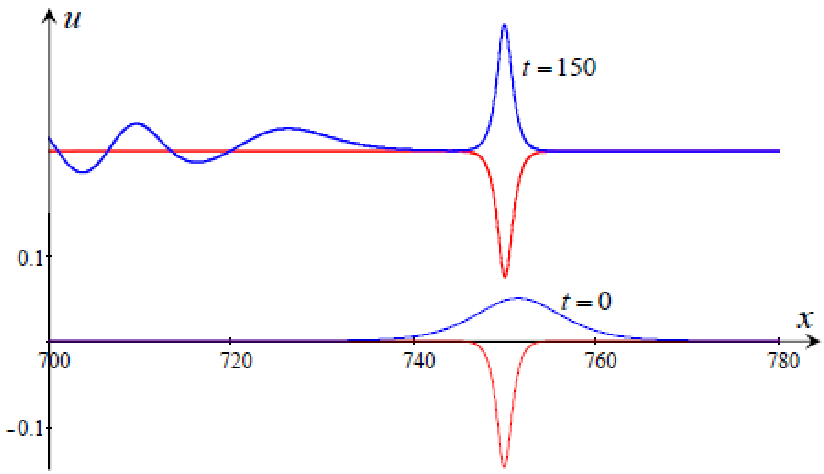


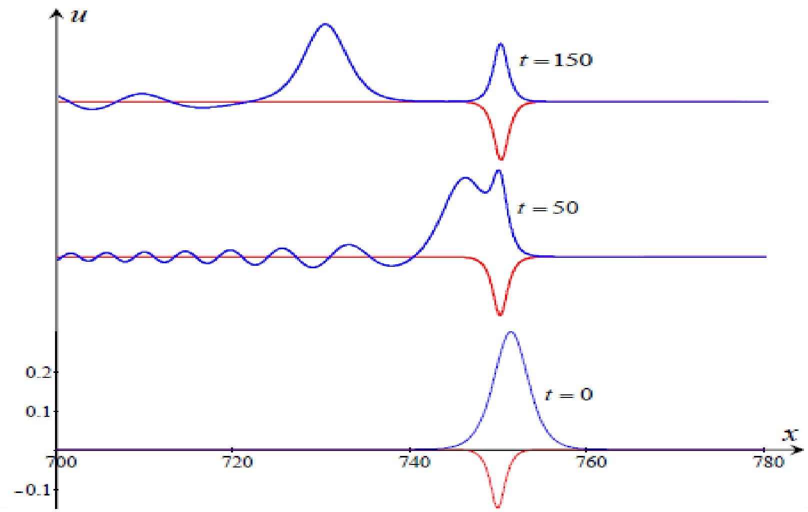


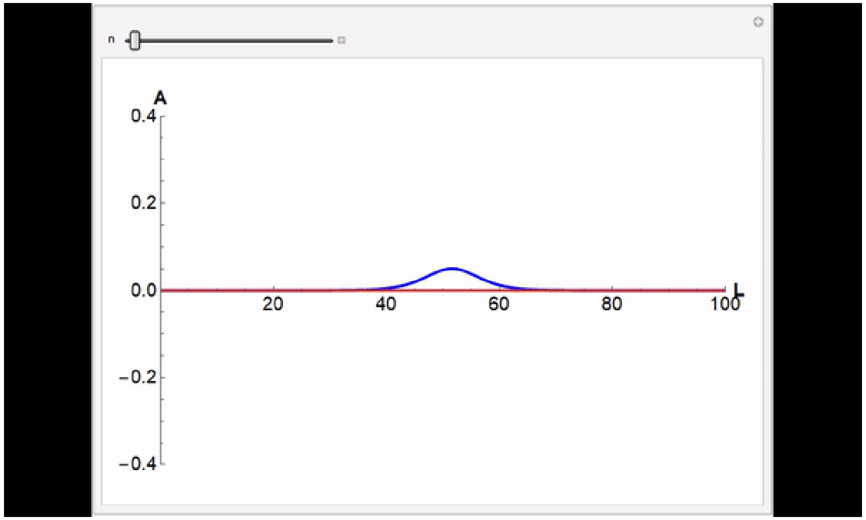
← Soliton generation from small random noise

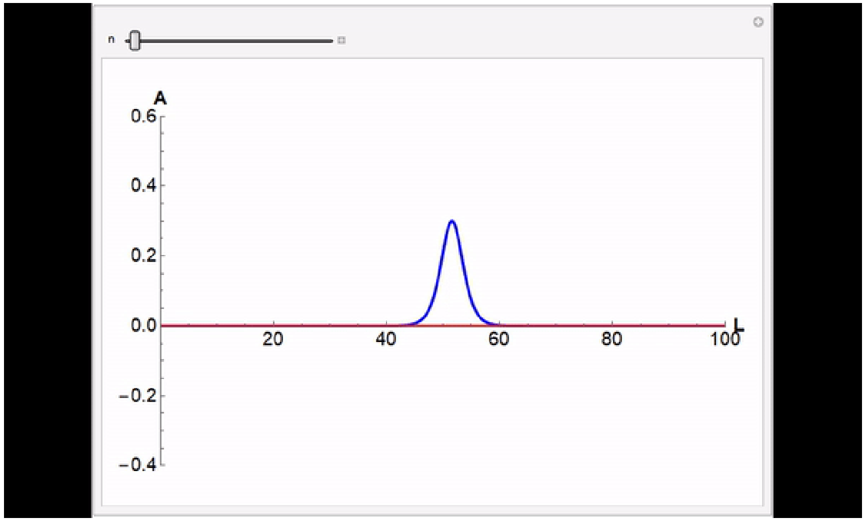
Formation of stationary soliton from a small-amplitude KdV soliton.

Formation of stationary soliton from a big-amplitude KdV soliton.

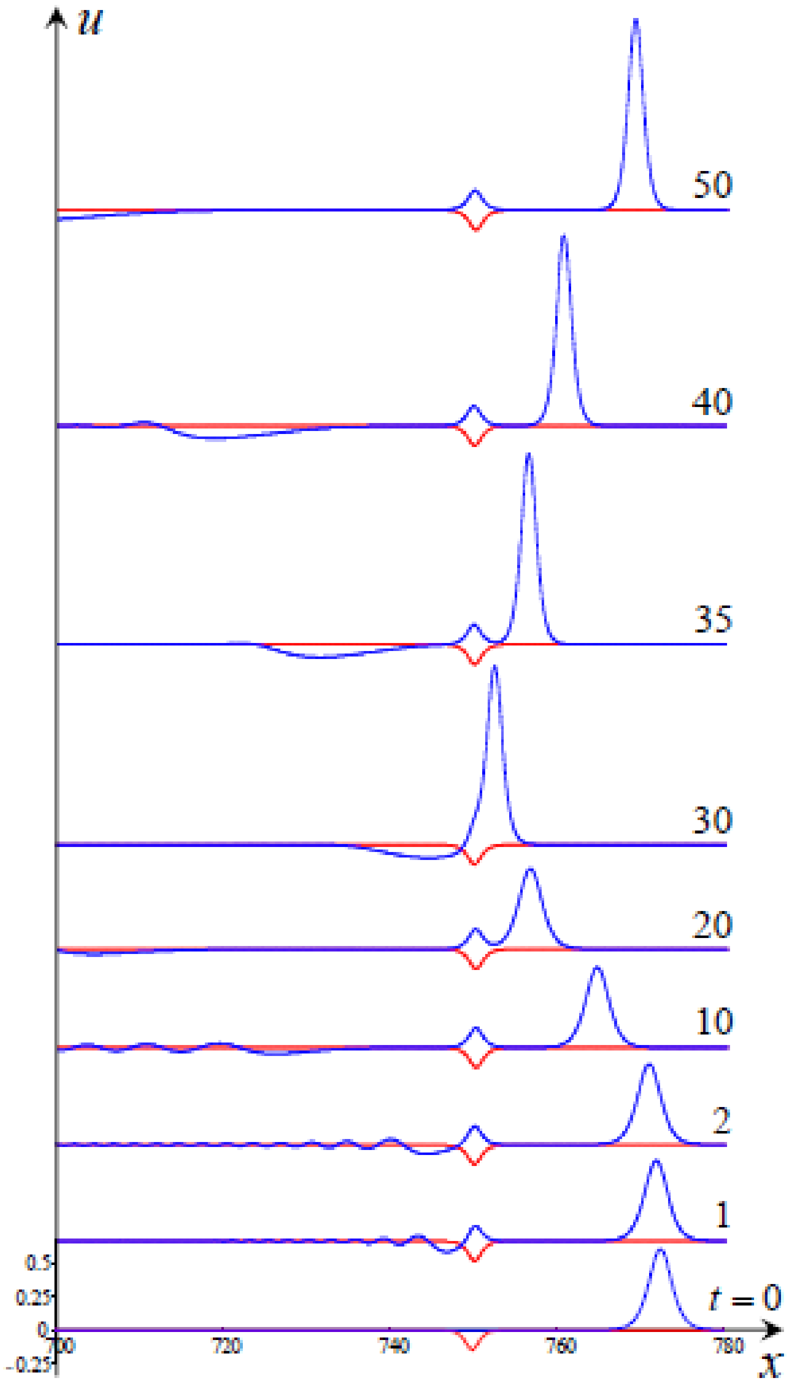


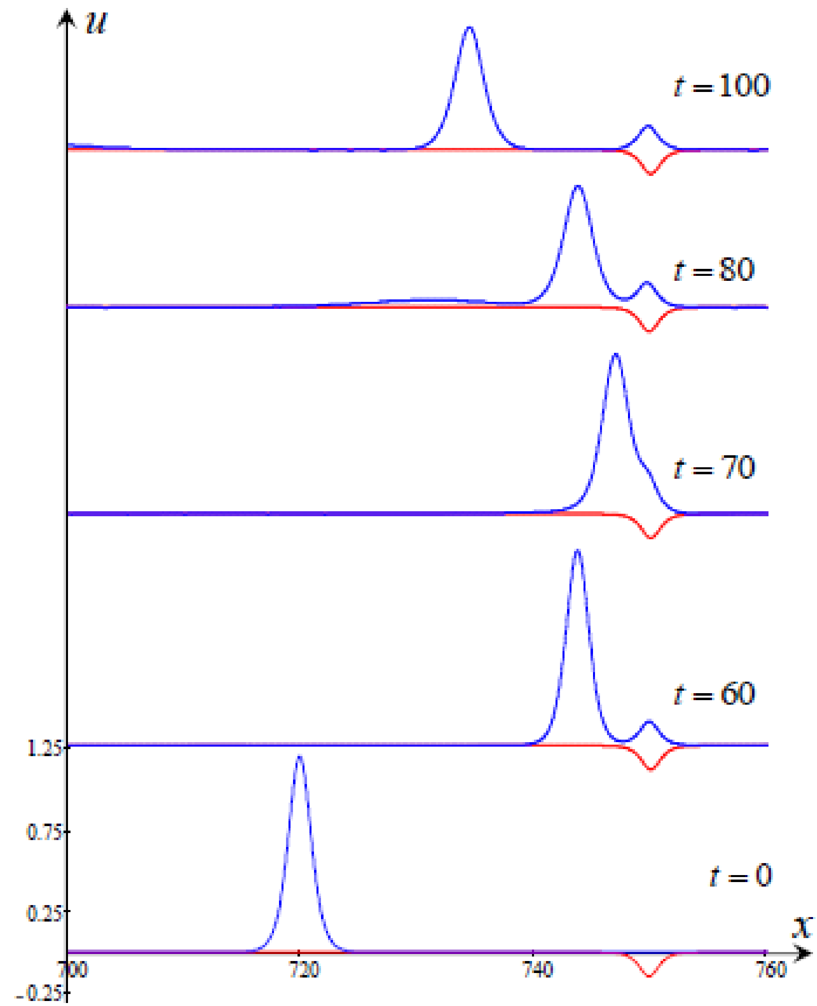




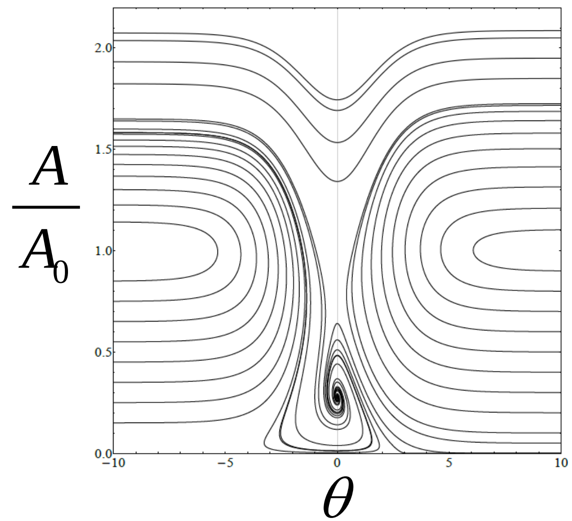


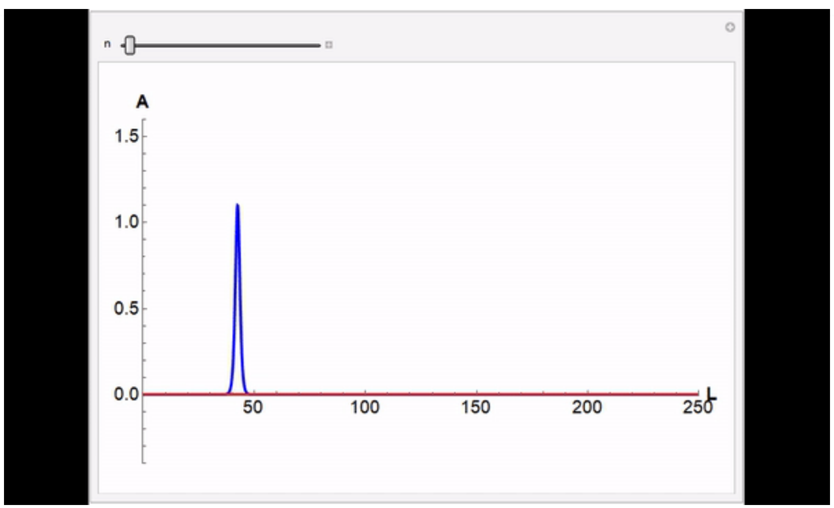
Bouncing of external KdV solitons from the forcing potential

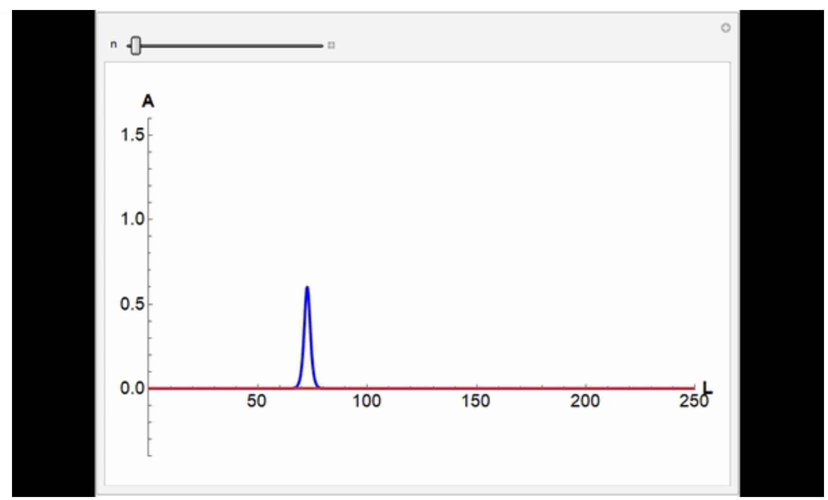


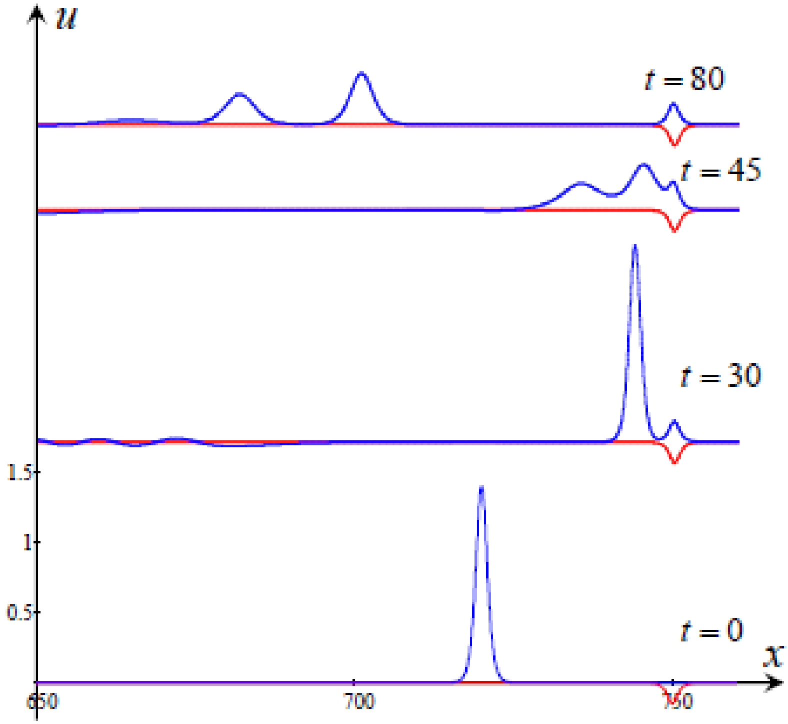


Bouncing of external KdV solitons from the forcing potential

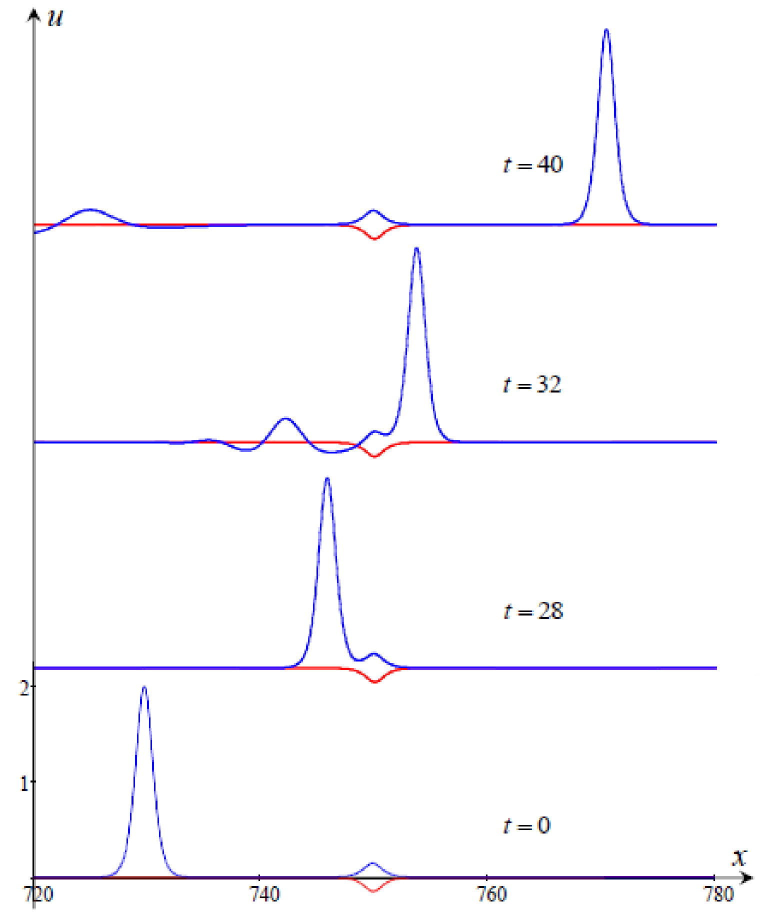






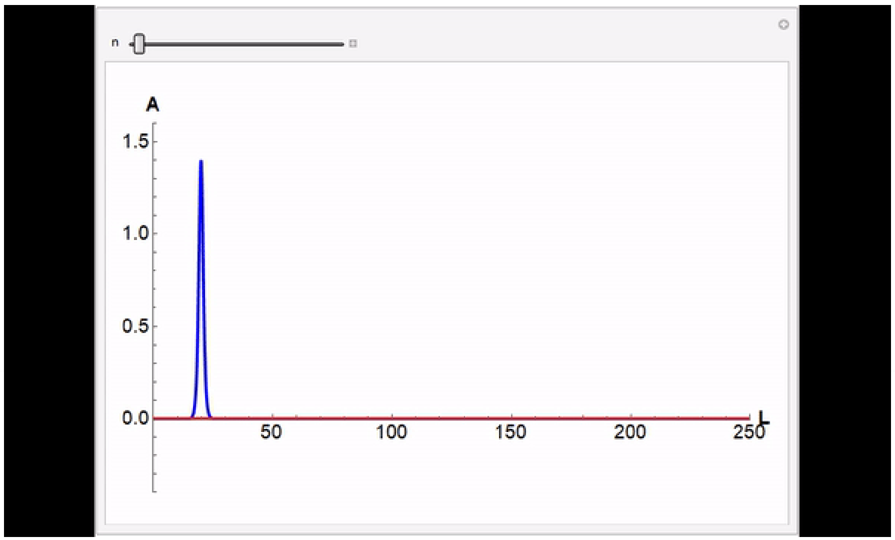


Breakdown of external KdV soliton onto three secondary solitons after bouncing from the forcing potential

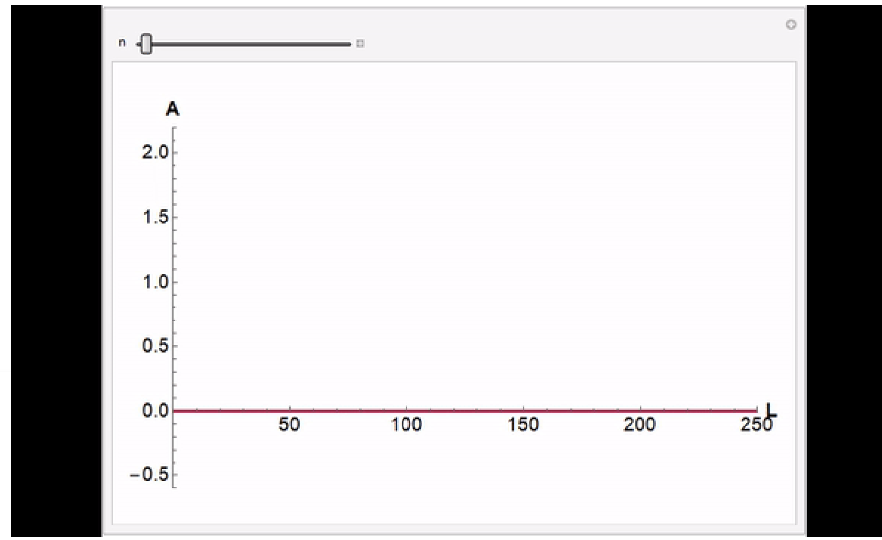


Transition of external KdV soliton through the forcing potential

Breakdown of external KdV soliton onto three secondary solitons after bouncing from the forcing potential



Transition of external KdV soliton through the forcing potential



5. A periodic forcing

Using the results obtained in (Lin, Zeng, Ma, 2001), one can construct a big variety of solutions describing interactions of KdV solitons with external stationary and non-stationary potentials.

Consider, in particular, the following forcing:

$$f(x,t) = \sigma F(t)\operatorname{sech}^2 \frac{x - \int S(t)\,dt}{\Delta_f}$$

where $F(t)$ and $S(t)$ are arbitrary functions.

$$u(x,t) = \frac{12\beta}{\alpha\Delta_f^2}\operatorname{sech}^2 \frac{x - \int S(t)\,dt}{\Delta_f}; \quad \sigma = \frac{12\beta}{\varepsilon\alpha\Delta_f^2}$$

$$S(t) = c + \frac{4\beta}{\Delta_f^2} - F(t)$$

Let us choose

$$F(t) = \frac{\varepsilon\alpha}{12\beta}\Delta_f^2\left(1+\tilde{V}\sin\varepsilon\omega t\right),$$

then we obtain a soliton moving with a variable velocity

$$u(x,t) = \frac{12\beta}{\alpha\Delta_f^2}\operatorname{sech}^2\frac{x-\int S(t)\,dt}{\Delta_f},$$

$$V_{tot}(t) = c + \frac{4\beta}{\Delta_f^2} - \frac{\varepsilon\alpha\Delta_f^2}{12\beta}\left(1+\tilde{V}\sin\varepsilon\omega t\right)$$

For a small-amplitude forcing, $\varepsilon << 1$, and external KdV soliton we can construct again an approximate asymptotic solution describing interaction of a soliton with the periodic forcing.

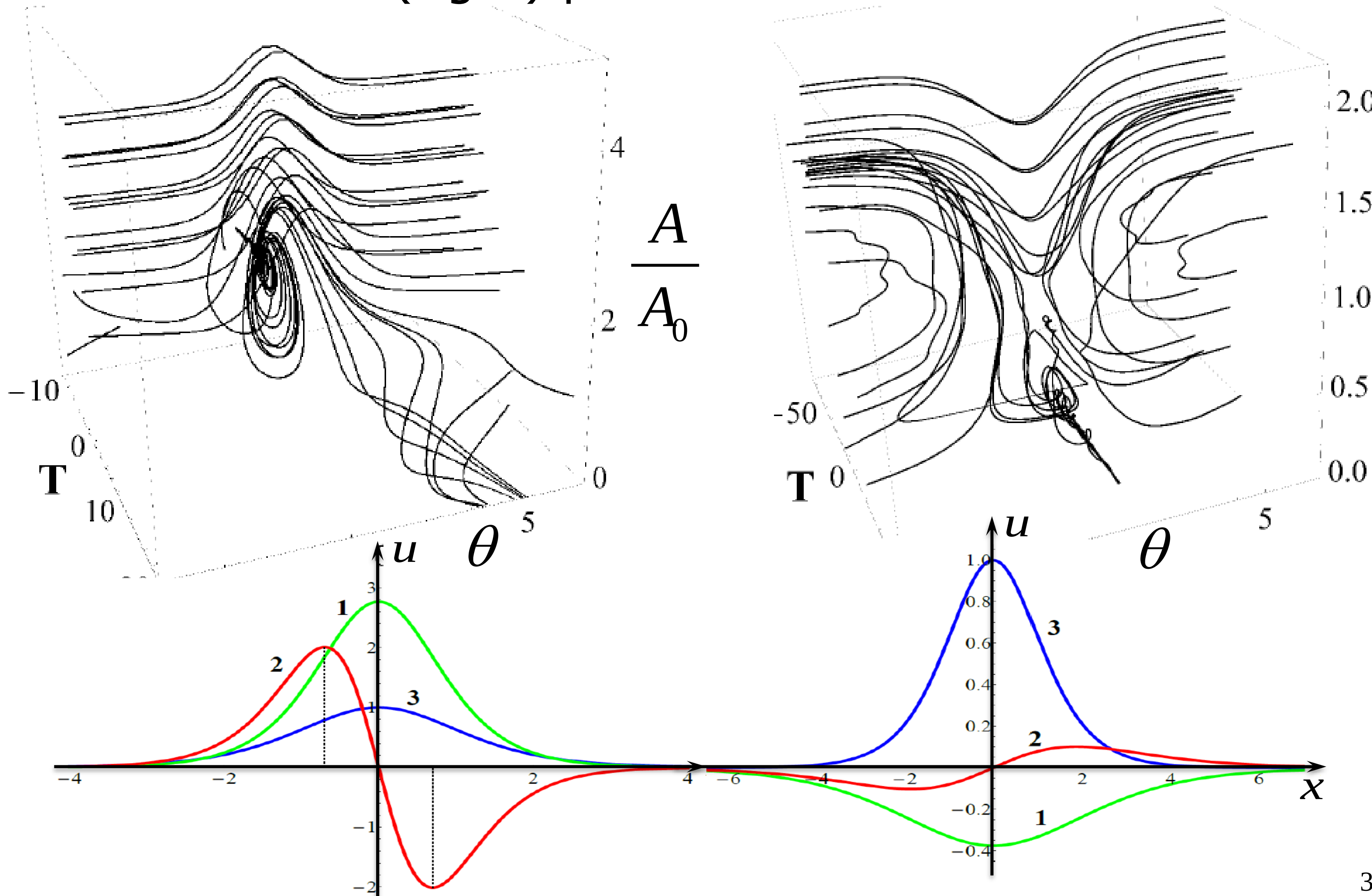
The governing set of equations now is:

$$\boxed{\frac{d\gamma}{dT}=\frac{2\varepsilon\alpha}{3\beta}e^{2\theta}\left(1+\tilde{V}\sin\omega T\right)\int_0^{+\infty}\frac{q^K}{\left(e^{2\theta}+q^K\right)^2}\frac{q-1}{\left(q+1\right)^3}dq,}$$

$$\boxed{\begin{aligned}\frac{d\theta}{dT}&=\Delta V\gamma+4\beta\gamma^3\\&\quad-\frac{\varepsilon\alpha}{3\beta\gamma}\left(1+\tilde{V}\sin\omega T\right)\int_0^{+\infty}\frac{e^{2\theta}+3q^K+q^K\left(2\theta-K\ln q\right)}{\left(e^{2\theta}+q^K\right)^2}\frac{q-1}{\left(q+1\right)^3}dq,\end{aligned}}$$

$$\Delta V\left(T\right)=c-V_{tot}\left(T\right)=-\frac{4\beta}{\Delta_f^2}+\frac{\varepsilon\alpha\Delta_f^2}{12\beta}\left(1+\tilde{V}\sin\omega T\right);\quad T=\varepsilon t$$

3D phase space for the positive (left) and negative (right) potential functions



References

1. R. Grimshaw, E. Pelinovsky, X. Tian, Interaction of a solitary wave with an external force, Physica D, 1994, v. 77, 405–433.
2. R. Lin, Y. Zeng, W-X. Ma, Solving the KdV hierarchy with self-consistent sources by inverse scattering method, Physica A, 2001, v. 291, 287–298.
3. M. Wang, Exact solutions for a compound KdV-Burgers equation, Phys. Lett. A, 1996, v. 213, 279–287.
4. L.A. Ostrovsky, E.N. Pelinovsky, V.I. Shrira, Y.A. Stepanyants, Beyond the KdV: Post-explosion development, Chaos, 2015, v. 25, 097620.
5. A. Ermakov, Y. Stepanyants, Soliton interactions with external forcing within the Korteweg–de Vries equation, Chaos, 2018, in press.

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