# Solitary wave trains and undular bores

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**Toowoomba, 2018**

# 1. Internal solitary waves: Ocean

## Internal tide and solitary waves, Pacific North West

An Atlas of Oceanic Internal Solitary Waves (May 2002)
by Global Ocean Associates
Prepared for the Office of Naval Research - Code 322PO

**Oregon Coast - Colombia River**

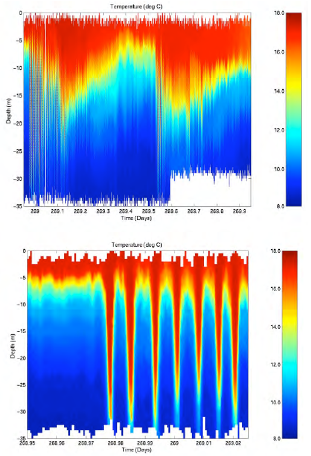



Figure 2. (Upper) A color contour time series of temperature profiles from the surface to 35m depth measured by the LMP over a one-day period. The 10°C span color contour scale is shown the right of the time series panel. The low frequency, semidiurnal internal tide displacement can clearly be seen along the yellow isotherm. (Lower) A profile time series of the first 1.7 hours of the time series shown in Figure 1a. White areas indicate times with no data. [From Stanton and Ostrovsky, 1998]

# 2. Internal solitary waves: Ocean

## Satellite view, Pacific North West



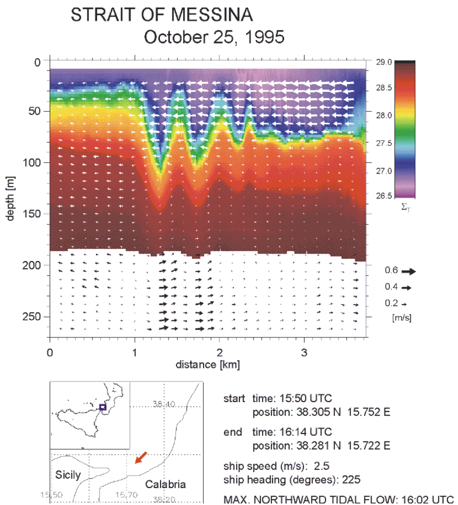
### Oregon Coast - Colombia River

with co-authors T. P. Stanton; Department of Oceanography, Naval Postgraduate School and
L. A. Ostrovsky CIRES, University of Colorado, NOAA ETL, CO 80303


Figure 1. Radarsat-1 image of the Colombia River and Oregon coast showing internal waves. The river plume induces seaward propagating internal waves. The shoreward propagating waves were generated at the coastal shelf break.

# 3. Internal solitary waves: Ocean

## Strait of Messina



# 4. Internal solitary waves: Ocean

## Strait of Messina, satellite view

An Atlas of Oceanic Internal Solitary Waves (February 2004)
by Global Ocean Associates
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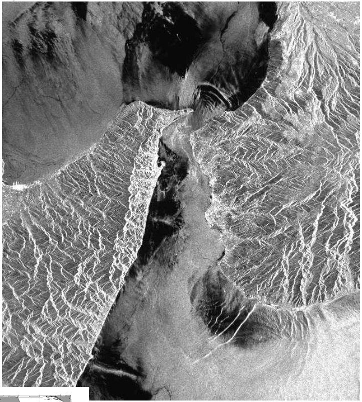
**Strait of Messina**



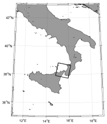



Figure 5. ERS-1 (C-band VV) SAR image of the Strait of Messina acquired on 11 July 1993 at 0941 UTC (orbit 10387, frame 2835). The image shows internal wave signatures radiating out of the strait in both the northern and southern directions. Northwards propagating internal waves are less frequently observed than southward propagating ones. Imaged area is 65 km x 65 km. ©ESA 1993. [From The Tropical and Subtropical Ocean Viewed by ERS SAR http://www.ifm.uni-hamburg.de/ers-sar/]

# 5. Internal solitary waves: Ocean

## Strait of Gibraltar satellite view



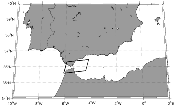
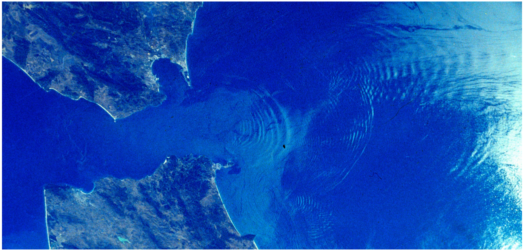



Figure 12. Astronaut photograph (STS41G-34-81) of Gibraltar region acquired on 11 October 1984 at 1222 UTC. The image shows three packets of solitons to the east of Gibraltar. Older packets are visible along Spanish and Moroccan coasts. [Image courtesy of Earth Sciences and Image Analysis Laboratory, NASA Johnson Space Center (http://eol.jsc.nasa.gov)]

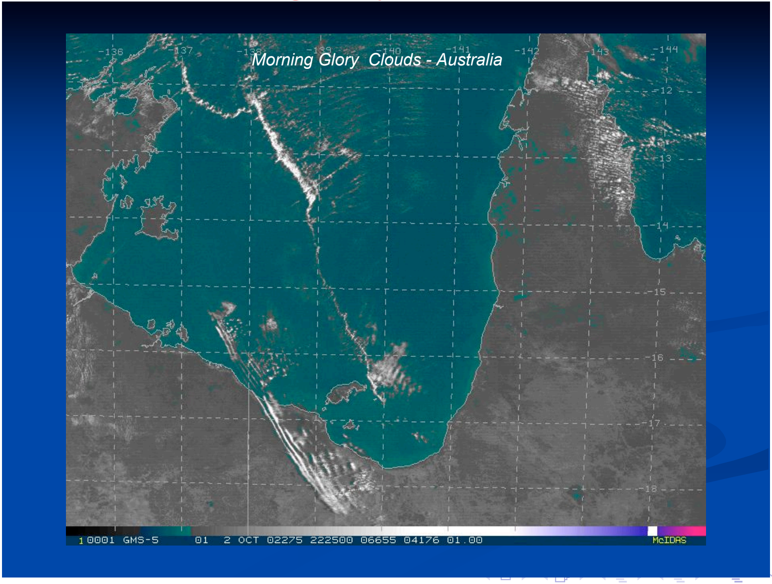


# 6. Internal solitary waves: Atmosphere

## Morning Glory Waves

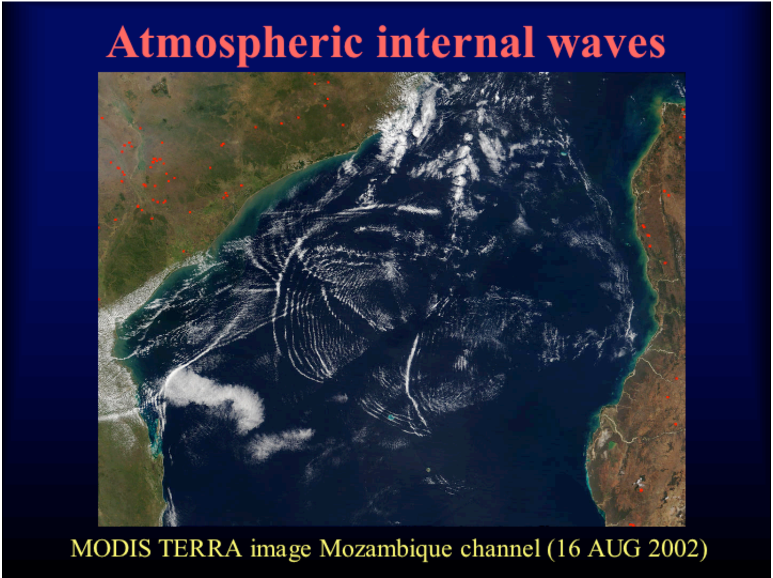
# 7. Internal solitary waves: Atmosphere

## Satellite view of Morning Glory Waves



# 8. Internal solitary waves: Atmosphere

**Satellite view of Mozambique waves**

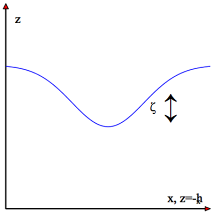


# 9. Internal solitary waves: Atmosphere

**Sable Island, Canada**

# 10. Weakly nonlinear model

Typically internal solitary waves are **long nonlinear** waves. Here we sketch the theory for the oceanic case.




The background state is the density $\rho_0(z)$ and horizontal current $u_0(z)$, while the **buoyancy frequency** is given by

$$\rho_0 N^2 = -g\rho_{0z}\,. \tag{1}$$

Then seek an asymptotic solution for the vertical particle displacement $\zeta = \zeta(x, z, t)$, based on the assumption that the waves are **long and weakly nonlinear**.

# 11. Korteweg-de Vries (KdV) equation

In a uniform medium this is given asymptotically by $\zeta \sim \eta(x,t)\phi(z)$, where

$$\eta_t + c\eta_x + \mu\eta\eta_x + \lambda\eta_{xxx} = 0\,. \tag{2}$$

$$\left\{\rho_0(c-u_0)^2\phi_z\right\}_z + \rho_0 N^2\phi = 0\,, \quad \text{for} \quad -h < z < 0\,, \tag{3}$$

$$\phi = 0 \quad \text{at} \quad z = -h\,, \quad (c-u_0)^2\phi_z = g\phi \quad \text{at} \quad z = 0\,. \tag{4}$$

The coefficients $\mu, \lambda$ are given by

$$I\mu = 3\int_{-h}^{0} \rho_0\,(c-u_0)^2\,\phi_z^3\,dz\,, \quad I\lambda = \int_{-h}^{0} \rho_0\,(c-u_0)^2\,\phi^2\,dz\,,$$

$$I = 2\int_{-h}^{0} \rho_0\,(c-u_0)\,\phi_z^2\,dz\,. \tag{5}$$

The KdV equation (2) is integrable. We are concerned with the **undular bore** solution, the outcome of a step-like initial condition using the Whitham modulation equations.

# 12. Variable-coefficient KdV (vKdV) equation

Now suppose the depth $h$, and background current $u_0$, density $\rho_0$ vary slowly in the horizontal direction with $x$. The KdV equation (2) is replaced by a similar equation, but with two extra terms,

$$\eta_t + c\eta_x + \frac{cQ_x}{2Q}\eta + \mu\eta\eta_x + \lambda\eta_{xxx} + \sigma\eta = 0\,, \quad Q = c^2 I\,, \tag{6}$$

$$I\sigma = -\int_{-h}^{\eta_0} \phi\phi_z F_{0z} dz\,, \; F_0 = \rho_0(u_0 u_{0x} + w_0 u_{0z}) + p_{0x}\,. \tag{7}$$

The modal equation now depends on $x$ **parametrically**, that is $\phi = \phi(z; x), c = c(x)$, and hence the coefficients $\mu, \lambda, Q$ also depend (slowly) on $x$. In (7) $p_0$ is the basic pressure, such that $p_{0z} = -g\rho_0$, and $w_0$ is the basic vertical velocity such that $u_{0x} + w_{0z} = 0$. The term $\sigma\eta$ arises when the basic state is not a solution of the inviscid Euler equations.

# 13. Canonical vKdV equation

The "spatial" evolution form, asymptotically equivalent to (6),

$$A = Q^{1/2}\eta\,, \quad A_T + \frac{\nu}{Q^{1/2}} A A_X + \delta A_{XXX} + \sigma A = 0\,. \tag{8}$$

$$X = \int_{x_0}^{x} \frac{dx}{c} - t\,, \quad T = \int_{x_0}^{x} \frac{dx}{c}\,, \quad \nu = \frac{\mu}{c}\,, \quad \delta = \frac{\lambda}{c^3}\,. \tag{9}$$

A further transformation yields the canonical form

$$U_\tau + \alpha U U_X + U_{XXX} = 0\,, \tag{10}$$

$$\text{where} \quad A = RU\,, \quad R = \exp\left(-\int_0^\tau \beta\, d\tau'\right),$$

$$\tau = \int_0^T \delta\, dT\,, \quad \alpha = \frac{R\nu}{\delta Q^{1/2}}\,, \quad \beta = \frac{\sigma}{\delta}\,. \tag{11}$$

$R, \alpha, \beta$ vary with $\tau$. There are two conservation laws, (10) for mass and wave action flux,

$$\{U^2\}_\tau + \{2\alpha U^3/3 + 2UU_{XX} - U_X^2\}_X = 0\,. \tag{12}$$

# 14. Solitary wave train

A modulated solitary wave is

$$U_{sol} = a\,\mathrm{sech}^2(\gamma\theta)\} + d\,, \quad \theta_X = k\,, \quad \theta_\tau = -kV\,, \tag{13}$$

$$V - \alpha d = \frac{\alpha a}{3} = 4\gamma^2 k^2\,. \tag{14}$$

with **three parameters**, the amplitude $a$, the pedestal $d$ and the wavenumber $k$. This is an exact solution if these parameters are constant. Here, especially when $\alpha(\tau)$ **is slowly varying,** $a, d, k$ **are slowly varying**. To find expressions for that variation, we use the Whitham modulation theory, developed for periodic waves, adapted for the solitary wave train, where (13) is regarded as sitting on a periodic lattice. Or one can use a direct multi-scale asymptotic expansion. Here we use the Whitham approach, using (10) and (12) for mass and wave action flux, together with the conservation of waves,

$$k_\tau + (kV)_X = 0\,. \tag{15}$$

# 15. Solitary wave train

The outcome is

$$d_\tau + \alpha d d_X = 0\,, \tag{16}$$

expressing conservation of mass. This is a Hopf equation for $d$ alone, and so $d$ can be regarded as a known quantity. Conservation of wave action leads to

$$\mathcal{A}_\tau + (\alpha d + \frac{\alpha a}{3})\mathcal{A}_X + \mathcal{A}\alpha d_X = 0\,, \quad \mathcal{A} = \{\frac{a^3}{\alpha}\}^{1/2}\,. \tag{17}$$

With $d$ already determined, this is a nonlinear hyperbolic equation for $\mathcal{A}$ alone, thus determining the amplitude $a$. Conservation of waves is the third equation,

$$k_\tau + (kV)_X = 0\,, \quad \text{where} \quad V = \alpha d + \frac{\alpha a}{3}\,, \tag{18}$$

can be regarded as known. This is a linear hyperbolic equation.

# 16. Whitham modulation theory: derivation

When the coefficient $\alpha$ in (10) is a constant the KdV equation supports a periodic travelling wave, the **cnoidal wave**,

$$U = a\{b(m) + \mathrm{cn}^2(\gamma\theta; m)\} + d\,, \quad \theta = k(X - V\tau)\,, \tag{19}$$

$$\text{where} \quad \alpha a = 12m\gamma^2k^2\,. \quad b(m) = \frac{1-m}{m} - \frac{E(m)}{mK(m)}\,, \tag{20}$$

$$V - \alpha d = \frac{\alpha a}{3}\left\{\frac{2-m}{m} - \frac{3E(m)}{mK(m)}\right\} = 4\gamma^2k^2\left\{2 - m - \frac{3E(m)}{K(m)}\right\}\,. \tag{21}$$

Here $cn(x; m)$ is the Jacobian elliptic function of modulus $m, 0 < m < 1$, and $K(m)$ and $E(m)$ are the elliptic integrals of the first and second kind, The expression (19) has period $2\pi$ in $\theta$ so that $\gamma = K(m)/\pi$, while the spatial period is $2\pi/k$ The (trough-to-crest) amplitude is $a$ and the mean value over one period is $d$. It is a three-parameter family with parameters $k, m, d$ say.

# 17. Whitham modulation theory: derivation

$$U = a\{b(m) + \mathrm{cn}^2(\gamma\theta; m)\} + d\,, \quad \theta = k(X - V\tau)\,,$$

where $\quad \alpha a = 12m\gamma^2 k^2\,. \quad b(m) = \dfrac{1-m}{m} - \dfrac{E(m)}{mK(m)}\,,$

$$V - \alpha d = \frac{\alpha a}{3}\left\{\frac{2-m}{m} - \frac{3E(m)}{mK(m)}\right\} = 4\gamma^2 k^2\left\{2 - m - \frac{3E(m)}{K(m)}\right\}\,.$$

it is three-parameter family with parameters $k, m, d$ say. As the modulus $m \to 1$, this becomes a solitary wave, since then $b \to 0$ and $\mathrm{cn}(x) \to \mathrm{sech}(x)$, while $\gamma \to \infty$, $k \to 0$ with $\gamma k = \Gamma$ fixed, producing the solitary wave train limit. As $m \to 0$, $b \to -1/2$, $\gamma \to 1/2$, $\mathrm{cn}(x) \to \cos(x)$, and it reduces to a sinusoidal wave $(a/2)\cos(\theta)$ of small amplitude $a \sim m$ and wavenumber $k$.

# 18: Whitham modulation theory: derivation

In the Whitham modulation theory **the cnoidal wave parameters, wavenumber $k$, modulus $m$ and mean level $d$, vary slowly with $\tau, X$**. Since $\alpha = \alpha(\tau)$ varies slowly with $\tau$, and the variable-coefficient KdV equation (10) is not integrable, we use the original Whitham method. That is, insert the cnoidal wave solution into the conservation laws (10, 12) and average over the phase $\theta$. The outcomes are

$$d_\tau + \alpha M_X = 0\,, \quad M = < \frac{U^2}{2} >\,, \tag{22}$$

$$M_\tau + P_X = 0\,, \quad P = < \frac{\alpha U^3}{3} - \frac{3U_X^2}{2} >\,, \tag{23}$$

where the $< \cdots >$ denotes a $2\pi$-average over $\theta$. The third equation is that for conservation of waves (15).

# 19: Whitham modulation theory: derivation

$$M = \frac{d^2}{2} + \frac{a^2}{2}\{C_4 - b^2\}, \tag{24}$$

$$P = \alpha\{-\frac{2d^3}{3} + 2dM +$$

$$a^3\{-\frac{2b^3}{3} + \frac{(1-m)b}{2m} + (b + \frac{1-2m}{2m})C_4 + \frac{5}{6}C_6\}\}. \tag{25}$$

Here the notation $C_4, C_6$ denote $< cn^4 >, < cn^6 >$ respectively, and like $b = -C_2 = - < cn^2 >$ depend on the modulus $m$ only.

$$C_4 = \frac{\{3m^2K(m) - 5mK(m) + 4mE(m) + 2K(m) - 2E(m)\}}{3m^2K(m)},$$

$$C_6 = \frac{1}{15m^3K(m)}\{15m^3K(m) - 34m^2K(m) + 23)m^2E(m)$$

$$+27mK(m) - 23mE(m) - 8K(m) + 8E(m)\}.$$

# 20: Solitary wave train

In the limit $m \to 1$, the cnoidal wave expression (19) becomes

$$U = a\,\mathrm{sech}^2(\gamma\theta; m)\} + d\,, \quad V - \alpha d = \frac{\alpha a}{3} = 12\gamma^2 k^2\,, \quad (26)$$

Also then $b \sim -1/K(m)$, $C_4 \sim 2/3K(m)$, and $C_6 \sim 8/15K(m)$. To leading order $M \sim d^2/2$ and $P \sim \alpha d^3/3$ and then both equations (22, 23) reduce to the same equation,

$$d_\tau + \alpha d d_X = 0\,. \quad (27)$$

The equation for conservation of waves (15) provides a second equation and the third equation is

$$\{\frac{a^2}{k\gamma}\}_\tau + V\{\frac{a^2}{k\gamma}\}_X + \frac{a^2}{k\gamma}\alpha d_X = 0\,, \quad (28)$$

This can be obtained by a more careful consideration of the limit $m \to 1$ in the modulation equations (22, 23) by retaining the terms in $1/K(m)$, or more directly by averaging the wave action conservation law (12) directly for a solitary wave.

# 21: Periodic wave train

In the general case for a modulated periodic wave train such as an undular bore the set (15, 22, 23) form three nonlinear hyperbolic equations for $k, m, d$. When $\alpha$ **is a constant** these equations can be cast into Riemann form, and using the Riemann variables the nonlinear hyperbolic system becomes diagonal and hence integrable. A similarity solution allows for a representation of an undular bore. **This describes an expanding wave train connecting a zero level at the front where $m \to 1$ to a mean level $U_0$ ($\alpha U_0 > 0$) at the rear where $m \to 0$. At the front the leading wave is a solitary wave of amplitude $2U_0$ and at the rear the waves are linear sinusoidal waves.**

However, this reduction is not available here when $\alpha = \alpha(\tau)$ as then the transformation from $k, m, d$ to the Riemann variables contains an explicit dependence on $\tau$, which generates extra terms and prevents a purely diagonal form being obtained.

# 22: Periodic wave train

If the periodic wave train is modulated only in $\tau$ then the solution of (15, 22, 23) is that $k, d, M$ are constants, which yield the formula

$$F(m) \equiv K^2\{(4-2m)EK - 3E^2 - (1-m)K^2\} = C\alpha^2 \,, \tag{29}$$

where $K = K(m)\,, E = E(m)$, and $C$ is a constant determined by the initial modulus $m_0$ at $\tau = 0$ where $\alpha = \alpha_0$. In normalised form

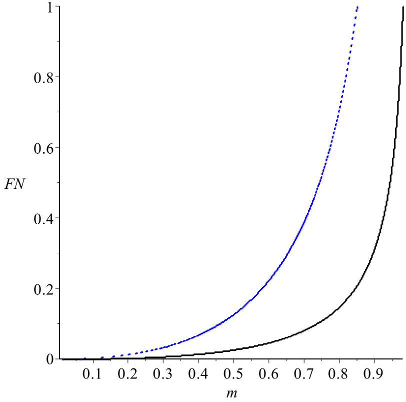
$$FN = \frac{F(m)}{F(m_0)} = \frac{\alpha^2}{\alpha_0^2}\,, \tag{30}$$

We plot when $m_0 = 0.98$ (a strongly nonlinear wave) and $m_0 = 0.85$. As $|\alpha|$ increases/decreases, then so does the modulus $m$

# 23: Periodic wave train

$$FN = \frac{F(m)}{F(m_0)} = \frac{\alpha^2}{\alpha_0^2}\,,$$

$$m_0 = 0.85\,(\textit{blue curve}),\, m_0 = 0.98\,(\textit{black curve})$$



# 24: Periodic wave train

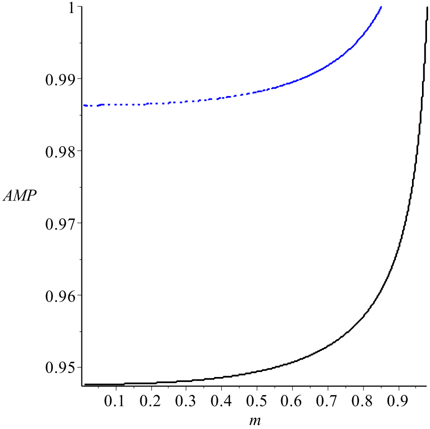
Then substitution into the expression (20) for the wave amplitude yields

$$\frac{a}{a_0} = \frac{mK(m)^2}{m_0 K(m_0)^2}\sqrt{\frac{F(m_0)}{F(m)}} \,. \tag{31}$$

This expresses the normalized amplitude in terms of the modulus $m$, which in turn varies with $\alpha$ according to (30). A plot of (31) is shown when $m_0 = 0.98, 0.85$, and these show that as $|\alpha|$ increases/decreases then so does the amplitude. In both cases there is a surprisingly small variation of the amplitude $a$ until $m \to 1$.

# 25: Periodic wave train

$$\frac{a}{a_0} = \frac{mK(m)^2}{m_0 K(m_0)^2} \sqrt{\frac{F(m_0)}{F(m)}}\,.$$



# 26: Single solitary wave

For a **single solitary wave** consider modulations in $\tau$ alone. Then (16) shows that $d$ is a constant, say $d = 0$, and (17, 18) show that $\mathcal{A}, k$ **are constants**, The former yields the well-known **adiabatic expression** $a^3 \propto \alpha$.

However, this single solitary wave limit is incomplete. A systematic multi-scale asymptotic expansion confirms the adiabatic expression $a^3 \propto \alpha$ due to conservation of wave action flux, but also reveals that the deforming solitary wave is accompanied by **a trailing shelf $d_S$, needed to conserve the total mass**. To leading order it depends on $\tau$ alone.

$$-Vd_s + \frac{\partial M_{sol}}{\partial \tau} = 0\,, \quad M_{sol} = \int_{-\infty}^{\infty} U_{sol}\, dX = \frac{2a}{\gamma k}\,. \qquad (32)$$

The difference between a single solitary wave and a solitary wave train is that, in the latter, the mass $d$ is an independent parameter, but the solitary wave has only one free parameter.

# 27: Undular bore

When $\alpha > 0$ **is a constant**, an undular bore can be found as an asymptotic solution of the KdV equation. This is an expanding wave train connecting a zero level at the front to a mean level $U_0 > 0$. **The leading wave is a solitary wave of amplitude** $2U_0$.

However, in a variable medium, $\alpha = \alpha(\tau)$, if an undular bore retains its structure as **a single-phase wave train**, then the **jump** $U_0$ **is preserved**, and so then the leading solitary wave would have a constant amplitude $2U_0$. But this is **inconsisten** with the result that the leading solitary wave amplitude should behave as $|\alpha|^{1/3}$. The resolution of this inconsistency is the formation of **a solitary wave train ahead of the undular bore**. This solitary wave train is described by (16, 17, 18), and there is a region where the rear of the solitary wave train interacts with the undular bore, forming a two-phase wave interaction.

# 28: Simulations

$$U_\tau + \alpha U U_X + U_{XXX} = 0\,,$$

$$\alpha = 1 + (\alpha_a - 1)\tanh(K\tau)\,,\ \ \alpha(\tau = 0) = 1\ \ \alpha(\tau \to \infty) = \alpha_a\,.$$

Either $\alpha_a > 1$ for propagation up a slope, or $\alpha_a < 0$ for propagation up a slope and through a **critical point of polarity change where** $\alpha = 0$. The initial condition $U(X, 0) = U_{ic}(X)$ is a modulated cnoidal wave representation of an undular bore in the constant coefficient KdV equation evolving from a step of height $U_0 > 0$, $\tau = \tau_1$,

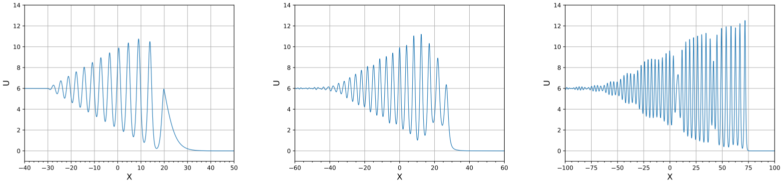
$$U_{ic}(X) = U_0\,\mathrm{ENV}(X)\{2m\mathrm{cn}^2(\kappa(X - V\tau_1); m) + 1 - m\}\,,$$

$$-U_0\tau_1 < X < \frac{2U_0\tau_1}{3}\,,\quad V = \frac{U_0}{3}\{1 + m\}\,,\quad U_0 = 6\kappa^2 k^2\,,$$

$$X = \frac{U_0\tau_1}{3}\{1 + m - \frac{2m(1-m)K(m)}{E(m) - (1-m)K(m)}\}\,.$$

$\mathrm{ENV}(X)$ is a box of height 1, containing the initial bore.

# 29: Simulation: Undular bore on a slope




A simulation of the vKdV equation (10) when $\alpha(\tau)$ varies from 1 to 1.5 for the undular bore initial condition with $U_0 = 6, \tau_1 = 5$ and $K = 0.3, \tau_a = 10$ $(K\tau_a = 3)$; the left panel is at $\alpha = 1$ , the middle panel is at $\alpha = 1.25$ and the right panel is at $\alpha = 1.5$.

For the leading waves, seek a similarity solution of (16, 17, 18),

$$d_\tau + \alpha d d_X = 0\,,$$

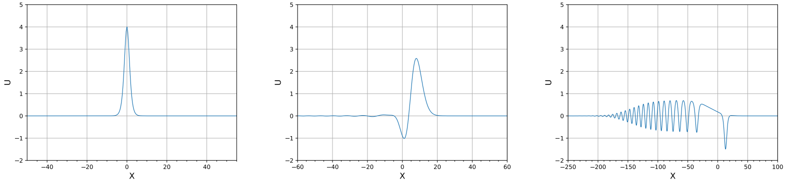
$$\mathcal{A}_\tau + (\alpha d + \frac{\alpha a}{3})\mathcal{A}_X + \mathcal{A}\alpha d_X = 0\,, \quad \mathcal{A} = \{\frac{a^3}{\alpha}\}^{1/2}\,.$$

$$k_\tau + (kV)_X = 0\,.$$

$$d = 0\,, \quad a = \frac{3\alpha^{1/3}X}{\chi}\,, \quad \chi = \int^\tau \alpha^{4/3}(\tau')\,d\tau'\,. \qquad (33)$$

This holds over the domain $X_m(\tau) < X < X_M(\tau)$ say. At the head the amplitude is $a_M = 3\alpha^{1/3}X_M/\chi$, and at the rear the amplitude is $a_m = 3\alpha^{1/3}X_m/\chi$. This must be matched to the following undular bore to determine $X_{M,m}(\tau)$. Note that if it is assumed that the leading wave in the following undular bore has amplitude $2U_0$ then we may expect that $a_m \approx 2U_0$ and $a_M \approx 2U_0\alpha^{1/3}$, since initially $\alpha = 1$.

# 31: Simulation of a polarity change for a single solitary wave




A simulation of the vKdV equation (10) when $\alpha(\tau)$ varies from 1 to $-1$ for a **solitary wave initial condition** with $U_0 = 4$ and $K = 0.03, \tau_a = 100$ ($K\tau_a = 3$); the left panel is at $\alpha = 1$ , the middle panel is at $\alpha = 0$ and the right panel is at $\alpha = -1$. Note the solitary waves moving forward along the rarefaction.

# 32: Polarity change for a solitary wave

There is a another similarity solution of (16, 17, 18) for this case of polarity change, **after the critical point**,

$$d_\tau + \alpha d d_X = 0\,,$$

$$\mathcal{A}_\tau + (\alpha d + \frac{\alpha a}{3})\mathcal{A}_X + \mathcal{A}\alpha d_X = 0\,, \quad \mathcal{A} = \{\frac{a^3}{\alpha}\}^{1/2}\,.$$

$$k_\tau + (kV)_X = 0\,.$$

$$d = \frac{X - X_0}{\eta}\,, \quad \eta = \int_{\tau_c}^{\tau} \alpha(\tau')\, d\tau'\,, \quad X < X_0\,, \tag{34}$$

$$\mathcal{A} = -\frac{1}{\eta}\{\frac{3(X - X_0)}{\eta\xi}\}^{3/2}\,, \quad \xi = \int_{-\infty}^{\eta} \frac{|\alpha(\eta')|^{1/3}\, d\eta'}{|\eta'|^{5/3}}\,, \tag{35}$$

Here $\alpha < 0$ and so $\eta < 0$, ensuring that the rarefaction wave $d > 0$ in $X < X_0$. The determination of $X_0$ requires a detailed matching with the solution at the critical point, omitted here.

# 33: Polarity change for a solitary wave

The rarefaction wave (34) can only extend to a point $X - X_0 = -L_r(\eta)$ where $L_r(\eta)$ is likewise undetermined. But the **mass of the rarefaction wave** is then $-L_r^2(\eta)/2\eta$ and this can be approximately equated to **the initial solitary wave mass** $2U_0/\kappa = 2(12U_0)^{1/2}$, thus giving an approximate expression for $L_r(\eta)$. The expression (35) for the solitary wave amplitude $a$ holds on the domain $-L_r(\eta) < X - X_0 < -L_s(\eta)$ where the upper bound $L_s(\eta)$ determines the amplitude of the leading solitary wave, that is $a_s = -3|\alpha|^{1/3}L_s/|\eta|^{5/3}\xi$. The values of $a_s, L_s$ are undetermined and requires matching with the solution at the critical point. However, an approximate estimate can be based on the assumption that since the emerging solitary wave train is the leading edge of an undular bore resolving the jump at the rear of the rarefaction wave, and then $a_s = 2L_r/\eta$, where in turn $L_r$ is estimated from conservation of mass, as above.

# 34: Simulation: Polarity change for an undular bore

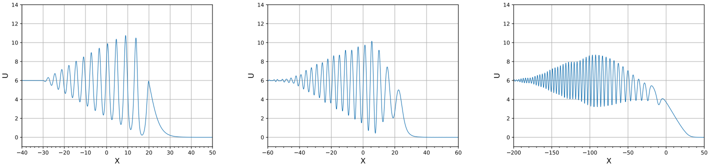



Figure: A simulation of the vKdV equation (10) when $\alpha(\tau)$ varies from 1 to $-1$ for the undular bore initial condition with $U_0 = 6, \tau_1 = 5$ and $K = 0.3, \tau_a = 10$ ($K\tau_a = 3$); the left panel is at $\alpha = 1$ , the middle panel is at $\alpha = 0$ and the right panel is at $\alpha = -1$.

# 35: Simulation: Polarity change for a periodic wave train

In contrast, the corresponding theory for a **periodic wave train** has only recently been developed. The periodic wave train deforms adiabatically according to the expression (30),

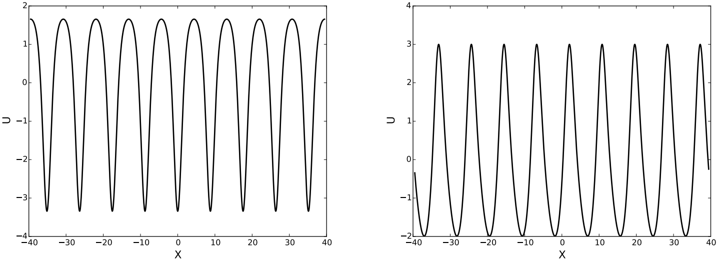
$$FN = \frac{F(m)}{F(m_0)} = \frac{\alpha^2}{\alpha_0^2},$$

As $\alpha \to 0$ it can be shown that $m \sim \alpha$ and so $m \to 0$. But from (31) **the amplitude $a$ tends to a finite value**.

$$\frac{a}{a_0} = \frac{mK(m)^2}{m_0 K(m_0)^2} \sqrt{\frac{F(m_0)}{F(m)}}.$$

The wave train then passes through the critical point as a **linear wave of finite amplitude**, but then reverses polarity, this being achieved by a phase change in the linear wave around the critical point.

# 36: Simulation: Polarity change for a periodic wave train




A simulation of the vKdV equation (10) when $\alpha(\tau)$ varies from $-1$ to 1 for a cnoidal wave initial condition with modulus $m = 0.95$ and $a_0 = -5$. The left panel is when $\alpha = -1$, the right panel is when $\alpha = 1$.

# 37: Summary

$$d_\tau + \alpha d d_X = 0\,,$$

$$\mathcal{A}_\tau + (\alpha d + \frac{\alpha a}{3})\mathcal{A}_X + \mathcal{A}\alpha d_X = 0\,, \quad \mathcal{A} = \{\frac{a^3}{\alpha}\}^{1/2}\,,$$

$$k_\tau + (kV)_X = 0\,, \quad \text{where} \quad V = \alpha d + \frac{\alpha a}{3}\,,$$

The solitary wave train equations are a very powerful tool for analysing solitary waves and undular bores in variable media, especially because they are each hyperbolic equations which can be solved in sequence for $d, \mathcal{A}, k$.

Although developed here for the KdV equation, I expect they can be found for other nonlinear wave equations.

# Thank you: References

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