Optimal Shear Instabilities of Large-Amplitude Internal Solitary Waves

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Workshop on Nonlinear Waves in Oceanography and Beyond

in honor of Roger Grimshaw's 80^{th}!

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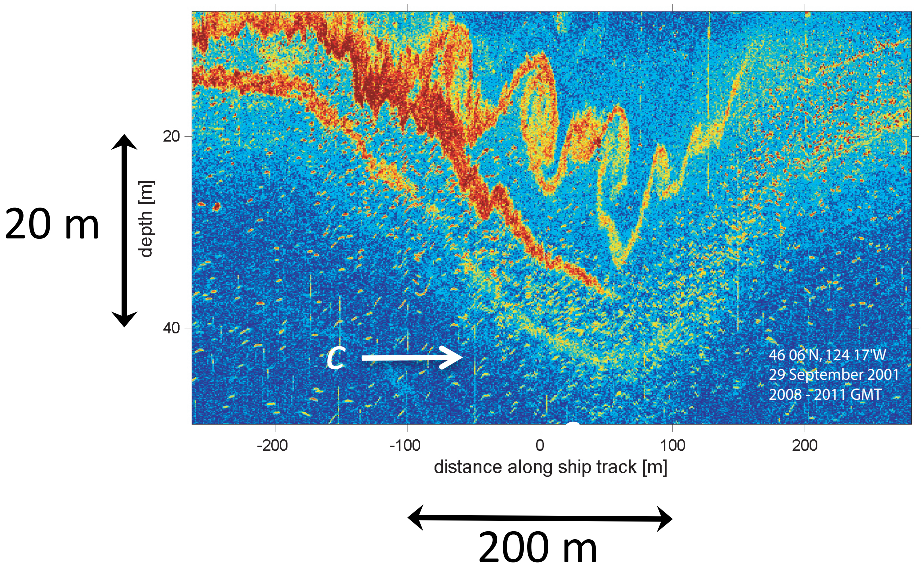
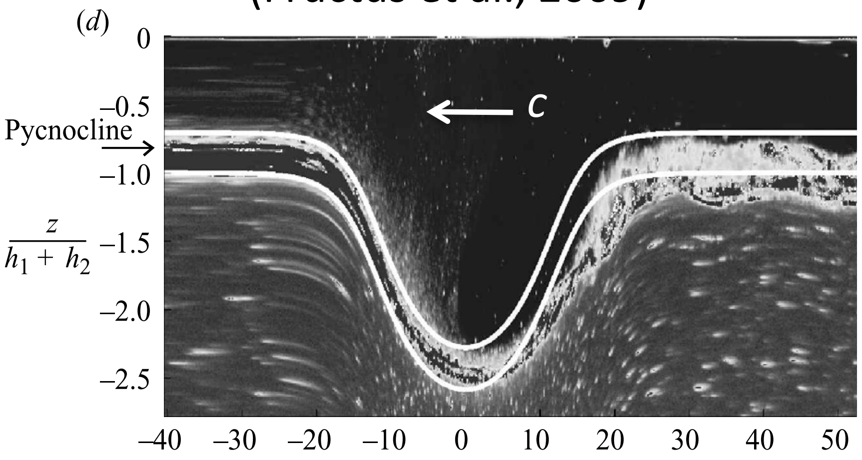
Examples of ISW shear instabilities

Acoustic backscatter of K-H billows in a large-amplitude ISW (Moum et al., 2003)

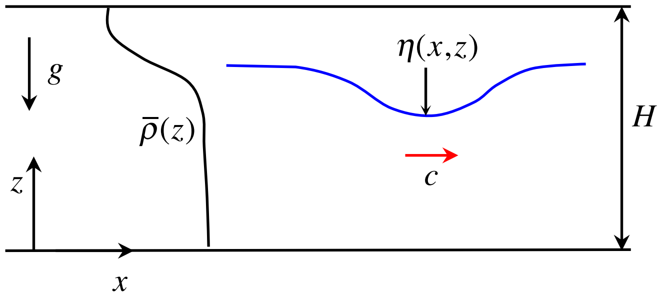


Laboratory experiments (Fructus et al., 2009)



These energetic wave are common in the world's coastal oceans and their instabilities of the type shown here may play a role in turbulent vertical mixing. Furthermore, energy loss to instabilities will control their lifetimes (i.e. propagation distances).

Dubreil-Jacotin-Long (DJL) theory for internal solitary waves (c.f. Stastna and Lamb, 2002)



$$\boxed{\nabla^2\eta + \frac{N^2(z-\eta)}{c^2}\eta = 0 \qquad \eta(x,0)=\eta(x,H)=0, \qquad \eta \to 0 \ \text{ for } |x| \to \infty}$$

$$N^2(z) = -\frac{g}{\rho_0}\frac{\partial\bar{\rho}}{\partial z}, \qquad \bar{\rho}(z) = \rho_0 + \Delta\rho S(z),$$

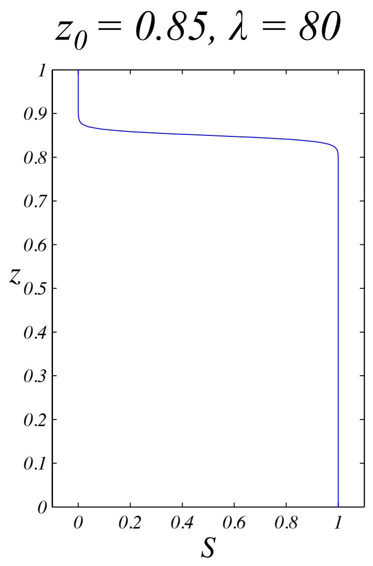
$$\psi = c(\eta - z), \qquad u(x,z) = \psi_z, \qquad w(x,z) = -\psi_x, \qquad s(x,z) = S(z-\eta)$$

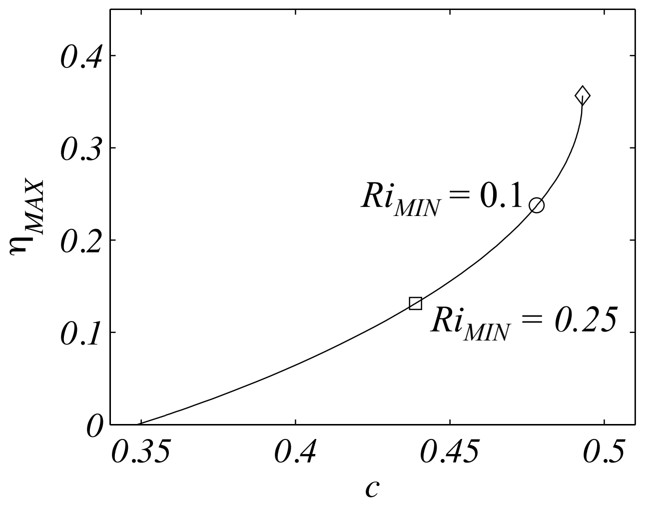
All variables scaled using: $H, \quad U = (g'H)^{1/2}, \quad H/U$

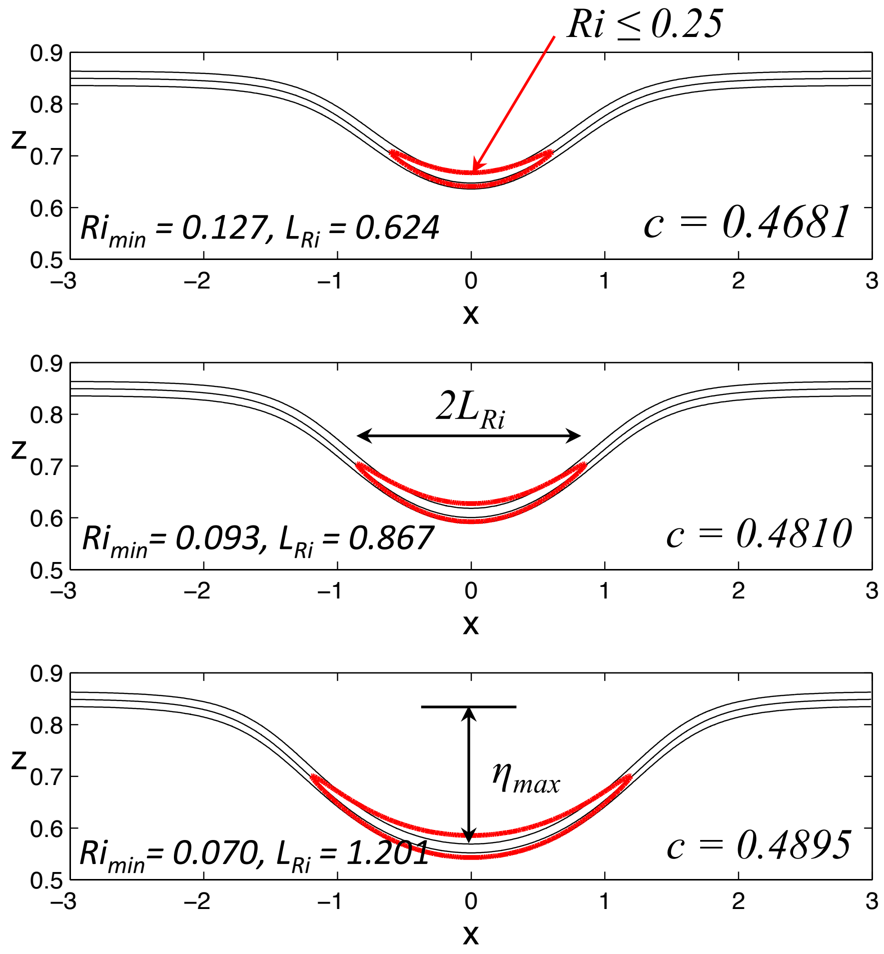
$g' = g\Delta\rho/\rho_0, \qquad \Delta\rho = \rho_b - \rho_t =$ bottom to top density difference

DJL solutions are obtained using Newton-Raphson method with pseudo arc-length continuation (Luzzatto-Fegiz and Helfrich, 2014)

DJL solutions with $S(z) = \frac{1}{2}(1 - \tanh[\lambda(z - z_0)])$







Semi-empirical stability criteria

Troy and Kosseff (2005), Fructus and Grue (2009), Barad and Fringer (2010) (from numerical and experimental studies)

$$Ri_{\min} \approx 0.1, \qquad \ln\left[\left(\frac{a_f}{a_0}\right)^2\right] = 2\bar{\omega}_i T_i \geq 10, \qquad \frac{L_{Ri}}{\xi} \geq 0.86, \qquad T_i \approx 2L_{Ri}/c, \qquad \eta(\xi) = \frac{\eta_{MAX}}{2}$$

Lamb and Farmer (2011) (spatial KH instability)

$$Ri_{\min} \approx 0.1, \qquad \ln\left[\left({a_f}/{a_0}\right)^2\right] = 4\bar{k}_i L_{Ri} \geq 8, \qquad \frac{L_{Ri}}{\xi} \geq 0.8$$

The instability is *spatial*, not temporal, (Camassa and Viotti, 2012). There is no point of absolute instability. It is a *signaling problem* that depends on the presence and properties of upstream disturbances.

- What are the bounds for disturbance growth?
- What disturbances lead to maximal growth?

- The zone of $Ri < 0.25$ is not a parallel flow and thus the standard normal-mode stability approach could be misleading.

- Could use WKB method for temporal or spatial growth (will consider later), but this will miss any non-normal disturbance growth (e.g., Farrell, 1984).

- Obtain initial *linear* disturbances that maximize the gain of perturbation energy, $G(T)$, over an evolution time T for a DJL solitary wave base state

$$G(T) = \frac{E(T)}{E_0}, \qquad E = \frac{1}{V}\int_V \left(\frac{\mathbf{u}\cdot\mathbf{u}}{2} + \frac{s^2}{2N_{max}^2} \right) dV, \qquad E_0 = E(0) = 1$$

- Do not want to impose anything about the initial disturbance structure and evolution other than it obey the Navier-Stokes equations linearized around the DJL internal solitary wave.

Optimal Transient Growth from a linear Direct-Adjoint-Loop (DAL) variational method (Schmid, 2007; Kaminski et al., 2014)

Maximize the the gain of perturbation energy $G(T)$ over the time T constrained by the LNS equations:

$$L = G(T) - \left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial U_i}{\partial x_j} + U_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} - \mathrm{Re}^{-1} \frac{\partial^2 u_i}{\partial x_j^2} + s\delta_{3i},\ \hat{u}_i \right] \quad \text{momentum}$$

$$- \left[\frac{\partial u_i}{\partial x_i},\ \hat{p} \right] \quad \text{continuity}$$

$$- \left[\frac{\partial s}{\partial t} + u_j \frac{\partial S}{\partial x_j} + U_j \frac{\partial s}{\partial x_j} - (\mathrm{Re}\,\mathrm{Pr})^{-1} \frac{\partial^2 s}{\partial x_j^2},\ \hat{s} \right] \quad \text{density}$$

$$- \langle u_i(0) - u_{0i},\ \hat{u}_{0i} \rangle - \langle s(0) - s_0,\ \hat{s}_0 \rangle \quad \text{initial conditions}$$

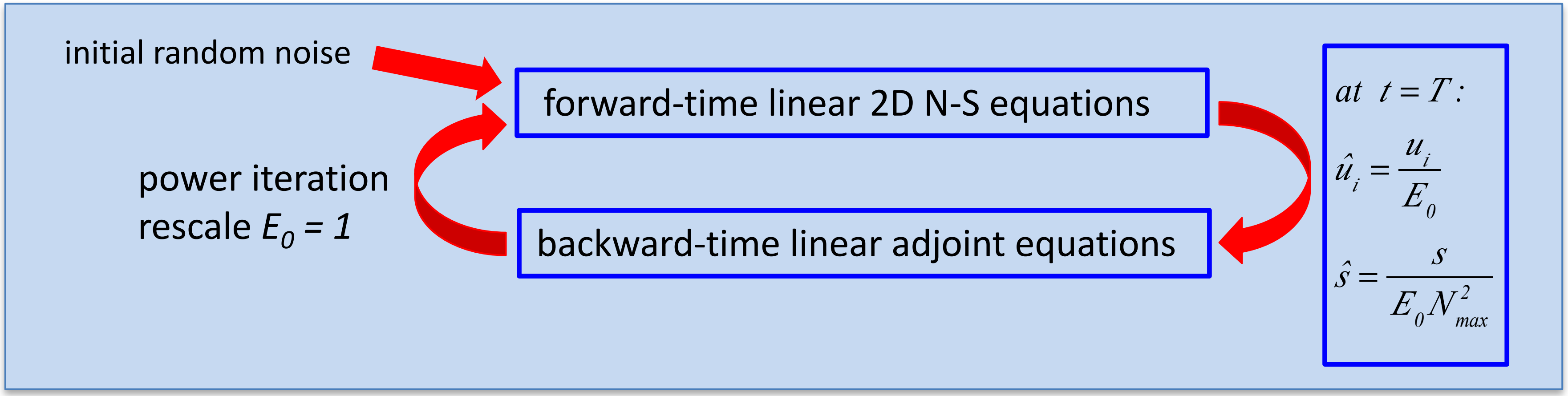
$$G(T) = \frac{E(T)}{E_0}, \qquad E = \frac{1}{2}\langle \mathbf{u},\mathbf{u} \rangle + \frac{1}{2N_{\max}^2}\langle s,s \rangle, \qquad \langle \mathbf{u},\mathbf{v} \rangle = \frac{1}{V}\int_V \mathbf{u}\cdot\mathbf{v}\, dV, \qquad [\mathbf{u},\mathbf{v}] = \int_0^T \langle \mathbf{u},\mathbf{v} \rangle dt$$

$\mathbf{U},\ S \ \rightarrow$ DJL internal solitary wave

$\mathbf{u},\ s,\ p \ \rightarrow$ perturbation velocity, density, pressure

$\hat{\mathbf{u}},\ \hat{s},\ \hat{p} \ \rightarrow$ Lagrange multipliers (adjoint variables)

DAL solution procedure



Numerical methods:

1. 6th-order compact finite-differences with 3rd-order RK integration in time (forward and adjoint equations in streamfunction-vorticity form)

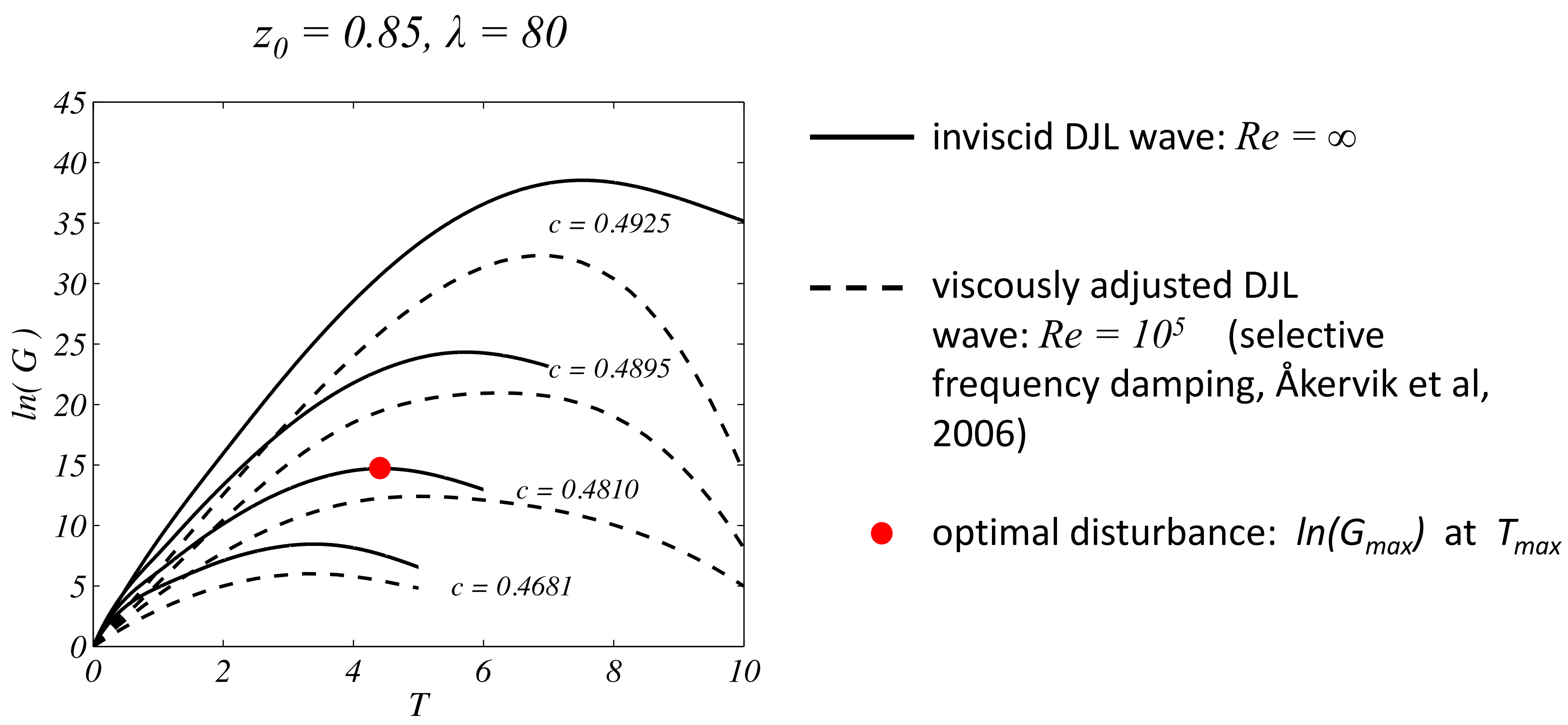
2. pseudo-spectral with RK3 in time for the viscously-adjusted DJL waves (see below)

Loop until $G(T) \longrightarrow constant$ and the changes in $u(0)$ and $s(0) \longrightarrow 0$

Typically takes 5-10 loops starting from random noise

Linear Optimal Perturbation Gain for: $\mathrm{Re} = \dfrac{\left(g' H^3\right)^{1/2}}{\nu} = 10^5$ and $\mathrm{Pr} = \dfrac{\nu}{\kappa} = 1$

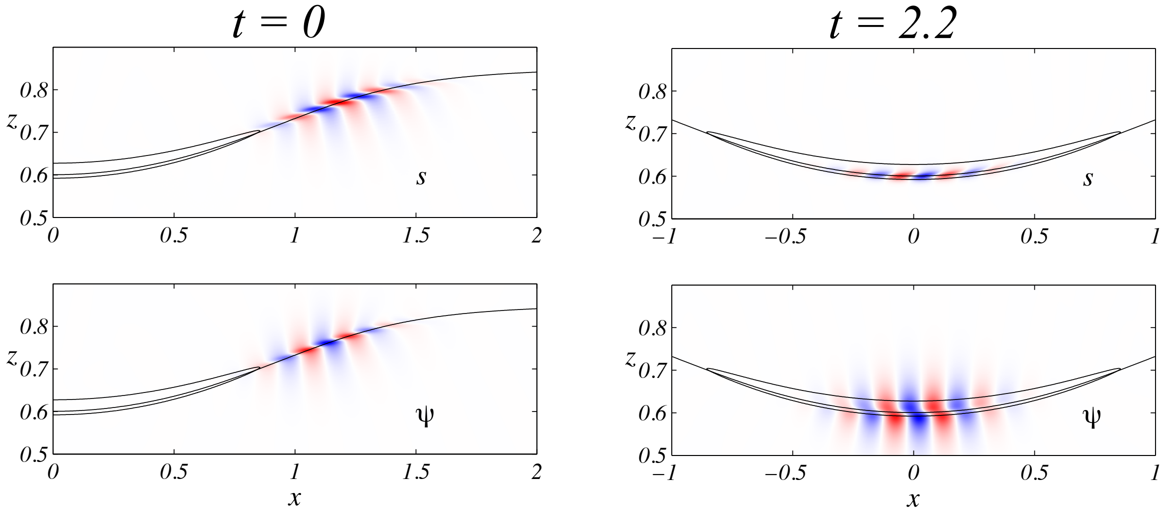
(Laboratory-scale conditions: $\Delta\rho/\rho_0 = 10^{-2}$, $H \approx 0.35$ m, $\nu = 10^{-6}$ m^2 s^{-1})



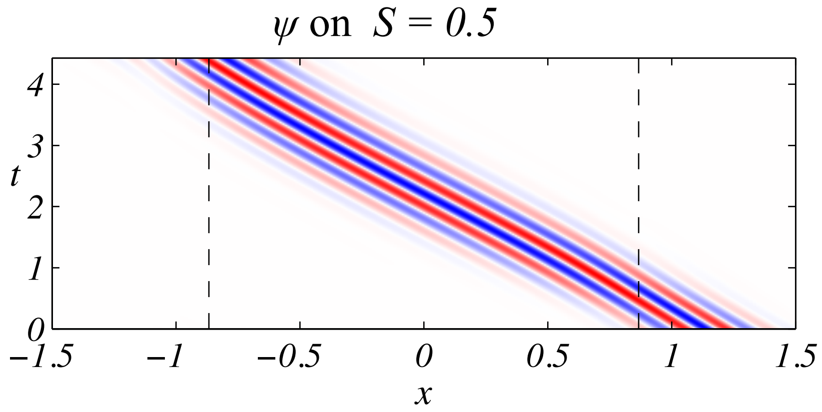
- Increasing Re and/or Sc causes $ln[G_{\max}]$ to increase some and then saturate.
- 2D mode is most dangerous (transverse modes ~ $e^{\beta y}$)

Linear evolution of the optimal disturbance

$c = 0.4810 \qquad T_{max} = 4.43 \qquad ln(G_{max}) = 14.65$



All fields normalized by maximum values at each time



Disturbance evolves as a coherent wave packet with carrier frequency $\omega \approx -14.85$ (Doppler shifted in ISW frame)

Hilbert transform methods are used to find ω, $k(x)$, $c_g(x)$.

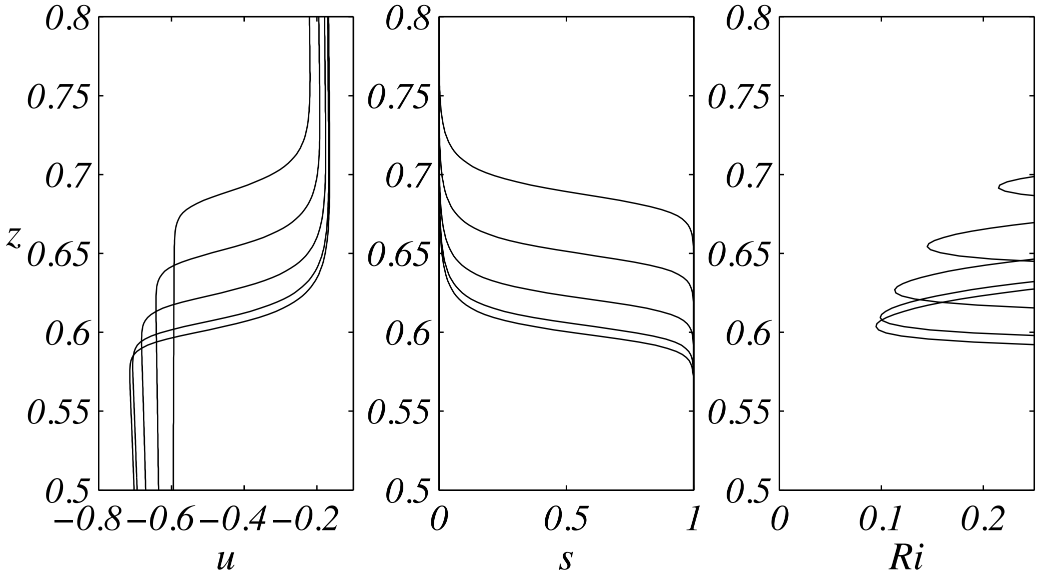
Slowly varying (WKB) growth: 1D Taylor-Goldstein equation at each $|x| \le L_{Ri}$
(see also: Lamb and Farmer, 2011; Camassa and Viotti, 2012)

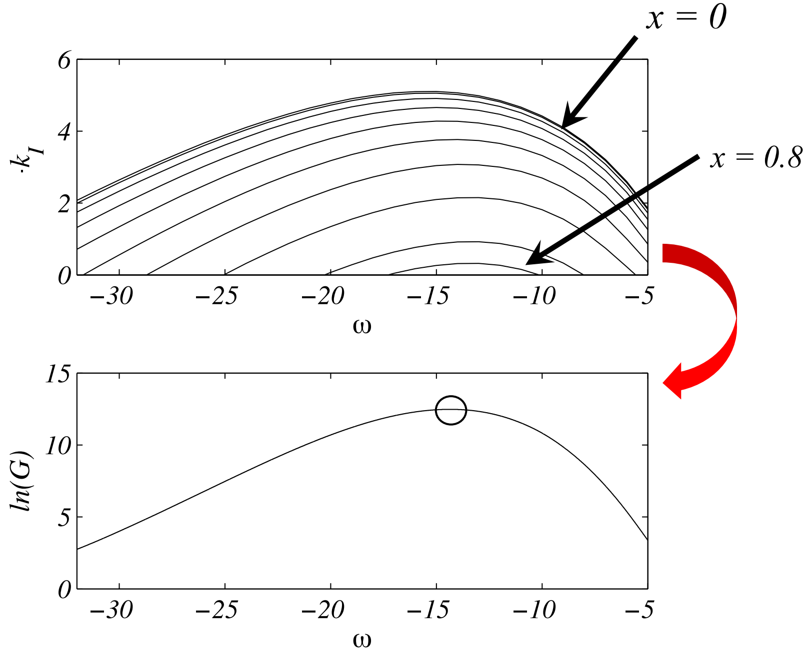
$$\frac{A(x)}{A(L_{Ri})} \sim \exp\left[-\left(\int_{L_{Ri}}^{x} k_I^-(\omega; x')\,dx'\right)\right], \qquad \text{with real} \quad \omega \quad \text{and} \quad k(\omega) = k_R + ik_I$$

$$G(\omega) = \left(\frac{A(-L_{Ri})}{A(L_{Ri})}\right)^2 = \exp\left[-2\left(\int_{L_{Ri}}^{-L_{ri}} k_I^-(\omega; x')\,dx'\right)\right], \qquad t(x) = \int_{L_{Ri}}^{x} \frac{dx'}{c_g(x')}, \qquad c_g(x) = \frac{\partial \omega}{\partial k_R}$$

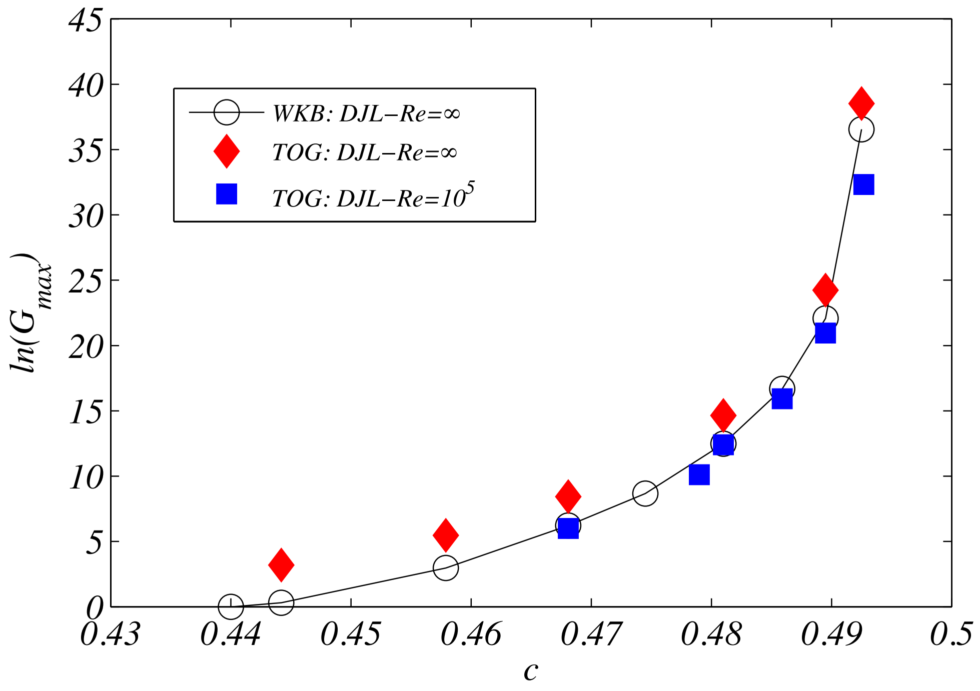
Profiles of *U(z), S(z)* and *Ri(z)* for $c = 0.4810$, $|x| \le L_{Ri} = 0.867$

$x = [0, 0.2, 0.4, 0.6, 0.8]$

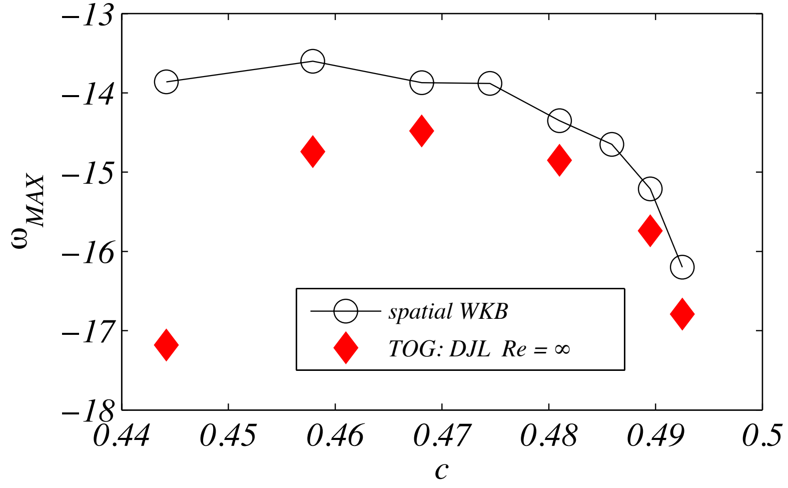




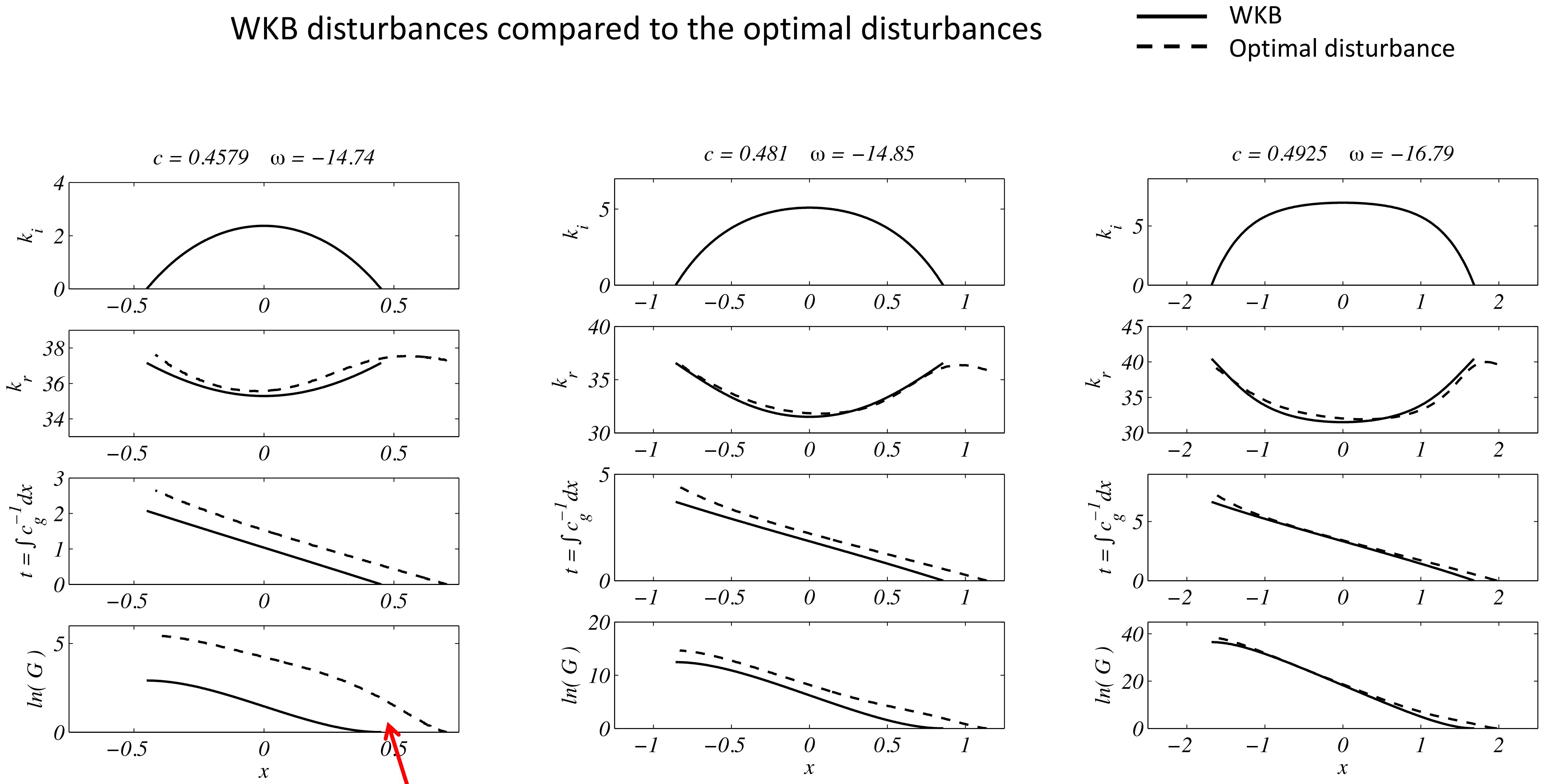
Optimal disturbances vs. slowly varying (WKB) K-H spatial growth



maximal growth frequency



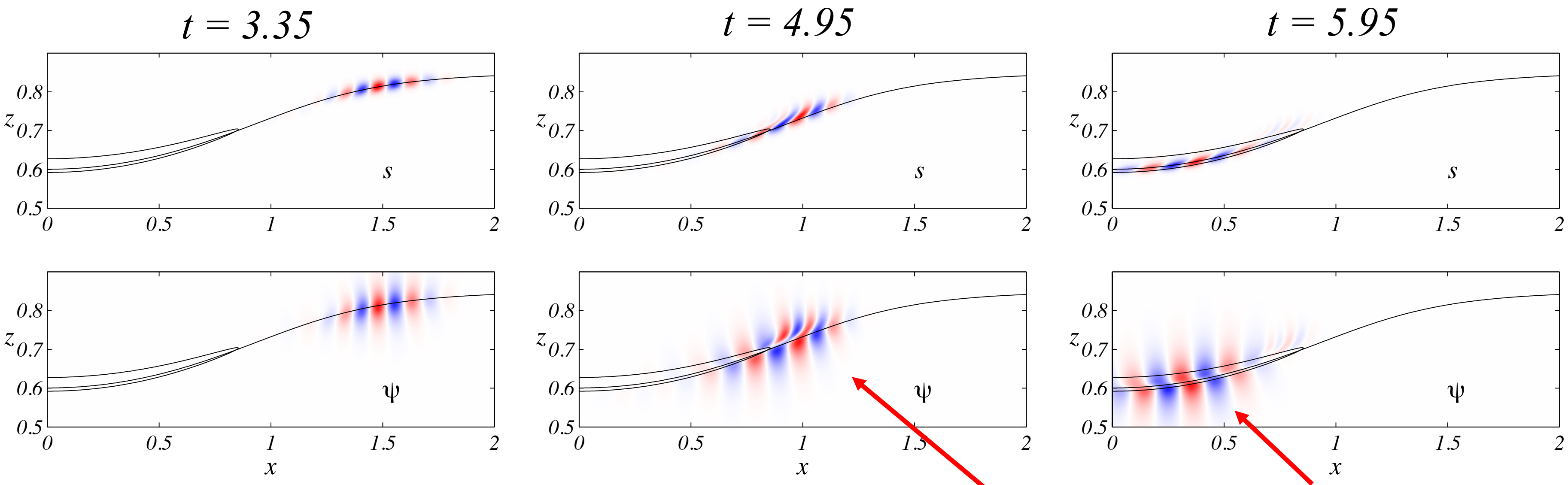
WKB disturbances compared to the optimal disturbances



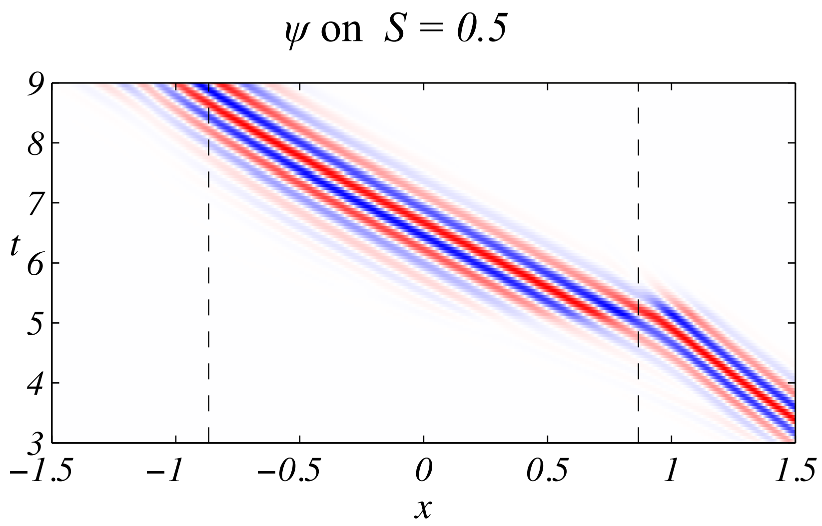
initial non-normal growth: $ln(G) \approx 1.9$ independent of c from an Orr-type mechanism (Orr, 1907; Schmid and Henningson, 2001)

Linear evolution of a Gaussian packet of free waves for *c = 0.4810*

carrier frequency $\omega \approx -14.85$ ($k_0 = 38.4$ and intrinsic frequency $\omega^i = 3.60$) initialized at $x = 3$

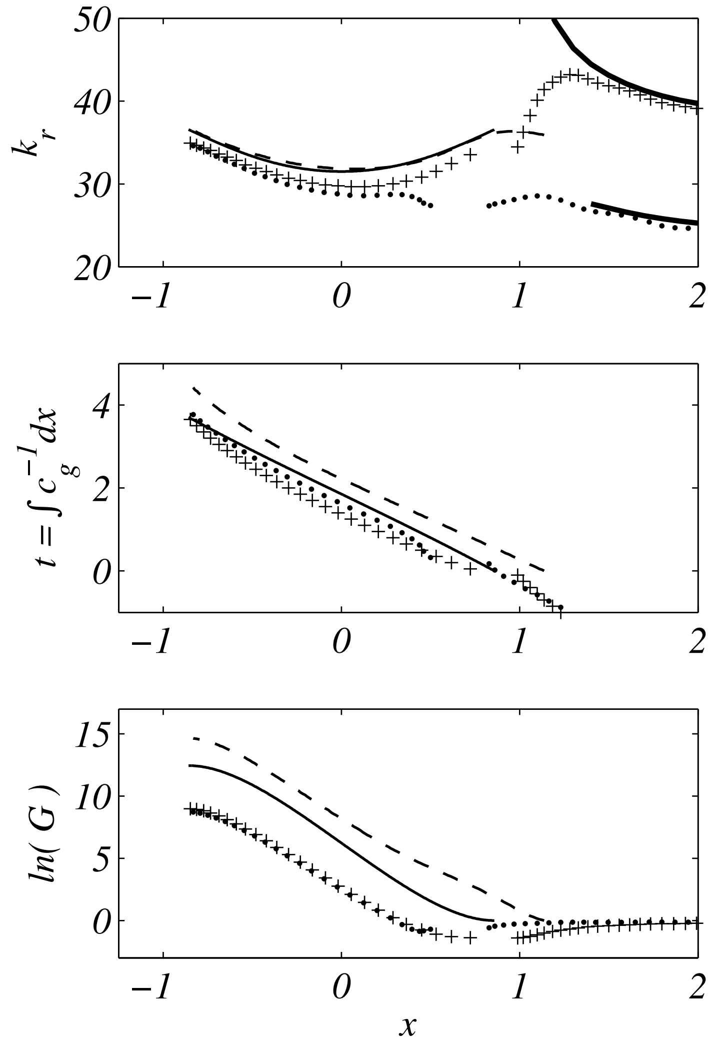


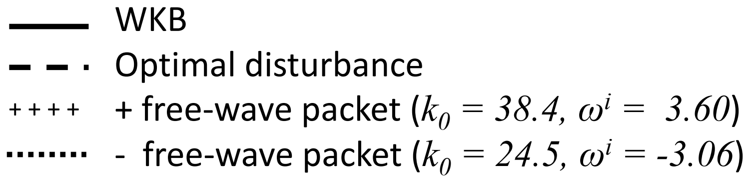
Disturbance takes K-H form



Disturbance is leaning with the shear prior to $|x| < L_{ri}$ leading to energy transfer from the disturbance to the ISW (c.f. Camassa and Viotti, 2012)

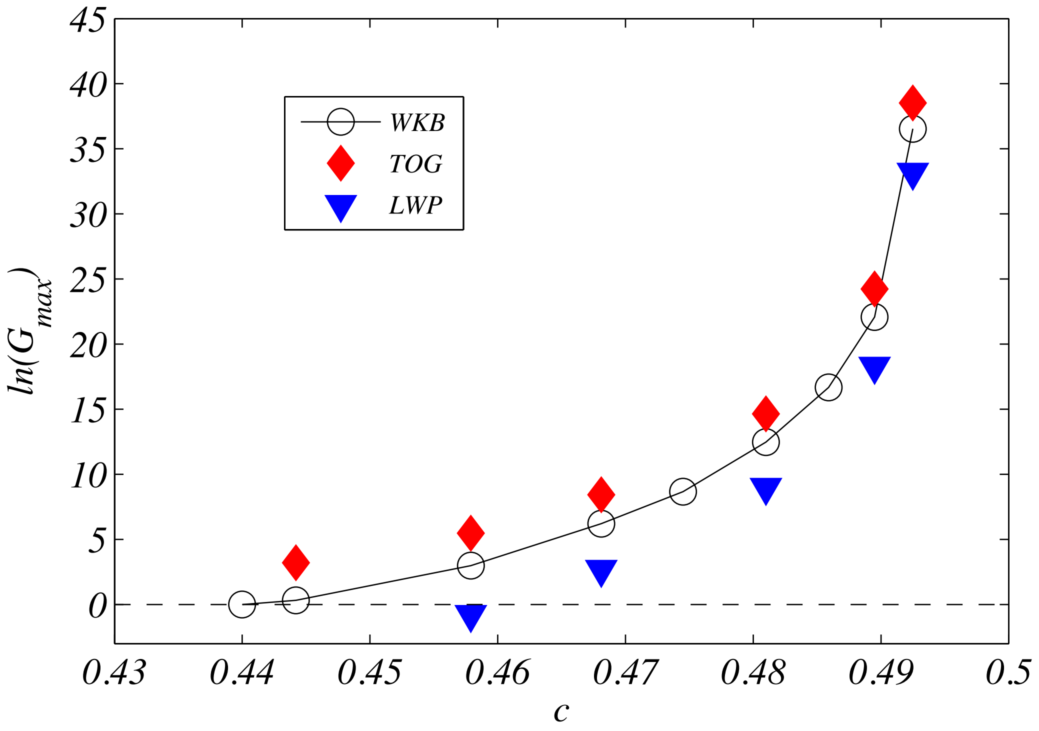
Optimal disturbance, WKB, and free-wave packets





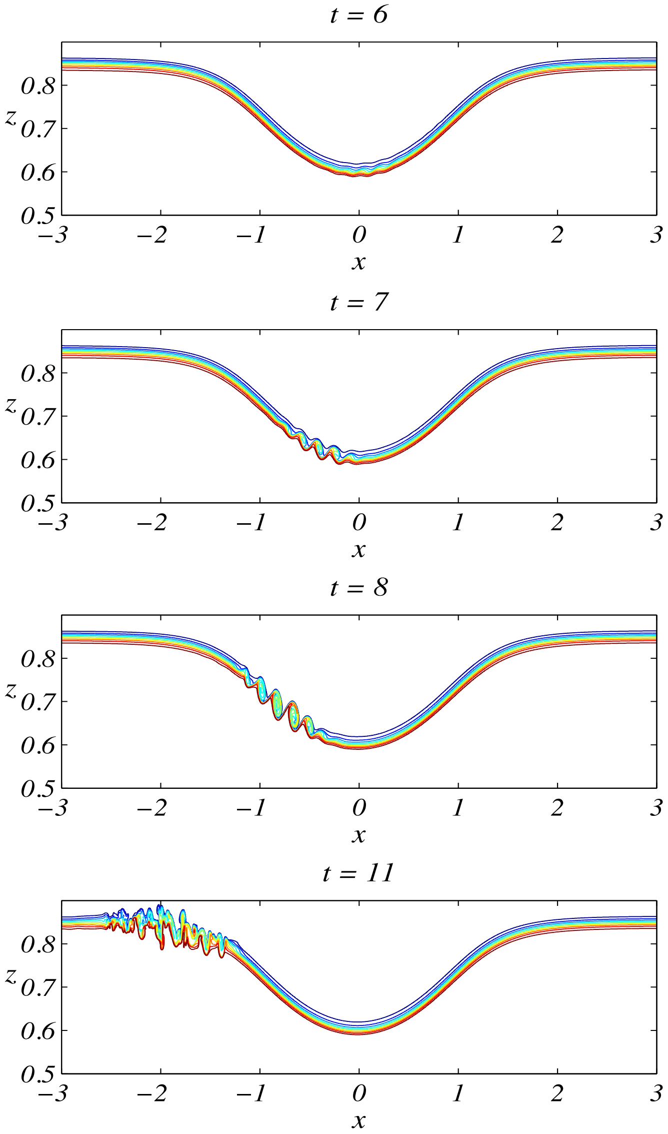
- Initial loss of free-wave disturbance energy as the free-wave packets enter the ISW
- Substantially less energy gain than the optimal disturbance.
- Similar gain for the + or – waves (at same Doppler-shifted frequency given by the optimal disturbance).

Optimal disturbance, WKB, and free-wave packet

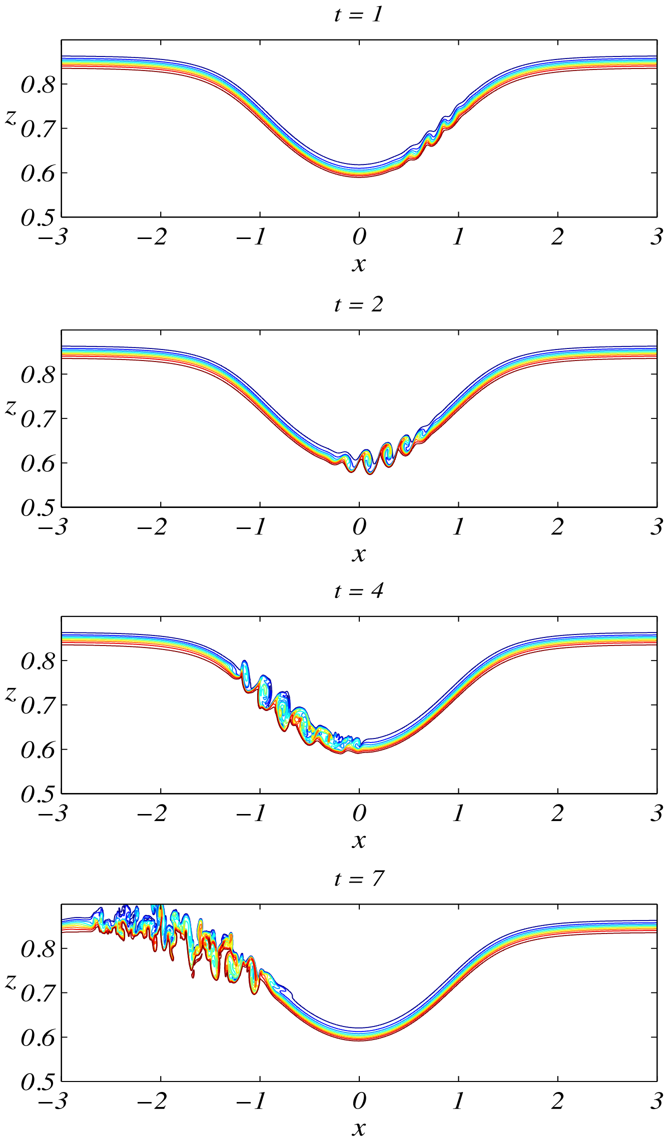


2D fully nonlinear development for $c = 0.4810$ and $E_{dist} / E_{ISW} = 10^{-5}$ at $t = 0$

Packet of free linear waves: $\omega = \omega_{opt}$



Optimal disturbance



$$E_{ISW} = E_{KE} + E_{APE}$$

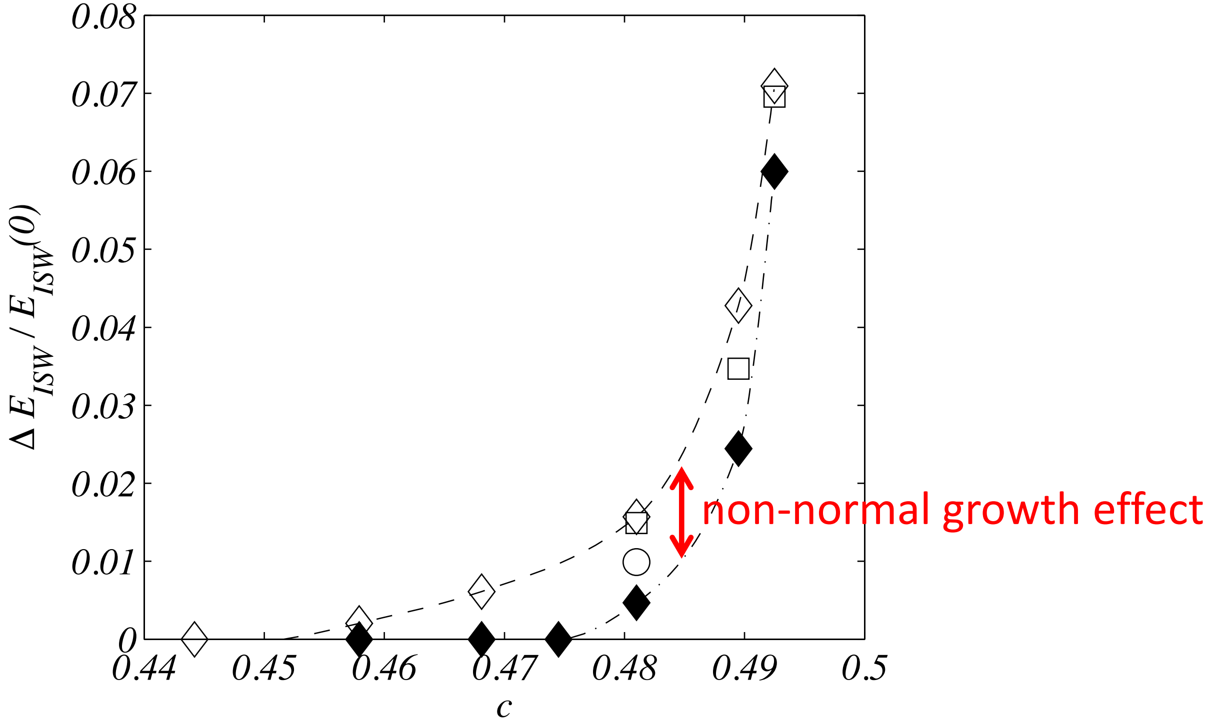
$$\frac{E_{ISW}(25) - E_{ISW}(0)}{E_{ISW}(0)} = \begin{cases} -0.005 & (flw) \\ -0.016 & (opt) \end{cases}$$

IAMR N-S numerical model: finite-volume projection method with adaptive refinement (Almgrem et al., 1998)

ISW energy loss after an encounter with ***one*** disturbance packet

$E_{dist} / E_{ISW} = 10^{-5}$

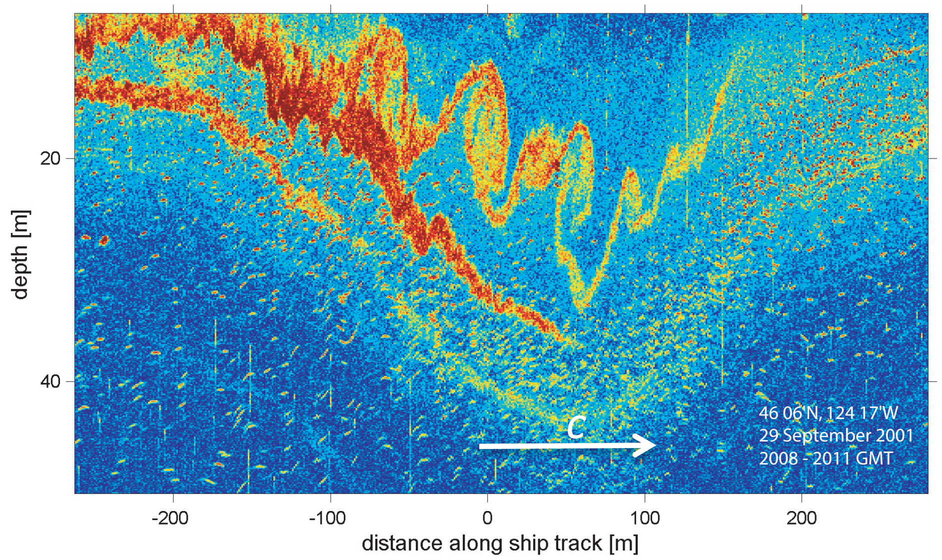
◇ TOG
◆ LWP



- substantially more energy loss for optimal disturbances
- TOG gives finite amplitude effects in smaller ISW (lower *c*)
- saturation in both cases for $E_{dist} / E_{ISW} \approx 10^{-5}$
- Basis for an adiabatic decay model

An oceanic example

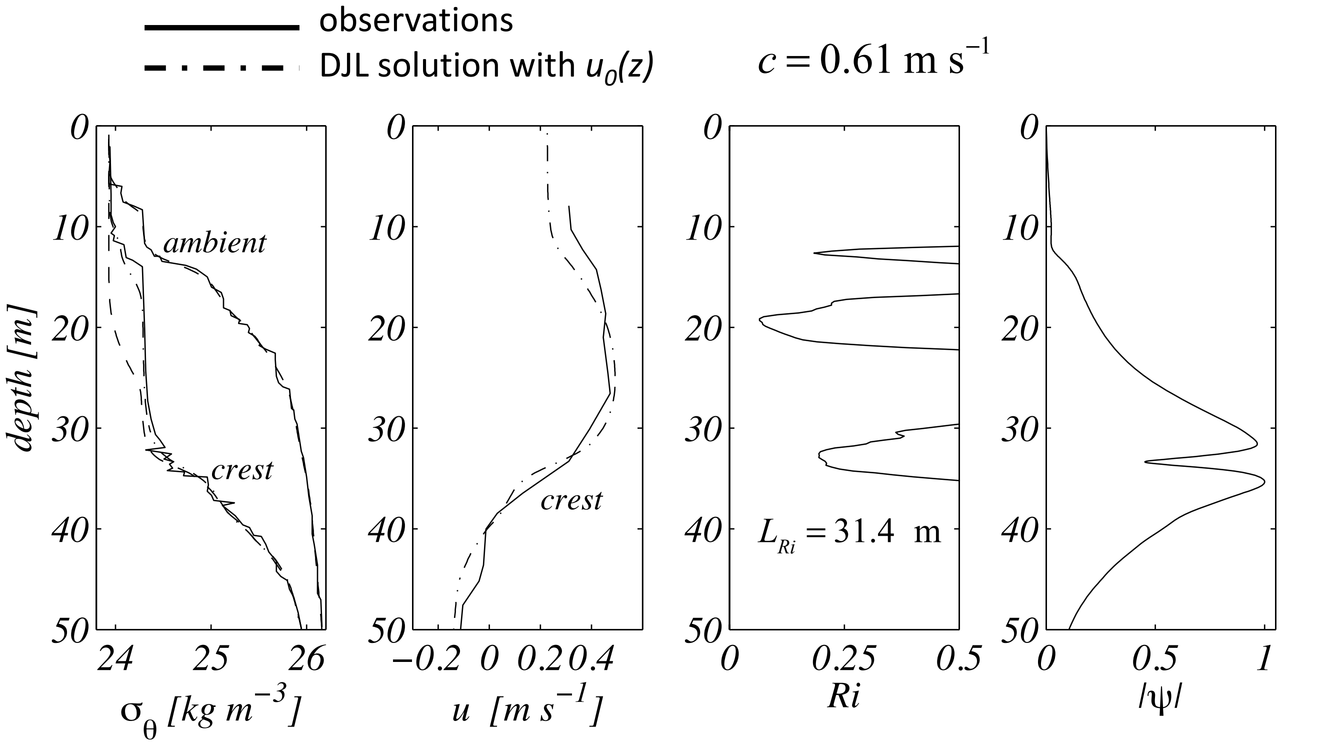
Acoustic image of K-H billows in a large ISW
(Moum et al., 2003)



Moum et al. (2003): K-H analysis found the wave was deeply unstable.

However, they used an adiabatic mapping between the upstream and *unstable* wave crest density profiles to estimate the wave crest velocity profile used in the T-G solution. This is problematic since the wave crest density field is itself affected by the instability.

Use ambient density field and DJL theory to obtain the ISW flow field



Spatial K-H analysis at the crest implies that the wave should be stable:

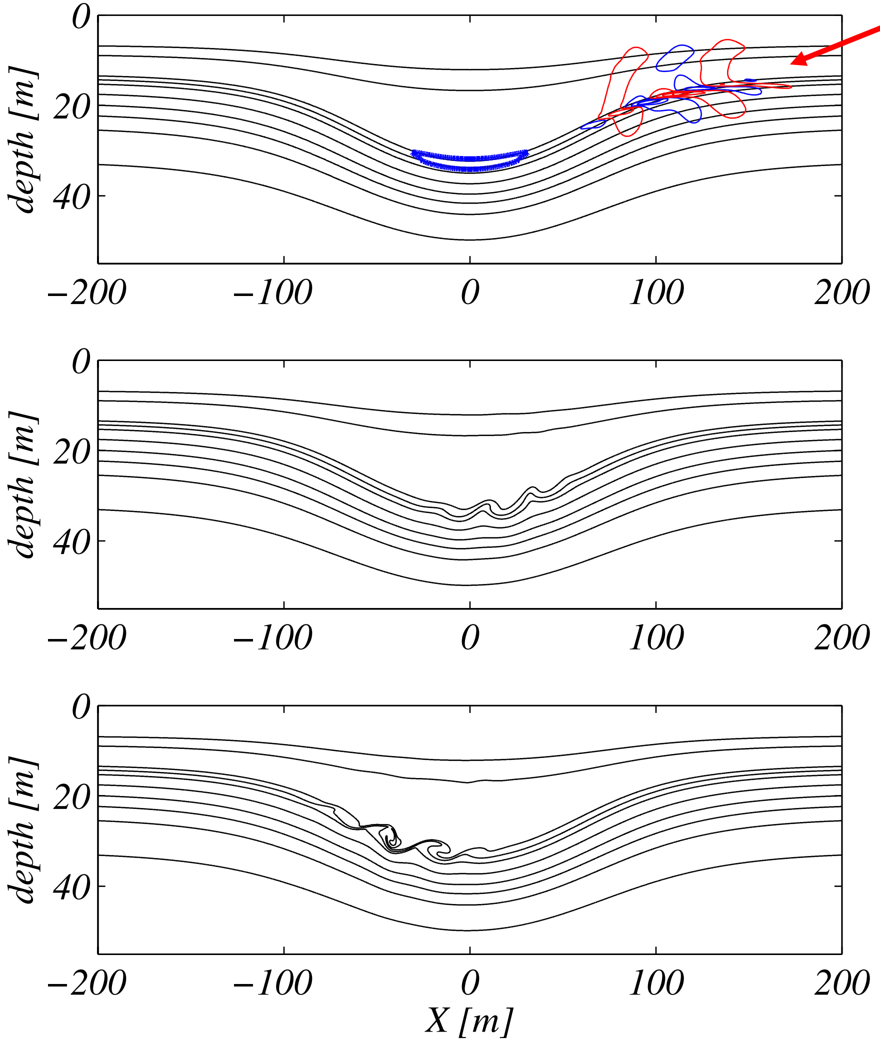
$$\left.\begin{aligned} \omega &= -6.53\times10^{-2}\ \text{s}^{-1} \\ k_R &= 0.185\ \text{m} \\ k_I &= 9.85\times10^{-3}\ \text{m}^{-1} \end{aligned}\right\} \rightarrow \underline{\ln(G) \le 4k_I L_{Ri} = 1.24}$$

nonlinear evolution of the optimal disturbance for $E_{opt}/E_{ISW} = 10^{-5}$

$t = 0$

$t = 3.96$ min

$t = 6.78$ min



optimal disturbance: $\ln(G_{max}) = 5.29$

$T_{max} = 6.78$ min

A packet of free linear waves (with the same total energy and carrier frequency) does not disturb the ISW

Summary

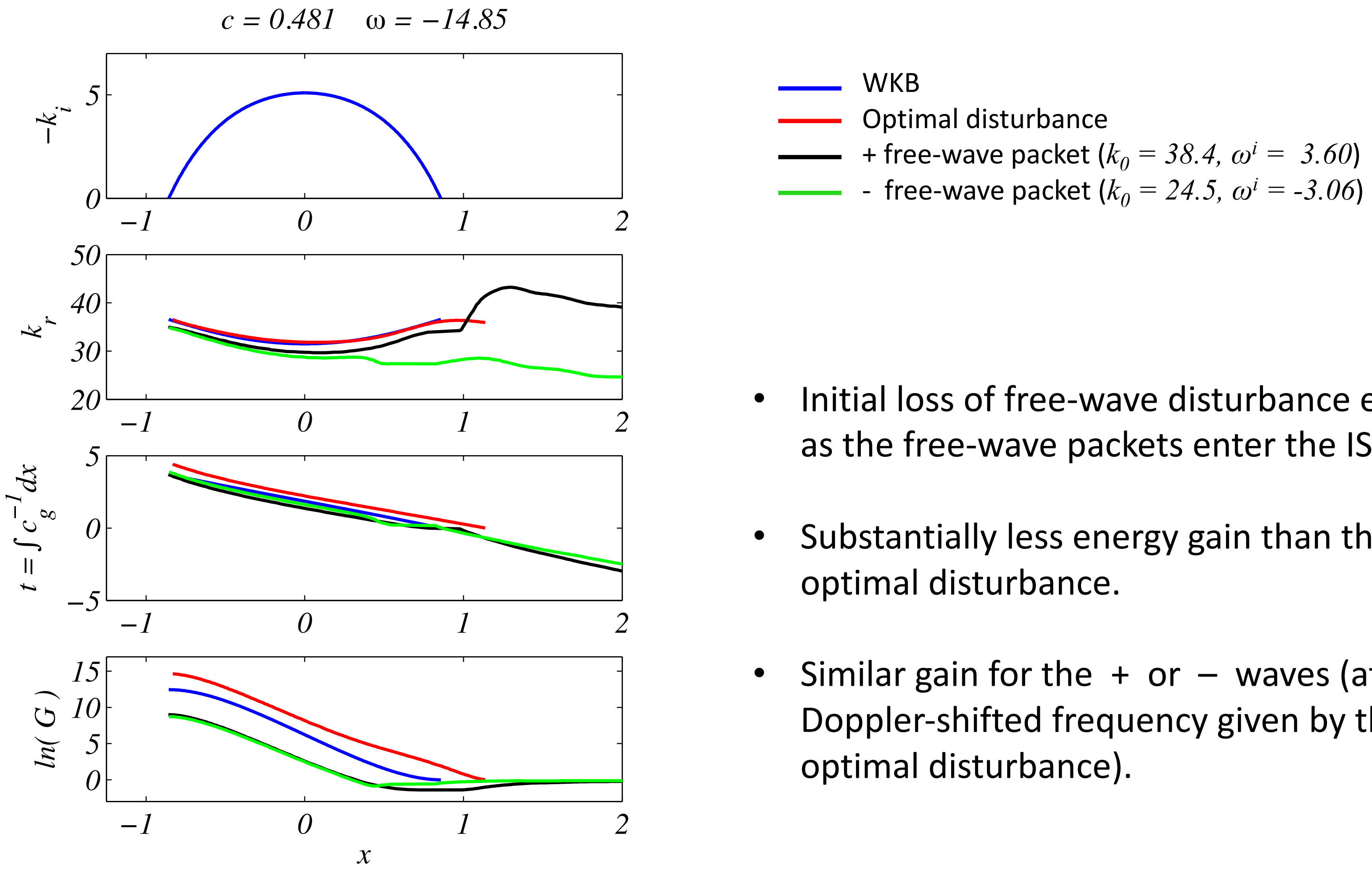
Thin interface waves (Passaggia, Helfrich and White, 2018 *JFM*):

- Optimal disturbances result in substantially more disturbance growth than free waves
- Spatial instability captured well by WKB approach with the addition of initial non-normal growth amplifying the response
- Saturation of growth in the nonlinear regime and production of finite-amplitude billows on upstream face of the ISW
- Observations of Moum et al. (2003) imply the importance of non-normal effects

Some questions (currently under investigation):

- How well does random noise (e.g. background turbulence) project on the optimal disturbances?
- Does they results extend to other more general background stratification and ambient shear?
- Waves with trapped cores appear to be susceptible to both transient disturbances and self-sustained (temporal) instabilities

Optimal disturbance, WKB, and free-wave packets



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