Martingale and mild solutions to stochastic Korteweg - de Vries - type equations

Anna Karczewska

Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra, Poland

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Deterministic hybrid Korteweg - de Vries - Burgers equation (hKdVB for short)

$$u_\tau + Auu_\xi + Bu_{3\xi} = Cu_{2\xi} - Du. \qquad (1)$$

Stretched coordinates $\xi = \epsilon^{\frac{1}{2}}(x - vt)$, $t = \epsilon^{\frac{3}{2}} t$, c - phase velocity of the wave.

$u(\tau, \xi)$ – (e.g.) electrostatic potential or electric pulse.

Indexes denote partial derivatives.

Constants A, B, C, D are related to parameters describing properties of plasma.

Misra, Adhikary, Shukla, Phys. Rev. E **86**, 056406 (2012)

Elkamash, Kourakis, Physics of Plasmas, **25**, 062104 (2018)

Deterministic hKdVB equation – particular cases

Particular cases of equation (1):

- the Korteweg - de Vries equation, when $C = D = 0$;
- the damped (dissipative) KdV equation, when $C = 0$;
- the Burgers equation, when $B = D = 0$;
- the KdV-Burgers equation, when $D = 0$;
- the damped Burgers equation, when $B = 0$.

The term with $A \neq 0$ introduces nonlinearity, that with $B \neq 0$ is responsible for dispersion, $C \neq 0$ supplies diffusive term and $D \neq 0$ introduces damping.

L. Ostrovsky, *Asymptotic Perturbation Theory of Waves*, Imperial College Press (2015)

Stochastic hKdVB equation

The presence of stochastic noise has physical grounds:

- waves in plasma – thermal fluctuations;
- water surface waves – air pressure fluctuations due to the wind.

$$\begin{cases} du(t,x) & + \; \big[Au(t,x)u_x(t,x) + Bu_{3x}(t,x) - Cu_{2x}(t,x) + Du(t,x)\big]\,dt \\ & = \; \Phi\left(u(t,x)\right) dW(t) \\ u(0,x) & = \; u_0(x), \quad t \geq 0, \quad x \in X, \quad X = \mathbb{R} \quad \text{or} \quad X = [x_1, x_2] \end{cases} \tag{2}$$

$W(t)$, $t \geq 0$ – a cylindrical Wiener process,

$u_0 \in L^2(X)$ – a deterministic real-valued function.

Stochastic hKdVB equation

There are two cases:

1. $\Phi(u) = \Phi$, independent of u – $\boxed{\text{additive}}$ case;
2. $\Phi(u)$ – dependent on u – $\boxed{\text{multiplicative}}$ case.

Two kinds of solutions:

1. $\boxed{\text{mild solution}}$;
2. $\boxed{\text{martingale solution}}$, respectively.

Definition of mild solution

Linear operator in (2):

$$\mathcal{A}u := -Bu_{3x} + Cu_{2x} - Du.$$

We can solve equation (2) in mild formulation

$$u(t) = S(t)\,u_0 - \int_0^t S(t-s)A\left(u(s)u_x(s)\right)ds + \int_0^t S(t-s)\Phi dW(s). \tag{3}$$

$S(t)\,u_0$ – the solution to the linear hybrid KdVB equation.

Definition

If u satisfies (3) we say that it is a **mild solution** to (2).

For particular cases of the operator $\mathcal{A}$, the family $\{S(t)\}_{t\geq 0}$ is different.

Definition of martingale solution

Definition

We say that the problem (2) has a **martingale solution** on the interval $[0, T]$, $0 < T < \infty$, if there exists a stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P}, \{W_t\}_{t\geq 0})$, where $\{W_t\}_{t\geq 0}$ is a cylindrical Wiener process, and there exists the process $\{u(t,x)\}_{t\geq 0}$ adapted to the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ with trajectories in the space

$$L^\infty(0,T;L^2(X)) \cap L^2(0,T;L^2(X)) \cap C(0,T;H^s(X)), \quad s<0, \quad \mathbb{P}-a.s.,$$

such that

$$\langle u(t,x);v(x)\rangle + \int_0^t \langle Au(t,x)u_x(t,x) + Bu_{3x}(t,x) - Cu_{2x}(t,x) + Du(t,x); v(x)\rangle ds$$
$$= \langle u_0(x); v(x)\rangle + \left\langle \int_0^t \Phi(u(s,x))dW(s); v(x)\right\rangle, \quad \langle \cdot\,,\cdot\rangle = \langle \cdot\,,\cdot\rangle_{L^2(X)},$$

for all $t \in [0,T]$ and $v \in H^1(X)$.

Stochastic solutions

MILD SOLUTION

Methods from the analysis of deterministic PDEs are adapted.

Equations are solved by a fixed point argument on the corresponding integral equation in the particular space.

The problem is to find a "good" function space in which the fixed point argument will be performed.

MARTINGALE SOLUTION

The Galerkin approximation of the problem is constructed and then the compactness methods are used. (Sometimes auxiliary problems are considered.)

Stochastic Korteweg-de Vries equation

Additive case: cylindrical Wiener process

$$A = B = 1, \qquad C = D = 0$$

$\mathcal{A}\,u := -u_{3x}; \qquad S(t)_{t\in\mathbb{R}\ (t\geq 0)}$ – unitary group

$$u(t) = S(t)u_0 = \mathcal{F}^{-1}\left(e^{it\xi^2}\mathcal{F}u_0\right)$$

$$X_\sigma(T) = \{u \in L^\infty(0,T; H^\sigma(\mathbb{R})) \cap L^2(\mathbb{R}; L^\infty([0,T])), \\ D^\sigma \partial_x u \in L^\infty(\mathbb{R}; L^2([0,T])),\ \partial_x u \in L^4([0,T]; L^\infty(\mathbb{R}))\},\ 0 < T < \infty$$

MILD SOLUTION

A. de Bouard and A. Debussche – (1998) on $X_\sigma(T)$

Stochastic Korteweg-de Vries equation

Multiplicative case: cylindrical Wiener process

A. de Bouard and A. Debussche – (1998)

J. Printems (1999)

Stochastic Korteweg-de Vries equation

Multiplicative case: Lévy noise

$$\begin{cases} du(t,x) & + & \big(u_{3x}(t,x) + u(t,x)u_x(t,x)\big)\, dt \\ & = & \int_Y F(t,u(t,x);y)\tilde{\eta}(dt,dy) + \Phi\,(t,u(t,x))\, dW(t) \\ u(0,x) & = & u_0(x) \end{cases} \tag{4}$$

A measurable function $F : [0,T] \times H \times Y \to H$; H, Y – function spaces.

F – fulfils conditions which guarantee the sense of integral.

Measure $\tilde{\eta}$ on $(Y, \mathcal{Y})$ – a time homogeneous Poisson random measure.

A. Karczewska and M. Szczeciński (2018 L)

Stochastic Burgers equation

Additive case: cylindrical Wiener process

$A = -1, \quad B = D = 0, \quad C = 1$

$\mathcal{A}\, u := -u_{2x}; \qquad S(t)_{t\geq 0} = e^{tA}$ – semigroup on $L^2(0,1)$

Local mild solution on

$$\Sigma_p(m, T^*) = \left\{ v \in C([0,T^*]); L^p(0,1) : |v(t)|_{L^p(0,1)} \leq m, \forall\, t \in [0,T^*] \right\},$$

for $p > 1$, and for some $T^* > 0$.

Global mild solution on

$$C([0,T]; L^p(0,1)) \quad \text{for some} \quad p \geq 2.$$

G. Da Prato, A. Debussche and R. Temam (1994)

Stochastic Burgers equation

Multiplicative case: cylindrical Wiener process $\Longrightarrow$ next screens.

Stochastic hybrid Korteweg-de Vries-Burgers equation

Additive case $\Longrightarrow$ in preparation for KdV-B, where $D = 0$.

Stochastic hybrid Korteweg-de Vries-Burgers equation

Multiplicative case

Condition

$$B, C, D \geq 0 \qquad \text{and} \qquad 3B \geq A + 1. \tag{5}$$

Condition (5) admits a broad class of physically meaningful equations which contain all particular cases listed previously.

MARTINGALE SOLUTION

A. Karczewska and M. Szczeciński (2018 H)

Resumé

	KdV	Burgers	KdV-Burgers	KdV-B hybrid
ADD mild	unitary group de Bouard & Debussche 1998	semigroup Da Prato, Debussche, & Temam 1994	semigroup Karczewska 2018 in preparation	?
MULT martingale solution	de Bouard & Debussche 1998 ——— + Lévy noise Karczewska & Szczeciński 2018 L	Ka & Sz 2018 H	Ka & Sz 2018 H	Ka & Sz 2018 H

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