TEL AVIV UNIVERSITY אוניברסיטת תל-אביב

Multidimensional solitons in optics and ultracold gases: Predictions and creation

Boris A. Malomed

Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering

and Center for Light-Matter Interaction

Tel Aviv University, Tel Aviv, Israel

also affiliated with **ITMO University, St. Petersburg, Russia**

and **School of Physics and Optoelectronic Engineering, Foshan University, Guangzhou, China**

Joint publications with Roger Grimshaw **(as per Web of Science, cited 175 times)**

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3. R. Grimshaw and B. A. Malomed. A new type of gap soliton in a coupled KdV-wave system. Phys. Rev. Lett. 72, 949-953 (1994).

4. R. Grimshaw, B. A. Malomed, and E. S. Benilov. Solitary waves with damped oscillatory tails: an analysis of the fifth-order Korteweg - de Vries equation. Physica D **77**, 473-485(1994).

5. R. Grimshaw and B. A. Malomed. Nonexistence of gap solitons in nonlinearly coupled systems. Phys. Lett. A **198**, 205-208 (1995).

6. R. Grimshaw, B. A.. Malomed, and Xin Tian. Gap-soliton hunt in a coupled Korteweg - de Vries system. Phys. Lett. A **201**, 285-292 (1995).

7. B. A. Malomed, R. Grimshaw, and X. Tian. Gap solitons in a coupled Korteweg - de Vries system. In: Structure and Dynamics of Nonlinear Waves in Fluids, ed. by A. Mielke and K. Kirchgaessner (World Scientific, Singapore, 1995), p. 324-334.

8. R. Grimshaw, J. He, and B. A. Malomed. Decay of a soliton in a periodically modulated nonlinear waveguide. Phys. Scripta **53**, 385-393 (1996).

9. **G. Gottwald**, R. Grimshaw, and B. Malomed. Parametric envelope solitons in coupled Korteweg -de Vries equations. Phys. Lett. A **227**, 47-54 (1997).

10. R. Grimshaw, J. He, and B. A. Malomed. Nonlinear analysis of instabilities produced by linear mode coupling. Physica D **113**, 26-42 (1998).

11. **G. Gottwald**, R. Grimshaw and B. A. Malomed. Stable two-dimensioı parametric solitons in fluid systems. Phys. Lett. A **248**, 208-218 (1998).

12. R. Grimshaw, J. He, and B. A. Malomed. Dissipative effects in a nonlinear wave system with an unstable linear spectrum. Physica D **132**, 63-86 (1999).

13. **S. R. Clarke**, J. Clutterbuck, R. H. J. Grimshaw, and B. A. Malomed. Passage of a wave pulse through a zero-dispersion point the nonlinear Schrödinger equation. Phys. Lett. A **262**, 434-444 (1999).

14. R. Grimshaw, **S. Clarke**, and B. A. Malomed. Soliton formation from a pulse passing the zero-dispersion point in a nonlinear Schrödinger equation. Phys. Rev. E **61**, 5794-5801 (2000).

15. **S. Clarke**, B. A. Malomed, and R. Grimshaw. " Dispersion management" for solitons in a Korteweg-de Vries system. Chaos **12**, 8-15 (2002).

16. R. Grimshaw, B. A. Malomed, and **G. A. Gottwald**. Cuspons, peakons, and regular gap solitons between three dispersion curves. Phys. Rev. E **65**, 066606 (2002).

17. R. Grimshaw, **G. A. Gottwald**, and B. A. Malomed. Cuspons and peakons vis-a-vis regular solitons and collapse in a three-wave system. Contempr. Math. **301**, 249-271 (2002).

In addition, joint publications with other participants of this Conference:

B. A. Malomed and **V. I. Shrira**. Solitons caustics. Physica D **53**, 1-12 (1991).

B. A. Malomed and **Yu. A. Stepanyants**. The inverse problem for the Gross-Pitaevskii equation. Chaos **20**, 013130 (2010).

(1) Introduction

Formation of ***localized structures*** in the form of ***solitons*** (***solitary waves***) in many physical settings is accounted for by the interplay of nonlinear ***self-attraction*** (alias ***self-focusing***) of physical fields (e.g., electromagnetic waves in photonics, or macroscopic wave functions, alias ***matter waves***, in ***Bose-Einstein condensates***, **BECs**) and basic linear effects, such as ***dispersion*** or ***diffraction*** (in **BEC**, the dispersion corresponds to the quantum-mechanical kinetic energy). The concept of solitons was introduced by Zabusky and Kruskal in 1965, in the context of the Korteweg – de Vries equation.

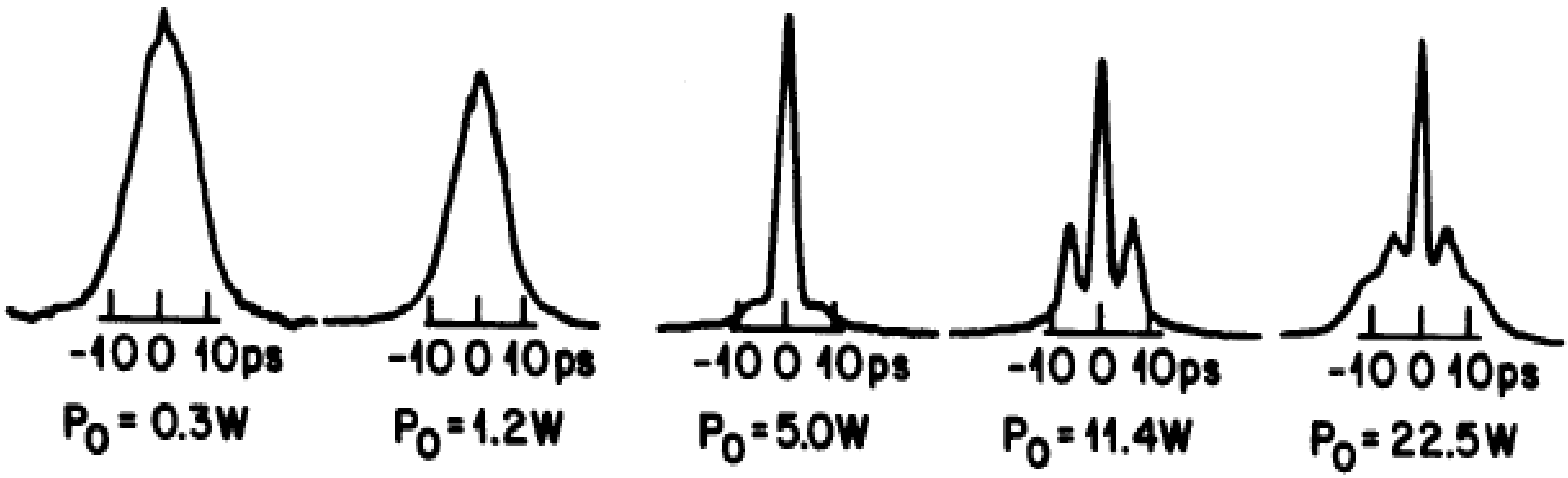
Arguably, the most important model which creates solitons is the ***nonlinear Schrödinger* (NLS) *equation:***

$$i\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2} - |\Psi|^2\,\Psi.$$

It generates a family of bright-soliton solutions with two free parameters - amplitude $\boldsymbol{\eta}$ and velocity $\boldsymbol{c}$:

$$\Psi(x,t) = e^{icx + i\left(\eta^2 - c^2\right)t/2}\,\frac{\eta}{\cosh(\eta(x-ct))}\,.$$

Such solitons in ***nonlinear optical fibers*** were predicted by Hasegawa and Tappert in 1973, and experimentally created by Mollenauer, Stolen and Gordon in 1980.
They observed self-trapping of an input pulse into a ***fundamental*** or ***higher-order*** soliton (***breather***) with the increase of the input's peak power:



Standard telecommunications fibers can carry ***soliton streams***, which may be used to transmit data in fiber-optical telecom networks. The bit-rate of up to **100 GB/s per channel** can be easily achieved, using currently available technologies. The **single** so far built soliton-based ***commercial*** telecom link, about **3,000** km long, was installed in Australia (between Adelaide and Perth) in **2003**. Later, interests of telecom developers have switched to other (non-soliton) technologies.

Another famous realization of solitons was demonstrated in **BEC** loaded in nearly one-dimensional ("cigar-shaped") trapping potentials. Such ***bright matter-wave solitons*** were first created in the condensate of ^{7}Li atoms:

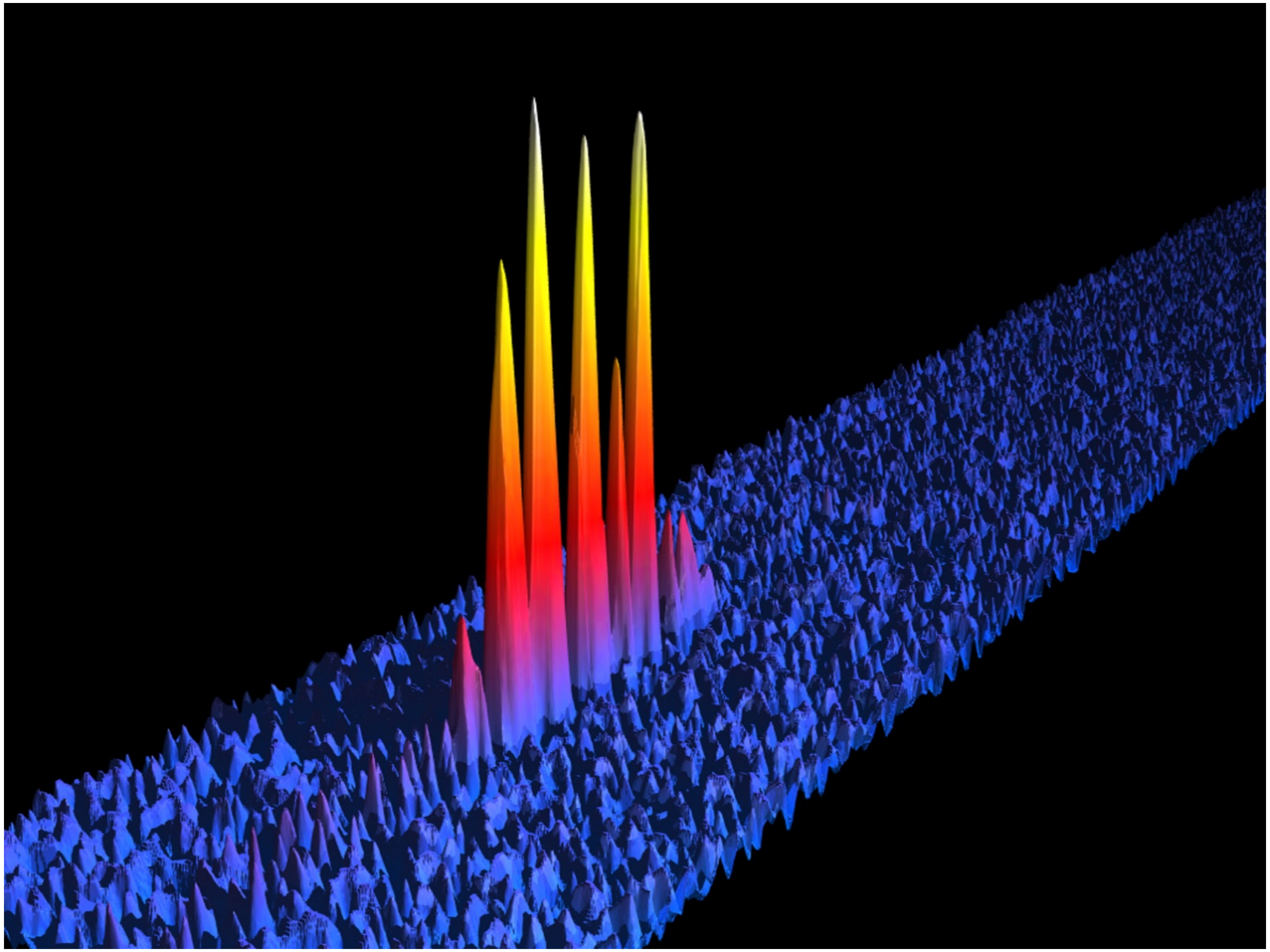
K.E. Strecker, G.B. Partridge, A.G. Truscott, and R. G. Hulet, Nature **417**, 150 (2002);

L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L.D. Carr, Y. Castin, and C. Salomon, Science **296**, 1290 (2002).

Then, bright solitons in a less anisotropic trap were created in ^{85}Rb:

S.L. Cornish, S.T. Thompson, and C.E. Wieman, Phys. Rev. Lett. **96**, 170401 (2006).

The famous experimental picture of the atomic density distribution in a chain of 7 matter-wave solitons with unequal amplitudes in ^{7}Li (from the work of R. Hulet *et al.*):



A natural extension of the one-dimensional (**1D**) **NLS** equation (a.k.a. the **Gross-Pitaevskii** (**GP**) equation, in the application to matter waves) is its **2D** version, which also finds important realizations in **nonlinear optics**:

$$i\frac{\partial u}{\partial t}=-\frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-|u|^2 u+W(x,y)u,$$

The equation may include the *trapping potential*, $W(x,y)$. In the case of axial symmetry, with $W=W(r)$, where (r,θ) are the polar coordinates in the (x,y) plane, **2D** soliton solutions, which may carry integer **vorticity** S, are looked for as

$$u=\exp\left(-i\mu t+iS\theta\right)U(r;\mu),$$

with $\mu<0$ and real function $U(r)$ generated by the stationary equation:

$$\mu U=-\frac{1}{2}\left(\frac{d^2U}{dr^2}+\frac{1}{r}\frac{dU}{dr}-\frac{S^2}{r^2}U\right)-U^3+W(r)U.$$

The objective of the talk is to present an overview of fundamental models which give rise to **self-trapped modes** in the form of **stable** two-and three-dimensional (**2D** and **3D**) solitons (including ones with ***intrinsic vorticity***), and physical realizations of the models. The realizations of multidimensional solitons are chiefly provided, like in the **1D** case, by **nonlinear optics** and matter waves in **BEC**.

An old (published in 2005) but still relevant review of the topic of multidimensional solitons (with a ***recent invited update***, published in 2016):

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF OPTICS B: QUANTUM AND SEMICLASSICAL OPTICS

J. Opt. B: Quantum Semiclass. Opt. **7** (2005) R53–R72 doi:10.1088/1464-4266/7/5/R02

REVIEW ARTICLE

Spatiotemporal optical solitons

Boris A Malomed[1], Dumitru Mihalache[2], Frank Wise[3] and Lluis Torner[4]

IOP Publishing Journal of Physics B: Atomic, Molecular and Optical Physics

J. Phys. B: At. Mol. Opt. Phys. **00** (2016) 000000 (3pp)

Viewpoint

CrossMark

Viewpoint on article ‘Spatiotemporal optical solitons’ by B A Malomed, D Mihalache, F Wise, and L Torner, 2006 *J. Opt. B: Quantum Semiclass. Opt.* 7 R53–R72

A more recent short review of the topic:

Eur. Phys. J. Special Topics **225**, 2507–2532 (2016)

DOI: 10.1140/epjst/e2016-60025-y

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Review

Multidimensional solitons: Well-established results and novel findings

Boris A. Malomed[1,2,a]

[1] Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel
[2] Laboratory of Nonlinear-Optical Informatics, ITMO University, St. Petersburg 197101, Russia

A brand-new review, submitted to ***Nature Physics Reviews***, is focused on novel theoretical and experimental findings:

New Frontiers in Multidimensional Self-Trapping of Nonlinear Fields and Matter

Yaroslav V. Kartashov,

Gregory E. Astrakharchik,

Boris A. Malomed,

and Lluis Torner

As the **stability** is the main issue for **2D** and **3D** settings (***unlike*** **1D** models, where solitons are generically stable), the rest of the talk is structured according to basic mechanisms which provide for **stabilization** of the multidimensional solitons.

The subsequent presentation is organized as follows:

(2) A brief review of solitons (fundamental and vortical ones) created by the **cubic** nonlinearity in the **2D** space, which are ***unstable***.

(3) Relatively old material: **Stable** **2D** and **3D vortex solitons** in models with the **cubic-quintic** (**CQ**) nonlinearity.

(4) A new model and novel results: Stable **2D** and **3D** composite solitons in **two-component spin-orbit (SO)-coupled BEC**.

(5) **Newest** theoretical and experimental results: the prediction and actual creation of **3D** and **2D** matter-wave solitons ("***quantum droplets***") stabilized by **quantum fluctuations** (represented by the so-called **Lee-Huang-Yang correction** to the **GP** equations).

(6) Conclusions.

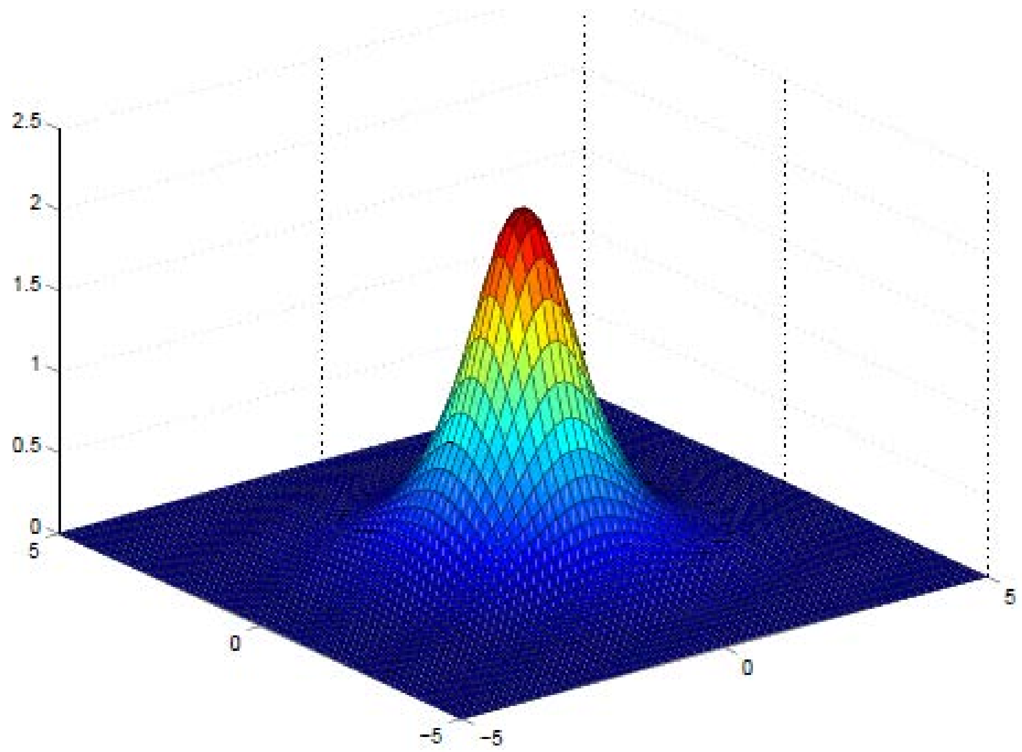
(2) 2D solitons produced by the NLS equation with the cubic nonlinearity,

$$iu_t + (1/2)(u_{xx} + u_{yy}) + |u|^2 u = 0.$$

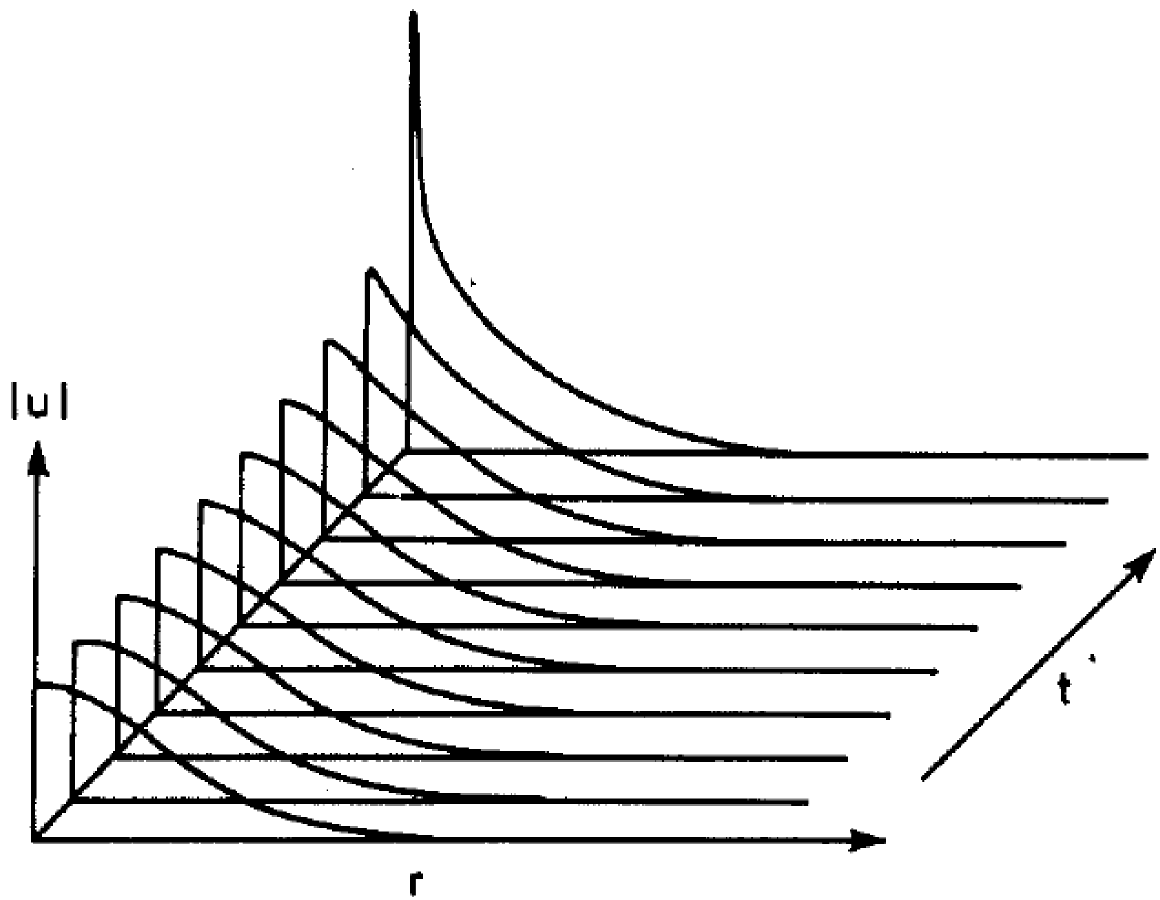
Recall that **2D** soliton solutions with **vorticity *S*** and chemical potential μ are looked for as

$$u = \exp(-i\mu t + iS\theta)\, U(r).$$

F**undamental (zero-vorticity, *S* = 0) solitons** [alias ***Townes' solitons***, R.Y. Chiao, E. Garmire & C. H. Townes, Phys. Rev. Lett . **13**, 479 (1964)] are ***unstable*** against the ***collapse*** (i.e., catastrophic self-compression of the wave function, which leads to formation of a singularity after a finite evolution time).



The numerically simulated ***collapse process*** (the evolution of the radial cross section of the collapsing ***Townes' soliton*** is displayed here):



The generalization in the form of ***Townes' solitons*** with **embedded vorticity**, alias the ***topological charge***, $\boldsymbol{S \geq 1}$, was introduced later:

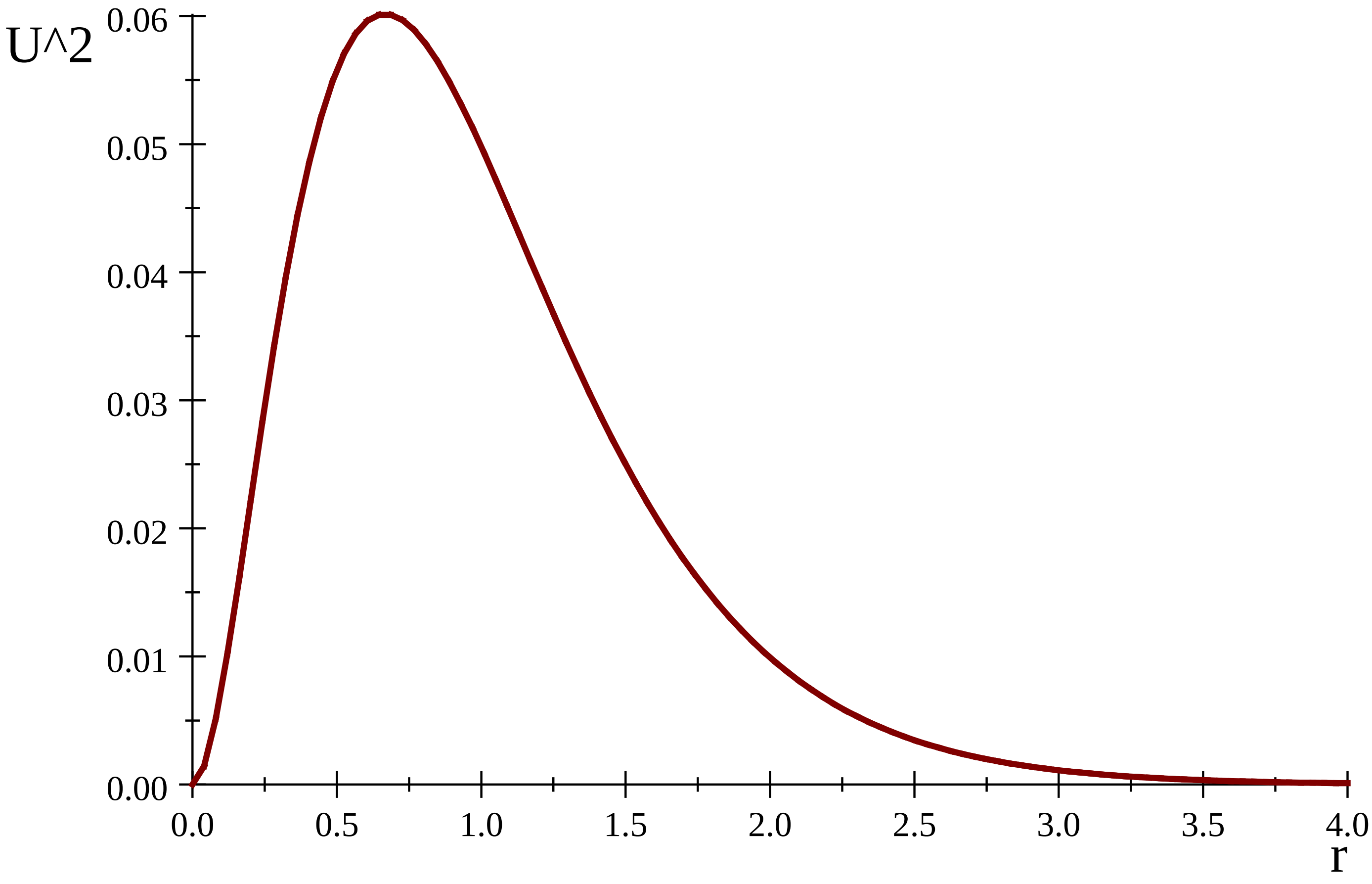
JOURNAL OF MODERN OPTICS, 1992, VOL. 39, NO. 11, 2277–2291

The theory of spiral laser beams in nonlinear media

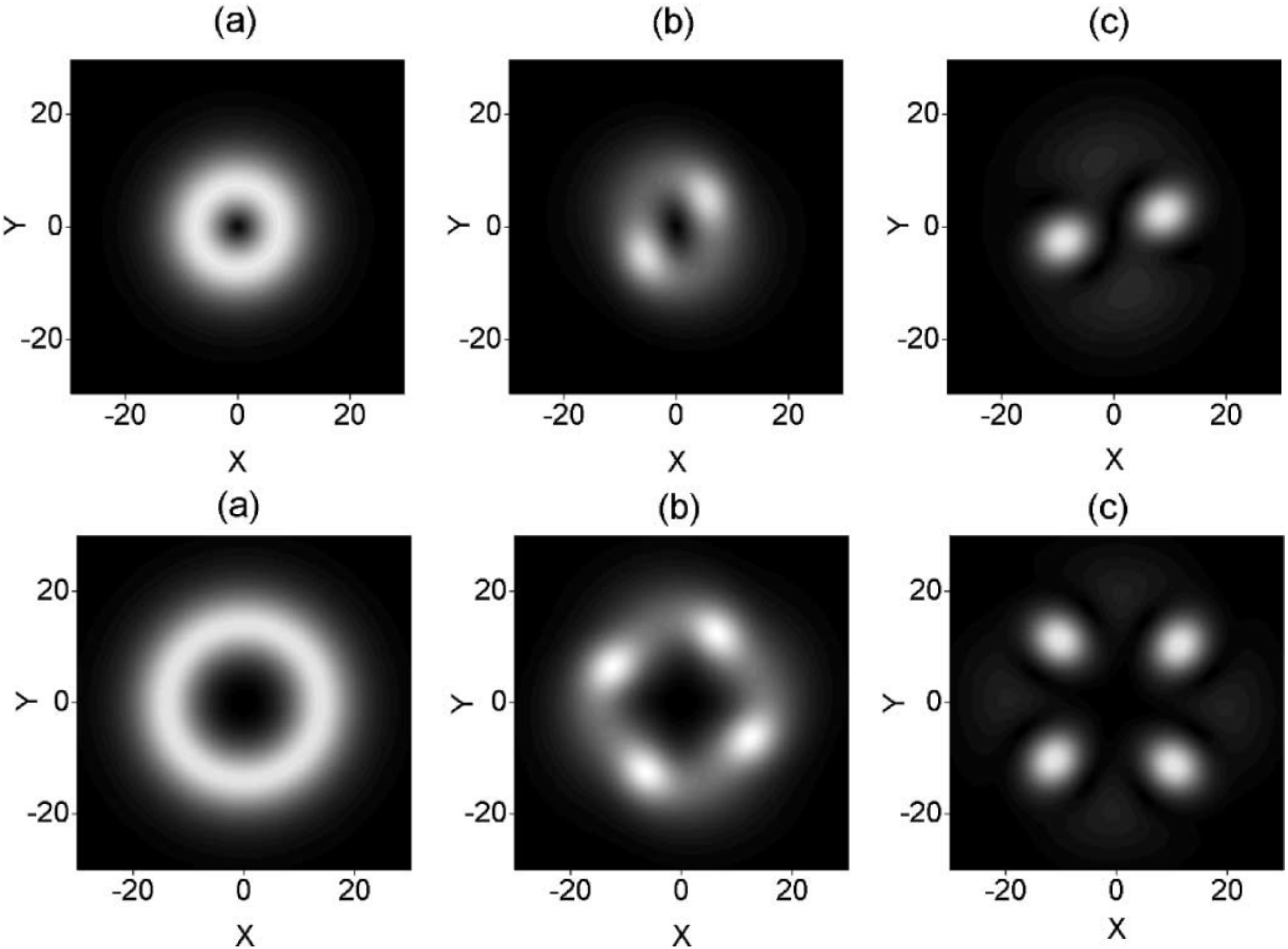
V. I. KRUGLOV, YU. A. LOGVIN
Institute of Physics, Byelorussian Academy of Sciences, 220602 Minsk, Republic of Belarus

and V. M. VOLKOV
Institute of Mathematics, Byelorussian Academy of Sciences, 220602 Minsk, Republic of Belarus

A schematic radial cross section of the ***vortex soliton*** with ***S* = 1**. They are subject to a strong ***azimuthal instability***, which splits them into a set of ***fundamental solitons***, that are later destroyed by the ***intrinsic collapse***.



Examples of the ***spontaneous splitting*** of unstable vortex solitons with ***S* = 1** and ***S* = 2**:



The families of the Townes' solitons and their vortex counterparts are ***degenerate***: due to the specific ***conformal symmetry*** of the **2D NLS** equation with the cubic nonlinearity, the norm of each family (with ***S* = 0,1,2**, *etc*.) takes a ***single value***, which ***does not*** depend on the soliton's ***chemical potential***, ***μ***:

$$N_S = 2\pi\int_0^\infty U^2(r;\mu)rdr.$$

For $S=0$ (the fundamental Townes' solitons),

$N_0 \approx \mathbf{5.85}$ [an analytical variational approximation,

M. Desaix, D. Anderson, and M. Lisak,

J. Opt. Soc. Am. B **8**, 2082 (1991), predicts $N_0 = \mathbf{2\pi}$].

This degenerate value of the norm of the ***Townes' soliton*** separates ***collapsing*** ($N > N_0$) and ***decaying*** ($N < N_0$) localized solutions of the **2D NLS** equation. As any ***separatrix*** solution, the Townes soliton is ***unstable*** against small perturbations.
Thus, the ***critical collapse*** sets in above a final ***threshold value*** of the norm, $N_{\mathrm{thr}} \equiv N_0$.
Any ***stabilizing mechanism***, added to the simplest **2D NLS** equation, acts by letting the norm of **2D** solitons take values ***below*** the ***threshold value***, hence they ***cannot*** undergo the ***collapse***.
However, in the **3D** case, the collapse is ***supercritical***, with ***zero threshold***.

(3) 2D and 3D Systems with the cubic-quintic nonlinearity

The stabilization of **2D** and **3D** fundamental and vortical solitons can be provided by a combination of ***competing self-focusing cubic*** and ***self-defocusing quintic*** nonlinear terms.

In optics, the **3D NLS** equation can be written as the equation governing the ***spatiotemporal evolution*** of the electromagnetic field:

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+|u|^2 u-|u|^4 u=0.$$

Stationary solutions with integer vorticity, $S \geq 0$, are looked for in the cylindrical coordinates as

$$u(x,y,z,t)=U(r,z)\exp(-i\mu t+iS\theta),\ \ r\equiv\sqrt{x^2+y^2},$$

with $U(r,z)$ satisfying the radial equation:

$$\mu R+\left(\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}-\frac{S^2}{r^2}U+\frac{\partial^2 U}{\partial z^2}\right)+U^3-U^5=0.$$

The stability of ***fundamental solitons*** (***S* = 0**) in the framework of this equation is obvious. A nontrivial problem is the ***stability of vortex solitons*** against splitting by azimuthal perturbations.

For **2D** ***vortex solitons***, this possibility was first reported in

J. Opt. Soc. Am. B/Vol. 14, No. 8/August 1997

Stable azimuthal stationary state in quintic nonlinear optical media

M. Quiroga-Teixeiro

Institute for Electromagnetic Field Theory, Chalmers University of Technology, S-412 96, Göteborg, Sweden

H. Michinel

Departamento de Física Aplicada, Escola Universitaria de Óptica, Universidade de Santiago de Compostela, E-157 06, Santiago de Compostela, Galicia, Spain

Accurate results for the same **2D** problem have been reported in the following paper:

J. Nonlinear Sci. Vol. 12: pp. 347–394 (2002)
DOI: 10.1007/s00332-002-0475-3

Journal of Nonlinear Science

Spectrally Stable Encapsulated Vortices for Nonlinear Schrödinger Equations

R. L. Pego[1] and H. A. Warchall[2,3]

Experimentally, the stability of **(2+1)D** ***fundamental*** (***S*** **= 0**) solitons in an optical ***cubic-quintic*** medium (actually, it is a **fluid**) was demonstrated relatively recently (stable ***vortex solitons*** supported by the cubic-quintic nonlinearity have not yet been created in the experiment):

PRL **110**, 013901 (2013) PHYSICAL REVIEW LETTERS week ending 4 JANUARY 2013

Robust Two-Dimensional Spatial Solitons in Liquid Carbon Disulfide

Edilson L. Falcão-Filho* and Cid B. de Araújo

Departamento de Física, Universidade Federal de Pernambuco, 50670-901 Recife, Pernambuco, Brazil

Georges Boudebs, Hervé Leblond, and Vladimir Skarka

LUNAM Université, Université d'Angers, Laboratoire de Photonique d'Angers, EA 4464, 49045 Angers, France

(Received 29 March 2012; published 2 January 2013)

The excitation of near-infrared (2 + 1)D solitons in liquid carbon disulfide is demonstrated due to the simultaneous contribution of the third- and fifth-order susceptibilities. Solitons propagating free from diffraction for more than 10 Rayleigh lengths although damped, were observed to support the proposed soliton behavior. Numerical calculations using a nonlinear Schrödinger-type equation were also performed.

The 3D setting with the cubic-quintic nonlinearity

A challenging problem is to construct **3D** vortex solitons in the cubic-quintic medium, and analyze their **stability**. Theoretically, this was done long ago in:

VOLUME 88, NUMBER 7 PHYSICAL REVIEW LETTERS 18 FEBRUARY 2002

Stable Spinning Optical Solitons in Three Dimensions

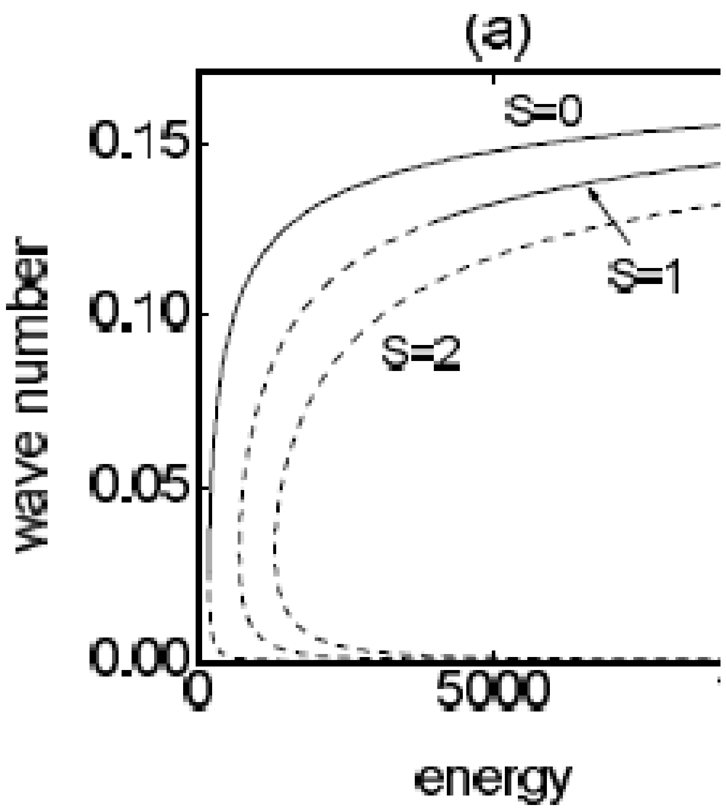
D. Mihalache,[1,2,5] D. Mazilu,[1,2] L.-C. Crasovan,[1,5] I. Towers,[3,4] A. V. Buryak,[3] B. A. Malomed,[4] L. Torner,[5] J. P. Torres,[5] and F. Lederer[2]

[1]*Department of Theoretical Physics, Institute of Atomic Physics, P.O. Box MG-6, Bucharest, Romania*
[2]*Institute of Solid State Theory and Theoretical Optics, Friedrich-Schiller University Jena, Max-Wien-Platz 1, D-07743, Jena, Germany*
[3]*School of Mathematics and Statistics, University of New South Wales at ADFA, Canberra, ACT 2600, Australia*
[4]*Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel*
[5]*Department of Signal Theory and Communications, Universitat Politecnica de Catalunya, ES 08034 Barcelona, Spain*

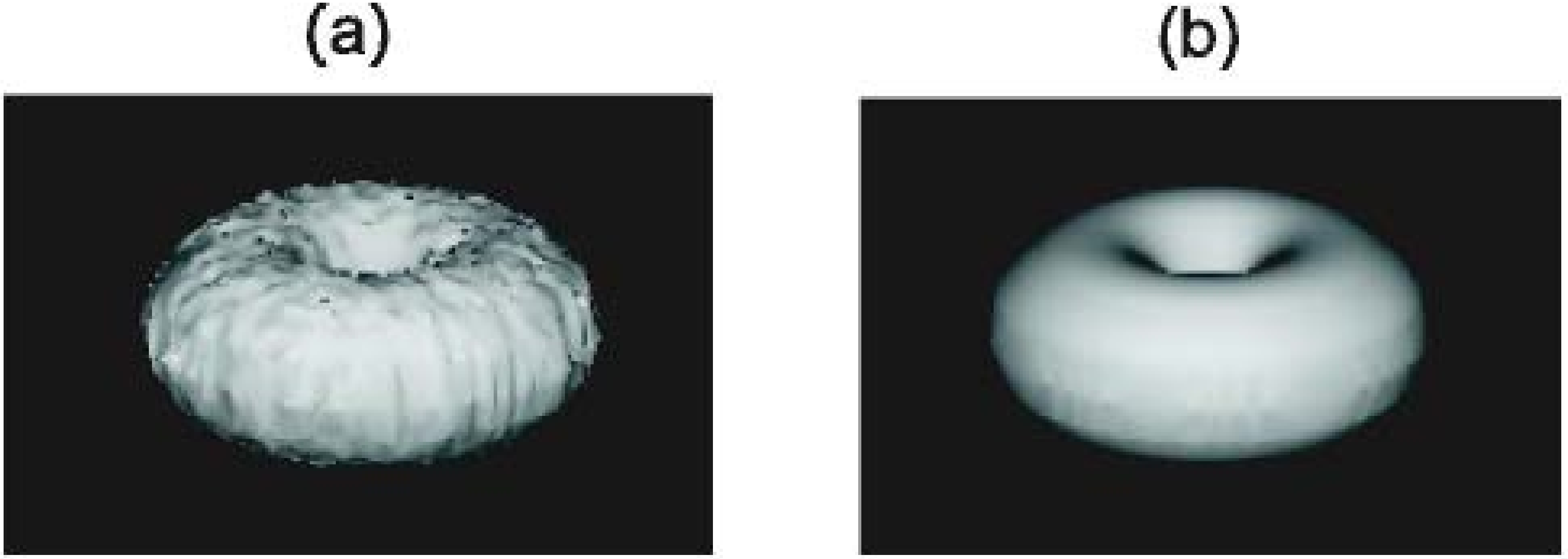
(Received 2 July 2001; published 4 February 2002)

The basic results for the stability are summarized by plots which show **wavenumber** ***-μ*** of the **3D** solitons vs. their total norm (energy, in terms of optics). Note that the ***Vakhitov-Kolokolov*** **(*necessary*)** ***stability criterion*** holds on ***top branches***, ***d*****(wave number)/*d*(energy) > 0**

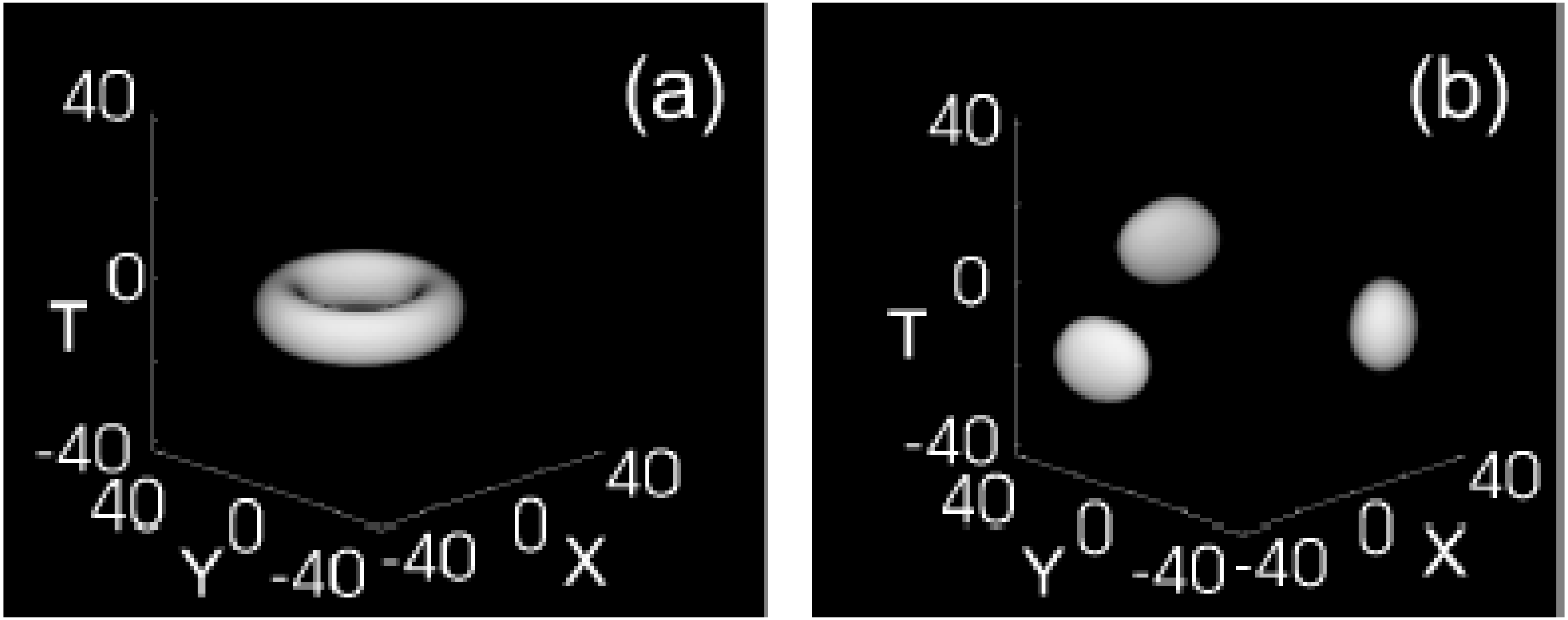
$$\left(E = \iiint U^2(x,y,t)\,dx\,dy\,dt\right):$$



An example of simulated ***recovery*** of a strongly perturbed ***stable*** soliton with intrinsic vorticity ***S* = 1** (***doughnut***):



An example of the simulated ***instability*** of the **3D doughnut soliton** with vorticity ***S* = 2**: ***splitting*** in ***three fragments***:



(4) Novel results: Stable two- and three-dimensional composite solitons in spin-orbit (SO)-coupled self-attractive BEC

Basic results are presented here for 2D solitons as per the following papers:

PHYSICAL REVIEW E **89**, 032920 (2014)

Creation of two-dimensional composite solitons in spin-orbit-coupled self-attractive Bose-Einstein condensates in free space

Hidetsugu Sakaguchi and Ben Li
Department of Applied Science for Electronics and Materials, Interdisciplinary Graduate School of Engineering Sciences, Kyushu University, Kasuga, Fukuoka 816-8580, Japan

Boris A. Malomed
Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel
(Received 12 December 2013; published 26 March 2014)

PHYSICAL REVIEW E **94**, 032202 (2016)

Vortex solitons in two-dimensional spin-orbit coupled Bose-Einstein condensates: Effects of the Rashba-Dresselhaus coupling and Zeeman splitting

Hidetsugu Sakaguchi
Department of Applied Science for Electronics and Materials, Interdisciplinary Graduate School of Engineering Sciences, Kyushu University, Kasuga, Fukuoka 816-8580, Japan

E. Ya. Sherman
Department of Physical Chemistry, University of the Basque Country UPV-EHU, 48940 Bilbao, Spain and IKERBASQUE, Basque Foundation for Science, Bilbao, Spain

Boris A. Malomed
Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel
(Received 18 February 2016; published 2 September 2016)

(4a) Introduction and objectives

The concept of ***emulation*** (alias ***simulation***) of complex physical effects, known in condensed-matter physics, by much simpler settings available in **BEC** (***matter waves***) and **photonics** (***optical waves***), has drawn a great deal of interest:

P. Hauke, F. M. Cucchietti, L. Tagliacozzo, I. Deutsch, and M. Lewenstein, Rep. Prog. Phys. **75**, 082401 (2012).

Lately, a new topic has emerged in the framework of this approach: the emulation of **spin-orbit** (**SO**) **interactions** in semiconductors, such as those accounted for by the ***Rashba*** and ***Dresselhaus*** Hamiltonians, by ***mapping*** the spinor wave function of electrons into the pseudo-spinor (two-component) wave function of a binary **BEC** gas:

Y. J. Lin, K. Jimenez-Garcia, and I. B. Spielman, Nature **471**, 83 (2011);

Y. Zhang, L. Mao, and C. Zhang, Phys. Rev. Lett. **108**, 035302 (2012);

A review: *H. Zhai*, Rep. Prog. Phys. **78,** 026001 (2015).

Here, our objective is to construct ***self-trapped*** (localized) ***stable*** **2D *vortical modes*** in the **SO-coupled BEC** with ***attractive*** nonlinearities.
At the first glance, this objective seems ***absolutely impossible***, as one may expect that such solitons are destabilized by the ***critical collapse***, similarly to the above-mentioned ***Townes' solitons*** of the **NLS** equation with the ***self-attractive*** cubic term.

(4b) The model

The system of **GP** equations for the two-component wave function **(*Φ*$_+$, *Φ*$_-$)** of the binary **BEC** coupled by the **SO** terms of the ***Rashba type*** with strength ***λ*** (it may be scaled to **1**), coefficient of the **SPM** self-attraction **≡ 1**, coefficient of the **XPM** inter-component attraction ***γ* ≥ 0** (the trapping potential ~ **Ω²** will be dropped, the aim being to construct ***stable solitons*** in ***free space***):

$$
\begin{aligned}
i\frac{\partial \phi_+}{\partial t} &= -\frac{1}{2}\nabla^2\phi_+ - (|\phi_+|^2 + \gamma|\phi_-|^2)\phi_+ \\
&\quad + \lambda\left(\frac{\partial \phi_-}{\partial x} - i\frac{\partial \phi_-}{\partial y}\right) + \frac{1}{2}\Omega^2(x^2+y^2)\phi_+, \\
i\frac{\partial \phi_-}{\partial t} &= -\frac{1}{2}\nabla^2\phi_- - (|\phi_-|^2 + \gamma|\phi_+|^2)\phi_- \\
&\quad - \lambda\left(\frac{\partial \phi_+}{\partial x} + i\frac{\partial \phi_+}{\partial y}\right) + \frac{1}{2}\Omega^2(x^2+y^2)\phi_-,
\end{aligned}
$$

(4c) Semi-vortex states

The coupled **GP** equations admit a family of solutions for ***semi-vortices***, with vorticities $\boldsymbol{m_+ = 0}$ in one component, and $\boldsymbol{m_- = 1}$ in the other. The ***exact ansatz*** for these solutions, compatible with the underlying equations ($\boldsymbol{\mu < 0}$ is the chemical potential):

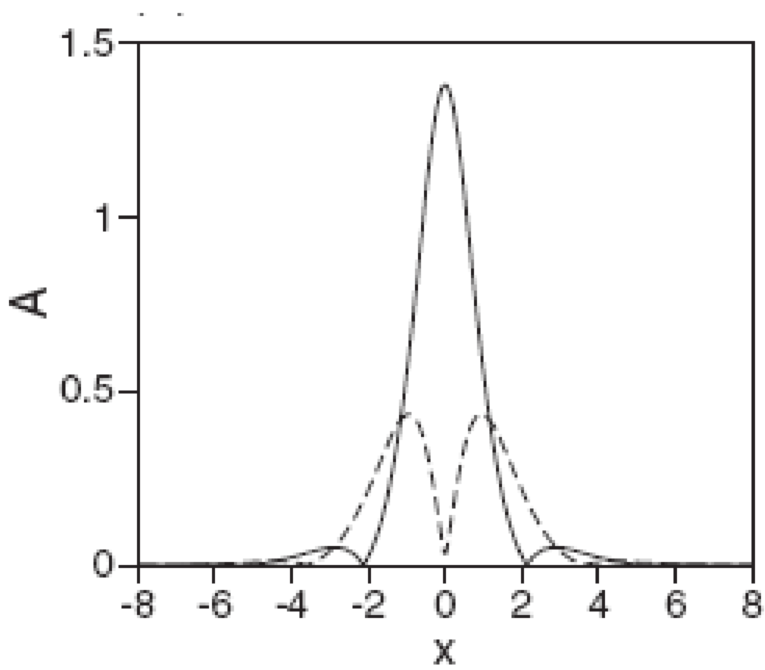
$$\phi_+(x,y,t) = \exp(-i\mu t)\, f_+(r),$$

$$\phi_-(x,y,t) = \exp(-i\mu t + i\theta)\, r f_+(r),$$

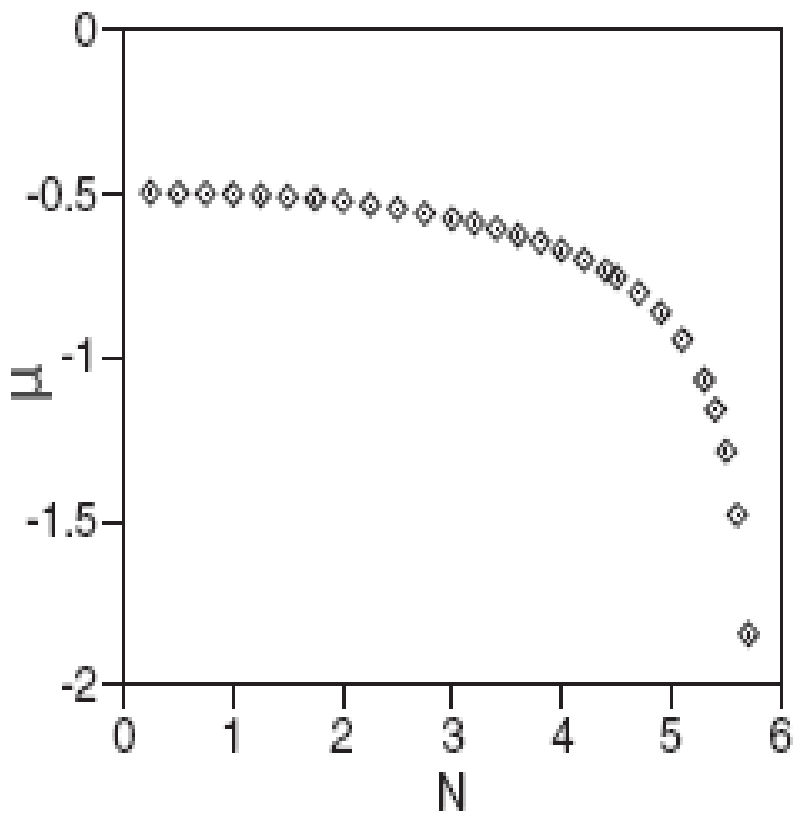
with $f_\pm(r)$ taking finite values $f_\pm(0)$ at $r = 0$,

and decaying $\sim \exp(-\sqrt{-2\mu}\, r)$ at $r \to \infty$.

A numerically found cross-section (along ***y* = 0**) of the two components, **A ≡ |*φ*$_{\pm}$|**, for a ***stable semi-vortex***, which was obtained, by means of the *imaginary-time integration*, as a stationary soliton in the ***free space***:



The numerically found dependence between the total norm of the semi-vortices and their chemical potential demonstrates that **(1)** the norm of the semi-vortex indeed falls ***below the threshold value*** necessary for the onset of the collapse: $\boldsymbol{N(\mu) < N_{thr} \equiv N(\mu \to -\infty) \approx 5.85}$; **(2)** there is ***no finite minimum value*** of the norm necessary for the existence of the semi-vortex; **(3)** the dependence satisfies the ***Vakhitov-Kolokolov*** criterion, $\boldsymbol{d\mu/dN < 0}$, which is a necessary condition for the stability:



In the limit of $N \to N_{thr}$ **≈ 5.85**, the semi-vortex ***degenerates*** into the usual (unstable) ***Townes' soliton*** in the first component,

with the ***chemical potential*** $\mu \to -\infty$, leaving the second (vortical) component ***empty***.

Direct simulations demonstrate that the ***entire family*** of the semi-vortices is ***completely stable***.

In this system, the **SO-coupling** terms break the ***confomal invariance*** of the **2D NLS** equations with the **cubic self-attraction**. This leads to ***lifting the degeneracy of the norm***, pushing the norm of the ***semi-vortices*** to $N < N_{thr}$, thus ***securing their stability against the onset of the collapse***. Actually, the semi-vortex solitons realize the system's **ground state** at $N < N_{thr}$, which ***does not exist*** in the absence of the **SO coupling**. The collapse still occurs at $N > N_{thr}$.

(4d) The stabilization of 3D solitons by the SO coupling

PRL **115**, 253902 (2015) PHYSICAL REVIEW LETTERS week ending 18 DECEMBER 2015

Stable Solitons in Three Dimensional Free Space without the Ground State: Self-Trapped Bose-Einstein Condensates with Spin-Orbit Coupling

Yong-Chang Zhang,[1] Zheng-Wei Zhou,[1,*] Boris A. Malomed,[2] and Han Pu[3,4,†]

[1]*Key Laboratory of Quantum Information, and Synergetic Innovation Center of Quantum Information and Quantum Physics, University of Science and Technology of China, Hefei, Anhui 230026, China*

[2]*Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, Tel Aviv University, 69978 Tel Aviv, Israel*

[3]*Department of Physics and Astronomy, and Rice Center for Quantum Materials, Rice University, Houston, Texas 77005, USA*

[4]*Center for Cold Atom Physics, Chinese Academy of Sciences, Wuhan 430071, China*

(Received 11 September 2015; published 17 December 2015)

The **3D** model with the **SO** coupling (in particular, of the ***Weyl type***) is based on the following system of **GP** equations for the ***spinor*** (two-component) ***wave function***, with the vector of **Pauli matrices**, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, and $\boldsymbol{g} \equiv +1$, $\boldsymbol{\eta} > 0$ (the ***self-attractive*** nonlinearity, which gives rise to the ***supercritical collapse*** in **3D**):

$$\left[i\frac{\partial}{\partial t} + \frac{1}{2}\nabla^2 + i\lambda\nabla\cdot\boldsymbol{\sigma} \right.$$
$$\left. +g\begin{pmatrix} |\psi_+|^2+\eta|\psi_-|^2 & 0 \\ 0 & |\psi_-|^2+\eta|\psi_+|^2 \end{pmatrix}\right]\Psi = 0.$$

These **GP** equations can be derived from the corresponding **Hamiltonian**:

$$E_{\rm tot} = E_{\rm kin} + E_{\rm soc} + E_{\rm int}\,, \tag{1}$$

$$E_{\rm kin} = \frac{1}{2}\int d^3r\,\Psi^\dagger \mathrm{p}^2\Psi,\; E_{\rm soc} = \lambda\int d^3r\,\Psi^\dagger\,(\mathrm{p}\cdot\boldsymbol{\sigma})\,\Psi,$$

$$E_{\rm int} = -\frac{g}{2}\int d^3r\,\left(|\psi_+|^4 + |\psi_-|^4 + 2\eta|\psi_+\psi_-|^2\right)\,,$$

A possibility of the existence of ***metastable*** **3D solitons** can be predicted by means of a simple qualitative consideration.

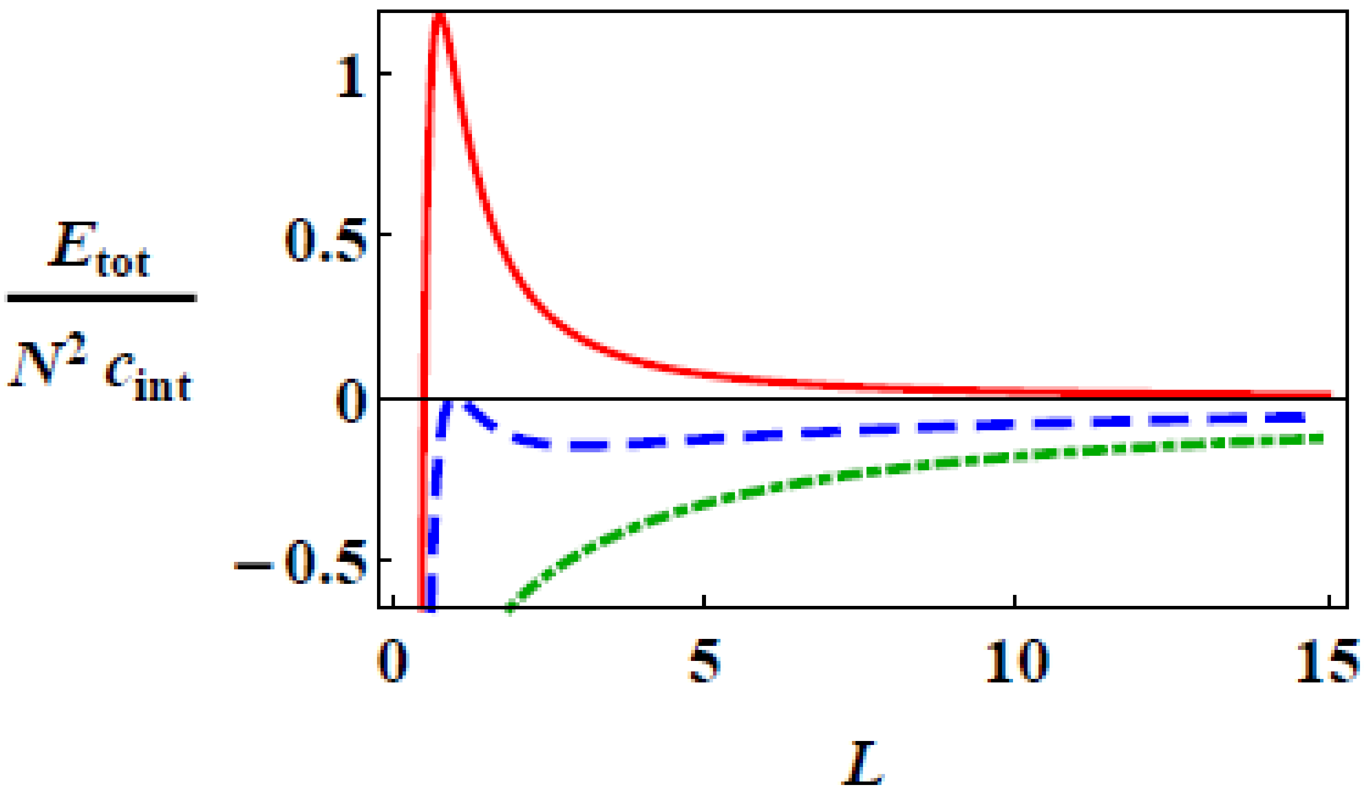
Dimensional analysis — If L is a characteristic size of the self-trapped condensate, an estimate for amplitudes of the wave functions with norm $N = \int d^3\mathbf{r}\,\left(|\psi_+|^2 + |\psi_-|^2\right)$ is $\left(|\psi_\pm|\right)_{\max} \sim \sqrt{N}L^{-3/2}$. Therefore, the three terms in Eq. (1) scale with L as

$$E_{\mathrm{tot}}/N \sim c_{\mathrm{kin}}L^{-2} - c_{\mathrm{soc}}\lambda L^{-1} - \left(c_{\mathrm{int}}^{(\mathrm{self})} + c_{\mathrm{int}}^{(\mathrm{cross})}\eta\right) N L^{-3}, \tag{2}$$

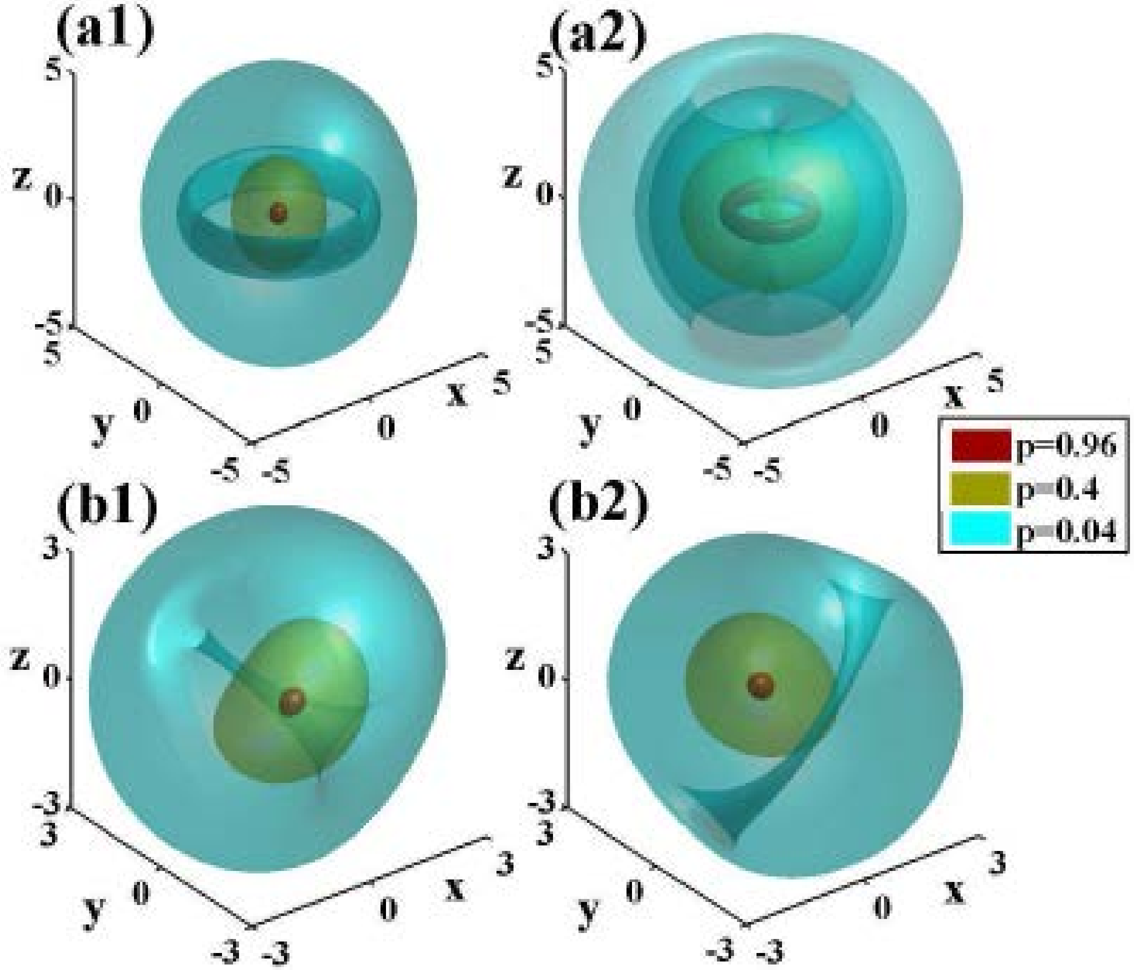
with positive coefficients c_{kin}, c_{soc}, and $c_{\mathrm{int}}^{(\mathrm{self/cross})}$. Evidently, Eq. (2) gives rise to a local minimum of $E_{\mathrm{tot}}(L)$ at finite L, provided that

$$0 < \lambda N < c_{\mathrm{kin}}^2 / \left[3\left(c_{\mathrm{int}}^{(\mathrm{self})} + c_{\mathrm{int}}^{(\mathrm{cross})}\eta\right) c_{\mathrm{soc}}\right], \tag{3}$$

An illustration of this: for **fixed *N*,** a ***metastable soliton*** may exist as a ***local minimum*** of the total energy (the blue line), but ***not*** as a ***ground state*** (red: no **SO** coupling; green: the **SO** coupling is too strong):



Examples of (***meta***) ***stable*** **3D** ***solitons***, with $\boldsymbol{N = 8}$ [(**a**), for $\boldsymbol{\eta = 0.3}$ – a semi-vortex;(**b**), for $\boldsymbol{\eta = 1.5}$ – a ***mixed mode***]:



(5) The newest addition: stabilization of 3D and 2D "superfluid droplets" by quantum fluctuations.

Corrections to the mean-field theory for **BEC** with self-repulsion, generated by quantum fluctuations around the mean-field state, were derived in 1957 by Lee, Huang, and Yang (**LHY**). Recently, a model was elaborated for a two-component **BEC** with the usual mean-field ***repulsive*** self-interaction in each component, and ***stronger attraction*** between the components. The ***collapse*** driven by the cross attraction is ***arrested*** by the **LHY** correction, which is effectively represented by the ***self-repulsive*** **quartic** terms, added to the usual mean-field cubic ones.

The theoretical elaboration of the model, in both **3D** and **2D** settings:

PRL **115,** 155302 (2015) PHYSICAL REVIEW LETTERS week ending 9 OCTOBER 2015

Quantum Mechanical Stabilization of a Collapsing Bose-Bose Mixture

D. S. Petrov
Université Paris-Sud, CNRS, LPTMS, UMR8626, Orsay, F-91405, France
(Received 28 June 2015; published 7 October 2015)

PRL **117,** 100401 (2016) PHYSICAL REVIEW LETTERS week ending 2 SEPTEMBER 2016

Ultradilute Low-Dimensional Liquids

D. S. Petrov[1] and G. E. Astrakharchik[2]
[1]*LPTMS, CNRS, Univ. Paris Sud, Université Paris-Saclay, 91405 Orsay, France*
[2]*Departament de Física, Campus Nord B4-B5, Universitat Politècnica de Catalunya, E-08034 Barcelona, Spain*
(Received 24 May 2016; revised manuscript received 28 July 2016; published 1 September 2016)

Assuming a symmetric configuration, with ***equal wave functions*** of both components, $\boldsymbol{\Psi}_1 = \boldsymbol{\Psi}_2 \equiv \boldsymbol{\Psi}$, the system of coupled **GP** equations may be reduced to the ***single*** one:

$$\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\nabla^2\Psi - |\Psi|^2\,\Psi + \gamma\,|\Psi|^3\,\Psi,$$

in **3D**, where $\gamma > 0$ is an effective strength of the **quartic** term which represents the **LHY** correction.

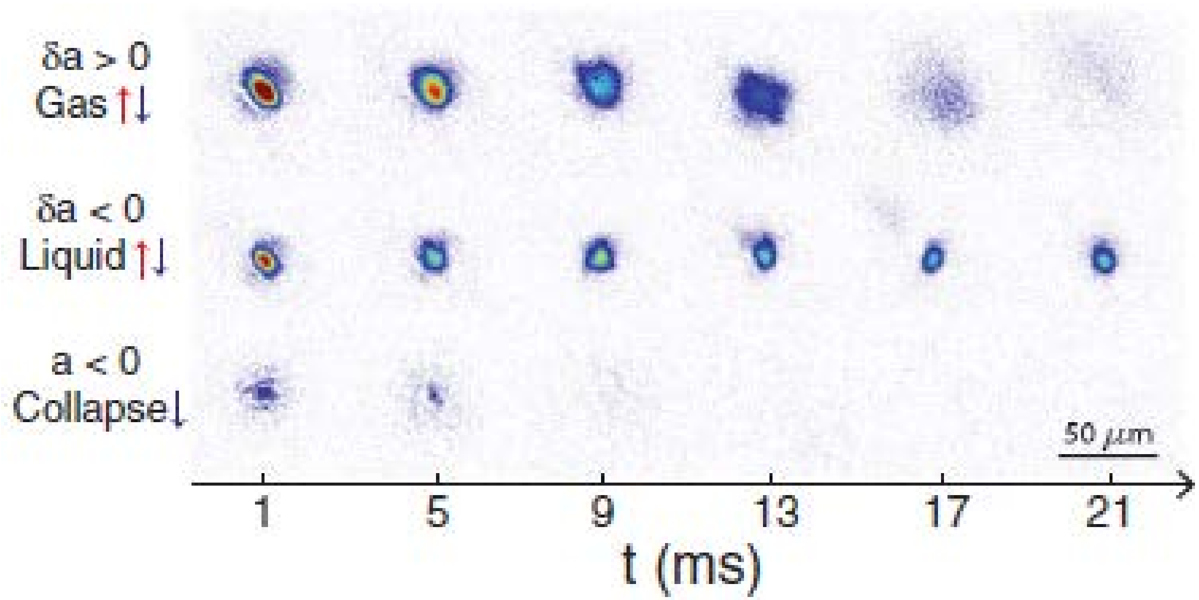
The dimension reduction **3D** $\rightarrow$ **2D** (under the action of **strong confinement** in the transverse direction) gives rise to the following **2D quasi-GP** equation:

$$\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\nabla^2\Psi + \ln\left(|\Psi|^2\right)|\Psi|^2\,\Psi.$$

Very recently, ***experimental creation*** of **quasi-2D** (***oblate***) "droplets", with ***aspect ratio*** **~ 10:1**, **zero vorticity**, and ~**10,000 ^{39}K** atoms per droplet, was reported in the following papers: **Science 359, 301 (2018)**,

Quantum liquid droplets in a mixture of Bose-Einstein condensates

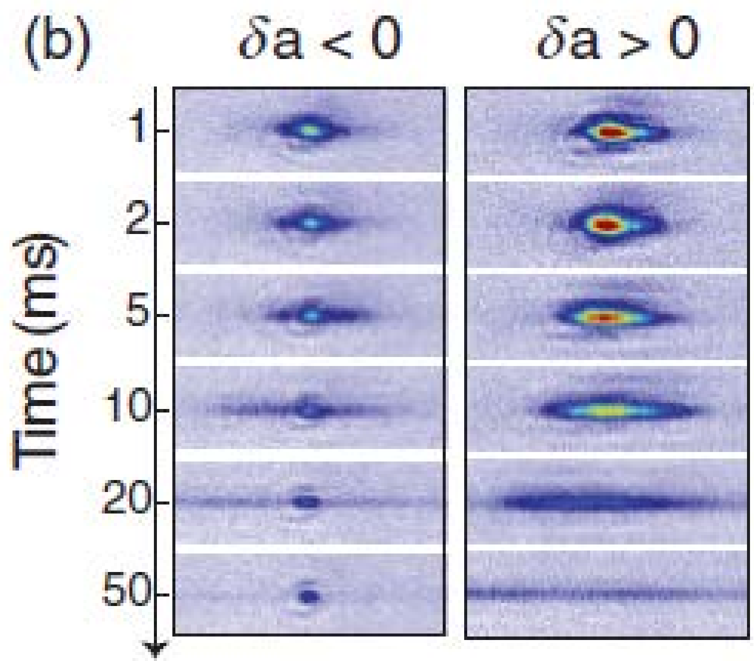
C. R. Cabrera,* L. Tanzi,* J. Sanz, B. Naylor, P. Thomas, P. Cheiney, L. Tarruell†



PHYSICAL REVIEW LETTERS **120,** 135301 (2018)

Bright Soliton to Quantum Droplet Transition in a Mixture of Bose-Einstein Condensates

P. Cheiney, C. R. Cabrera, J. Sanz, B. Naylor, L. Tanzi, and L. Tarruell*
ICFO—Institut de Ciencies Fotoniques, The Barcelona Institute of Science and Technology, 08860 Castelldefels (Barcelona), Spain

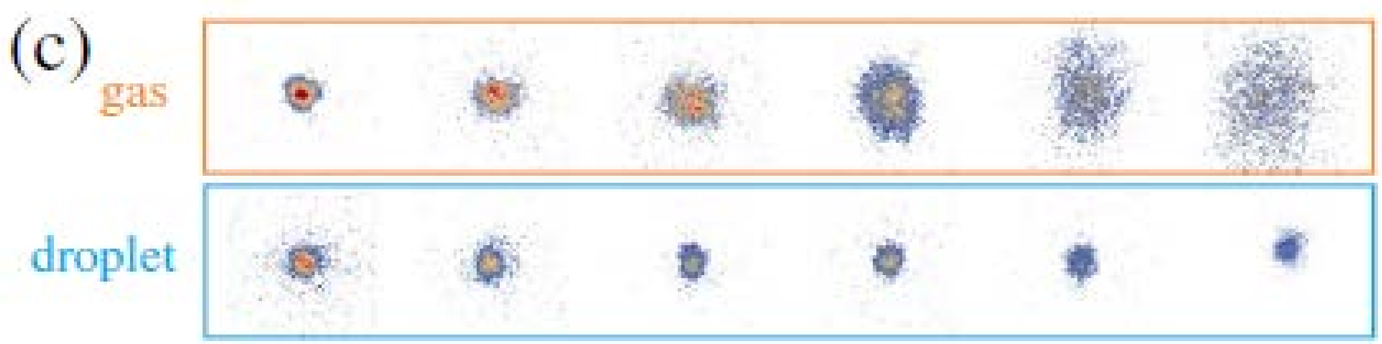


Another experimental result: the creation of ***nearly isotropic*** **3D quantum droplets** (with negligible confinement in any direction) in the same atomic species, ^{39}**K**:

PHYSICAL REVIEW LETTERS **120,** 235301 (2018)

Self-Bound Quantum Droplets of Atomic Mixtures in Free Space

G. Semeghini,[1,2,*] G. Ferioli,[1,2] L. Masi,[1,2] C. Mazzinghi,[1] L. Wolswijk,[1] F. Minardi,[2,1,3] M. Modugno,[4,5] G. Modugno,[1,2] M. Inguscio,[2,1] and M. Fattori[1,2]
[1]*LENS and Dipartimento di Fisica e Astronomia, Università di Firenze, 50019 Sesto Fiorentino, Italy*
[2]*CNR Istituto Nazionale Ottica, 50019 Sesto Fiorentino, Italy*
[3]*Dipartimento di Fisica e Astronomia, Università di Bologna, 40127 Bologna, Italy*
[4]*Departamento de Física Teórica e Historia de la Ciencia, Universidad del Pais Vasco UPV/EHU, 48080 Bilbao, Spain*
[5]*IKERBASQUE, Basque Foundation for Science, 48011 Bilbao, Spain*



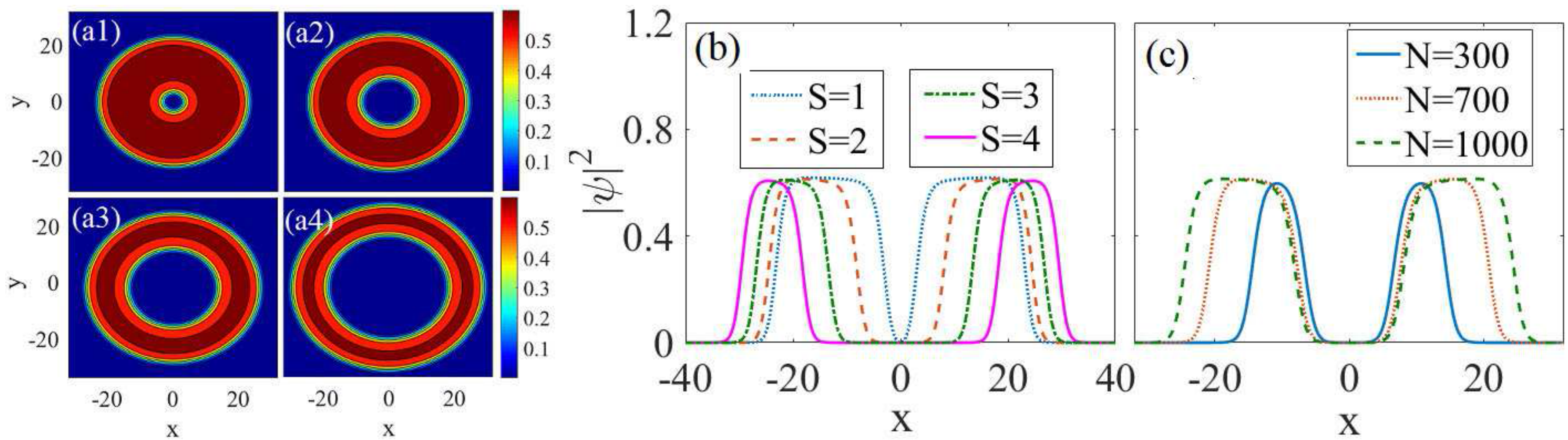
Getting back to the theory: ***vortex-droplet*** states, with **embedded vorticity *S***, were very recently constructed in the framework of the **2D** reduced model, $\boldsymbol{\Psi = \exp(-i\mu t + iS\theta)U(r)}$.

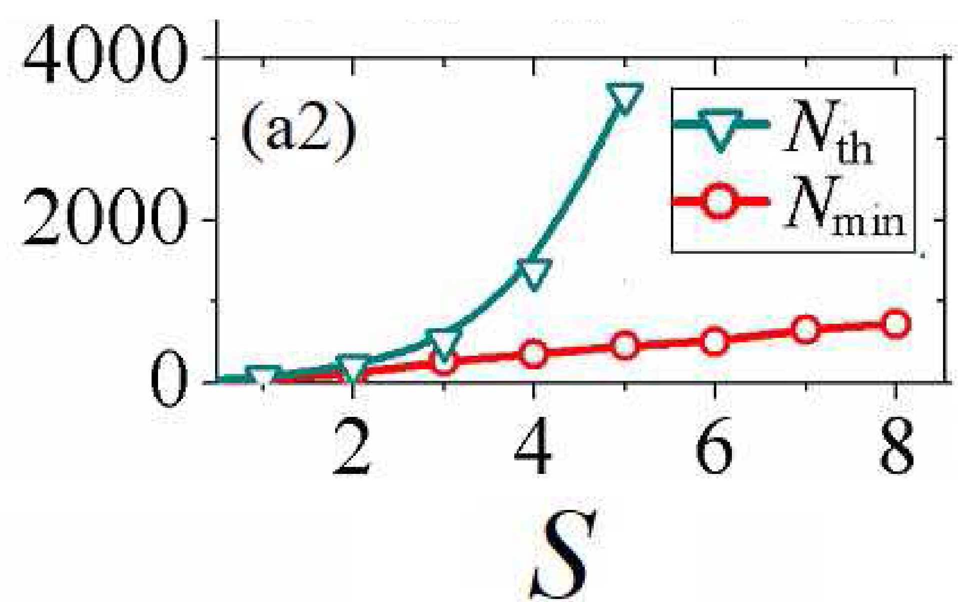
Y. Li, Z. Chen, Z. Luo, C. Huang, H. Tan, W. Pang, and B. A. Malomed,

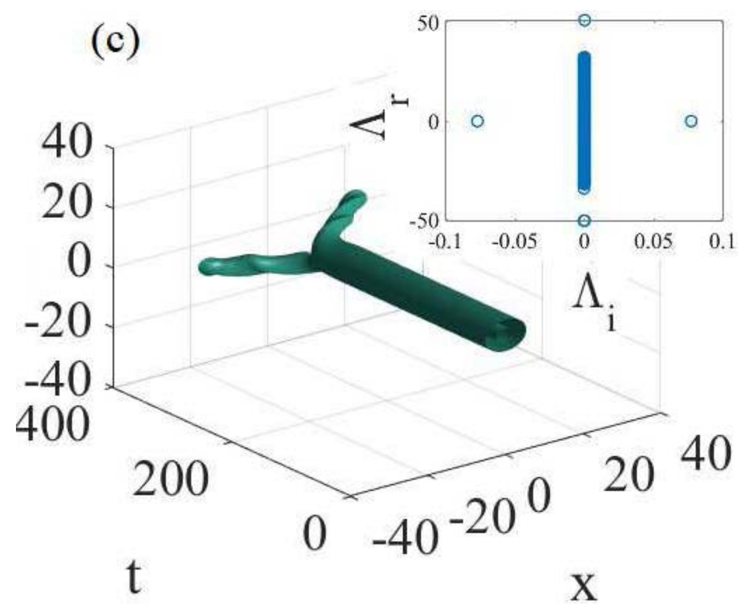
Two-dimensional vortex quantum droplets**,**

arXiv:1801.10274; Phys. Rev. A, in press.

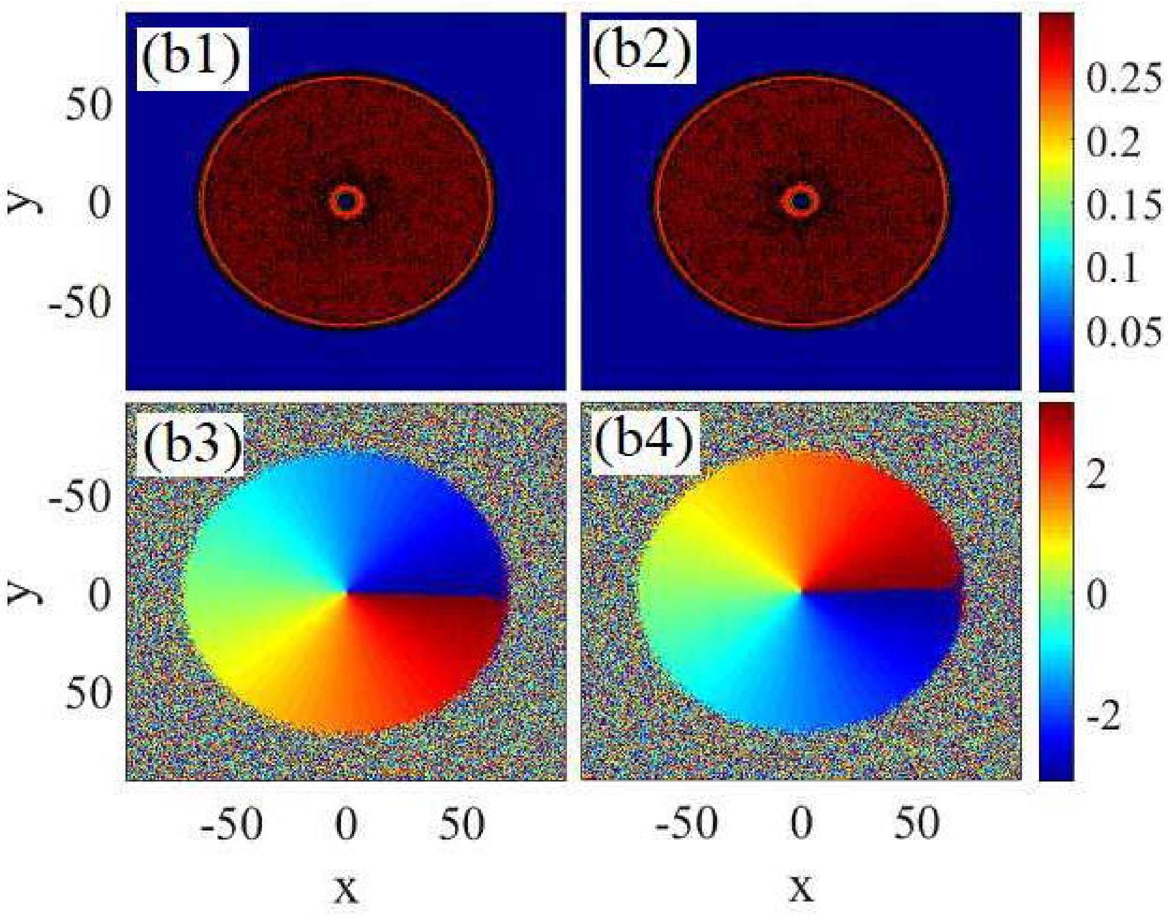
The ***vortex droplets*** have their stability regions up to $\boldsymbol{S = 5}$ (an example of the evolution **(*splitting*)** of an ***unstable*** vortex mode with $\boldsymbol{S = 1}$ is shown too):


(a1)
(a2)
(a3)
(a4)
y
x
(b)
S=1
S=2
S=3
S=4
$|\psi|^2$
x
(c)
N=300
N=700
N=1000
x


(a2)
N_{th}
N_{min}
S


(c)
Λ_r
Λ_i
t
x

Also found were (in a small parameter regions) ***stable*** modes with ***hidden vorticity***, $\mathbf{\Psi_{1,2} = exp(-i\mu t \pm i\theta)U(r)}$, in the system which ***does not coalesce*** into a singe **GP** equation:



Another recent theoretical result: prediction of **stability regions** (by means of a systematic numerical analysis, augmented by analytical approximations) for **fully 3D** droplets, with embedded vorticity ***S* = 1** and **2**:

PHYSICAL REVIEW A **98**, 013612 (2018)

Three-dimensional droplets of swirling superfluids

Yaroslav V. Kartashov,[1,2,*] Boris A. Malomed,[3] Leticia Tarruell,[1] and Lluis Torner[1,4]

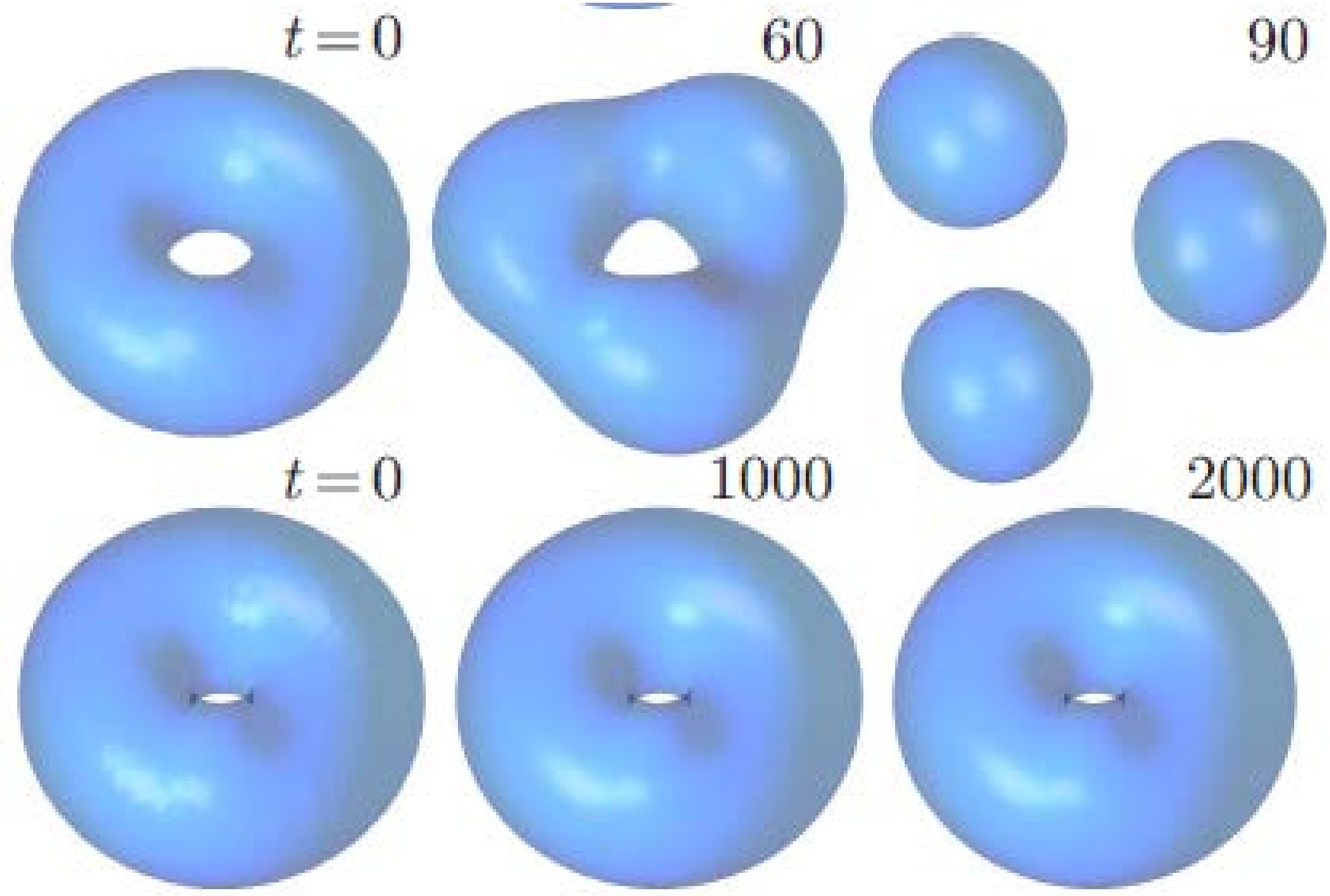
[1]*ICFO-Institut de Ciencies Fotoniques, The Barcelona Institute of Science and Technology, 08860 Castelldefels, Barcelona, Spain*

[2]*Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow, 108840, Russia*

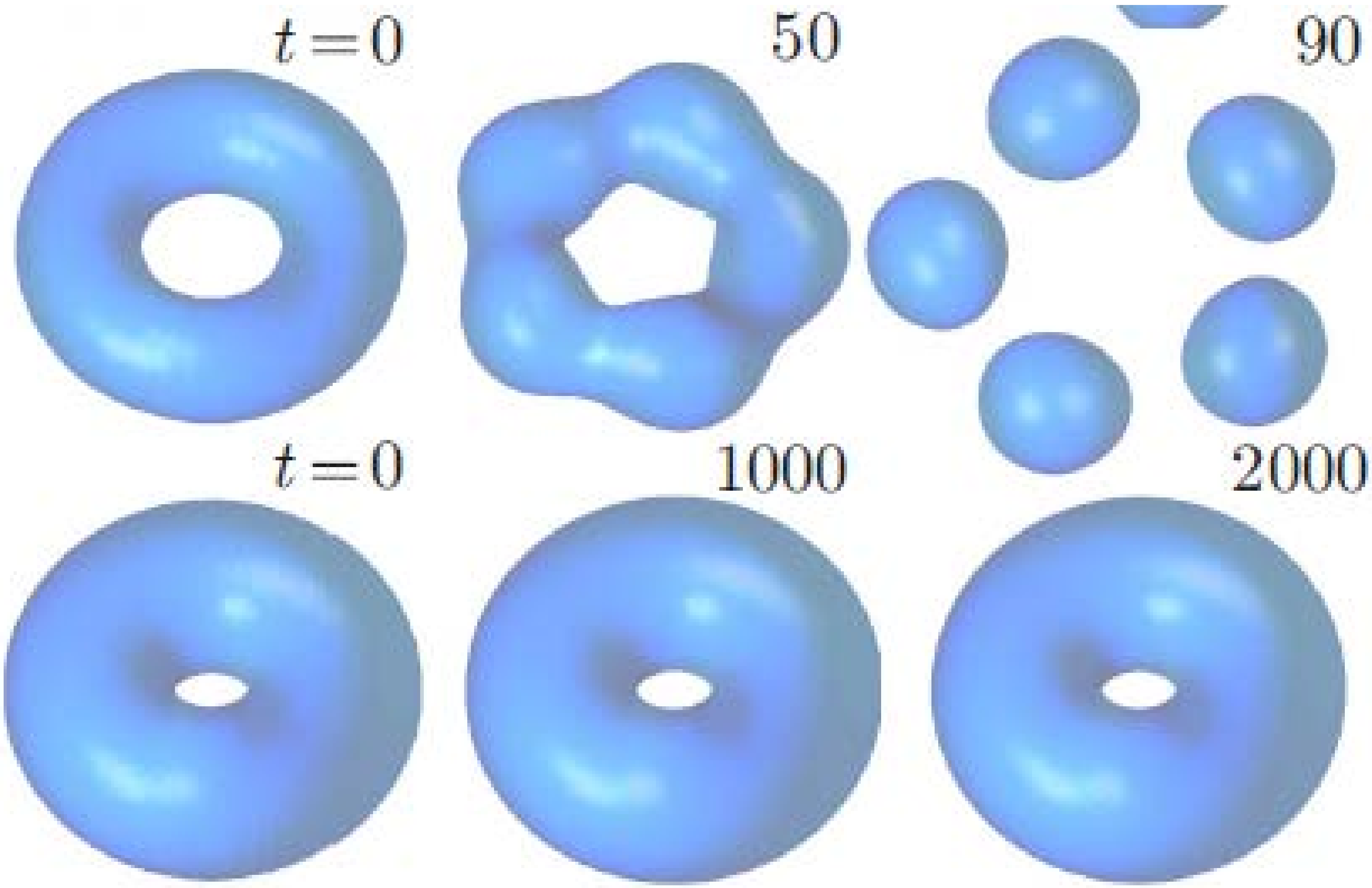
Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, and Center for Light-Matter Interaction, Tel Aviv University, 69978 Tel Aviv, Israel and ITMO University, St. Petersburg 197101, Russia

[4]*Universitat Politecnica de Catalunya, 08034, Barcelona, Spain*

Examples of ***splitting*** of an ***unstable vortical droplet***, with ***μ* = -0.04** (the **top row**), and of the evolution of a ***stable*** one (the **bottom row**), with ***μ* = -0.16**. In both cases, ***S* = 1**.



Vortical droplets with ***S* = 2** may be ***stable*** too, but only at ***very large values of the norm***. Examples of the splitting of an ***unstable droplet*** (***μ* = -0.04**) and evolution of a ***stable one*** (***μ* = -0.183**):



(6) Conclusions

Recent theoretical and experimental studies have led to the prediction and, in some cases, experimental creation of stable self-trapped modes in the form of **fundamental** and **vortex solitons** in **2D** and **3D** geometries. Especially interesting are the predicted possibilities for the creation of (meta)stable **semi-vortices** by means of the **SO (spin-orbit) coupling** for the binary **BEC** with cubic attractive interactions (something which was previously considered ***absolutely impossible***), as well as the

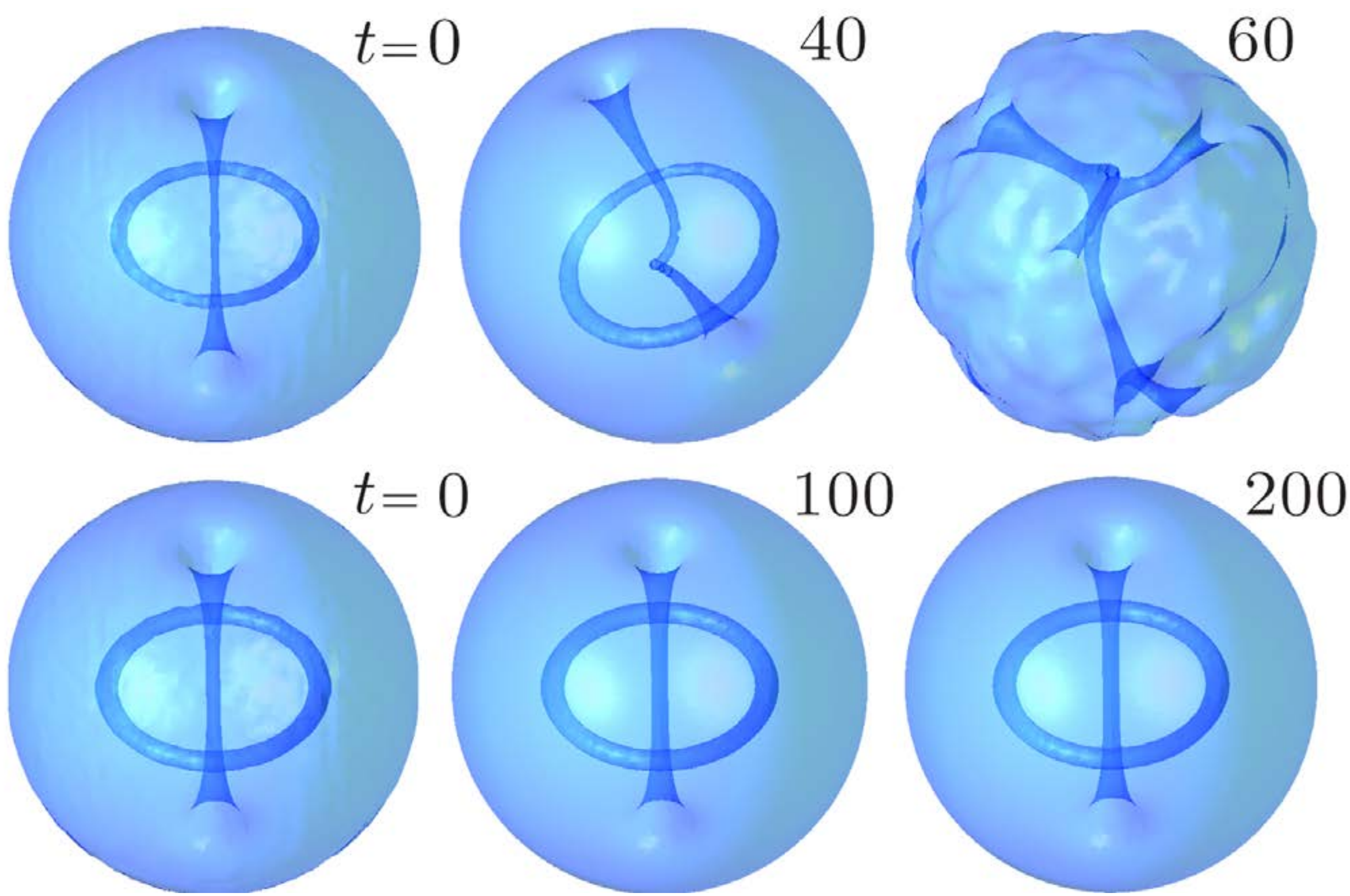
theoretically predicted and ***experimentally realized*** creation of **3D superfluid droplets**, stabilized by **quantum fluctuations**.

Generally, the theoretical studies of multidimensional solitons have advanced much farther than the experimental work. Creation of stable **2D** and **3D** solitons in ***real experiments*** remains a challenging objective.

As concerns further development of the ***theory***, an intriguing possibility is the elaboration of ***complex topological self-trapped states*** in **3D**, such as ***hopfions***, i.e, ***vortex tori*** with an ***intrinsic twist*** of the ***toroidal core***. Relatively simple **3D** models make it possible to produce ***hopfions***, both unstable and ***stable*** ones:

Twisted Toroidal Vortex Solitons in Inhomogeneous Media with Repulsive Nonlinearity

Yaroslav V. Kartashov,[1,2] Boris A. Malomed,[3] Yasha Shnir,[4,5] and Lluis Torner[1]



Another species of “***exotic***” self-trapped **topologically organized** states, which may be ***stable***: ***hybrids*** built of **mutually displaced** ***vortices*** and ***antivortices*** (**+*S*** and **-*S***), with ***broken axial symmetry*** (New J. Phys. **16**, 063035 (2014)).

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Three-dimensional hybrid vortex solitons

Rodislav Driben[1], Yaroslav V Kartashov[2,3], Boris A Malomed[4], Torsten Meier[1] and Lluis Torner[2]