Linear and nonlinear surface wave patterns for flow past a submerged source or doublet

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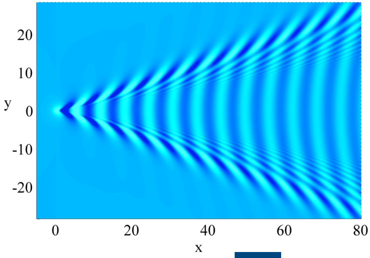
Ship wave patterns

moderately fast moving ships

- Transverse and divergent waves.
- Kelvin's angle $\arcsin(1/3) \approx 19.47^\circ$.
- Geometric arguments, method of stationary phase, geometric ray theory.



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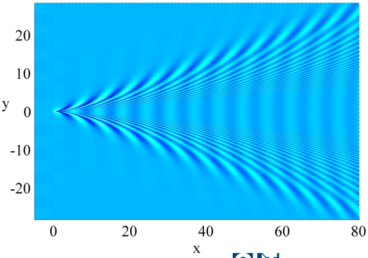
Ship wave patterns

fast moving ships

- Divergent waves dominate transverse.
- Apparent angle appears bit less than Kelvin's angle.



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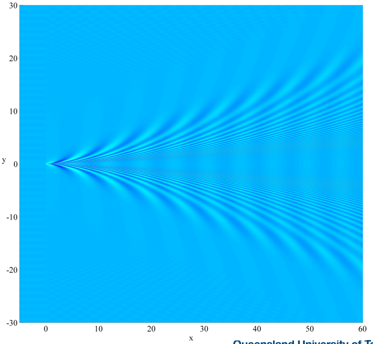
Ship wave patterns

very fast moving ships

- Divergent waves totally dominate.
- Apparent angle much less than Kelvin's angle.

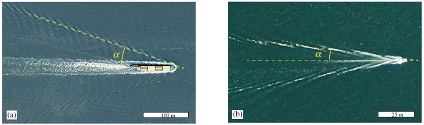


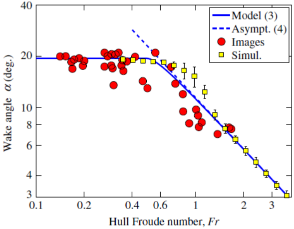
Apparent wake angle

measured angles less than Kelvin's angle

- Remember Kelvin angle independent of ship speed.

- Rabaud & Moisy (2013; *Phys. Rev. Lett.*), reported data from airborne images taken from Google Earth to show wake angle was roughly constant for moderately fast ships, then scaled as speed^{-1} for fast ships.





- Proposed theory that a ship cannot generate wavelengths much greater than its hull length.

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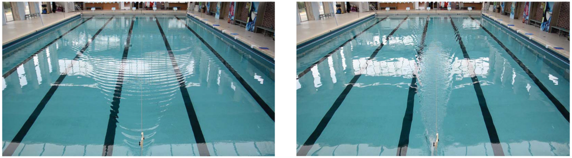
Apparent wake angle

measured angles less than Kelvin's angle

- See also Moisy & Rabaud (2014; *Phys Rev E*).



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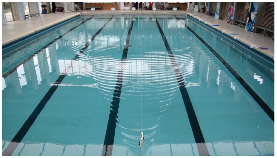
Apparent wake angle

measured angles less than Kelvin's angle

- See also Moisy & Rabaud (2014; *Phys Rev E*).



- Darmon *et al.* (2014; *J. Fluid Mech.*) and others suggest that Rabaud & Moisy were probably just measuring the angle provided by the highest peaks of the waves, which gives rise to the same scaling.
- Noblesse et al. (2014; *Euro. J. Mech. B/Fluids*) and others explain that smaller angles are due to interference between bow and stern waves.

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Flow past a submerged point source

A toy problem related to the Green's function

Velocity potential ϕ:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad \text{for } z < \zeta(x,y)$$

Conditions on free-surface:

$$\phi_x\zeta_x + \phi_y\zeta_y = \phi_z \text{ on } z = \zeta(x,y),$$

$$\frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + \frac{\zeta}{F^2} = \frac{1}{2} \text{ on } z = \zeta(x,y),$$

Near submerged source:

$$\phi \sim -\frac{\epsilon}{4\pi\sqrt{x^2+y^2+(z+1)^2}} \text{ as } (x,y,z) \to (0,0,-1).$$

Far-field conditions:

$$(\phi_x,\phi_y,\phi_z) \to (1,0,0), \quad \zeta \to 0 \quad \text{as} \quad x \to -\infty,$$

$$(\phi_x,\phi_y,\phi_z) \to (1,0,0), \quad \text{as} \quad z \to -\infty.$$

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Flow past a submerged point doublet

A toy for flow due to a submerged vessel

Velocity potential ϕ:

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Conditions on free-surface:

$$\begin{aligned} \phi_x\zeta_x + \phi_y\zeta_y &= \phi_z \text{ on } z = \zeta(x,y), \\ \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + \frac{\zeta}{F^2} &= \frac{1}{2} \text{ on } z = \zeta(x,y), \end{aligned}$$

Near submerged point doublet:

$$\phi \sim \frac{\mu x}{4\pi(x^2 + y^2 + (z+1)^2)^{3/2}} \text{ as } (x,y,z) \to (0,0,-1).$$

Far-field conditions:

$$\begin{aligned} (\phi_x, \phi_y, \phi_z) &\to (1,0,0), \quad \zeta \to 0 \quad \text{as} \quad x \to -\infty, \\ (\phi_x, \phi_y, \phi_z) &\to (1,0,0), \quad \text{as} \quad z \to -\infty. \end{aligned}$$

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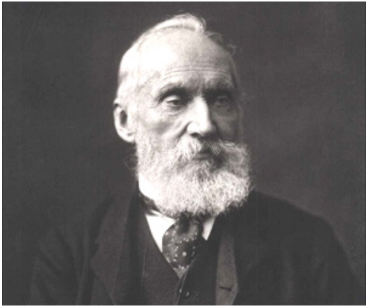
Part I

Linear problem of flow past a submerged source and doublet



Sir William Thomson, Lord Kelvin

LINEARISED FLOW PAST SUBMERGED SOURCE

A PROTOTYPE PROBLEM

- Governing equation: $\nabla^2\phi = 0$ for $z < 0$.
- Conditions on free-surface: $\zeta_x = \phi_z$, $\phi_x + \zeta/F^2 = 1$ on $z = 0$ $(F = U/\sqrt{gH})$
- Near submerged source: $\phi \sim -\epsilon/4\pi(x^2 + y^2 + (z+1)^2)^{1/2}$ as $(x, y, z) \to (0, 0, -1)$.
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Linearised flow past submerged source

a prototype problem

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- Far-field conditions: $(\phi_x, \phi_y, \phi_z) \to (1, 0, 0)$ as $x \to -\infty$ and $z \to -\infty$.
- Exact solution:

$$\zeta(x,y) = -\frac{\epsilon F^2 \operatorname{sgn}(x)}{\pi^2}\int_0^{\frac{\pi}{2}} \cos\theta \int_0^\infty \frac{k\mathrm{e}^{-k|x|}\cos(ky\sin\theta)g(k,\theta)}{F^4k^2 + \cos^2\theta}\,\mathrm{d}k\,\mathrm{d}\theta + \frac{\epsilon H(x)}{\pi}\int_{-\infty}^{\infty} \xi\mathrm{e}^{-F^2\xi^2}\cos(x\xi)\cos(y\xi\lambda)\,\mathrm{d}\lambda$$

where $g(k,\theta) = F^2k\sin(k\cos\theta) + \cos\theta\cos(k\cos\theta)$, $\xi(\lambda) = \sqrt{\lambda^2+1}/F^2$

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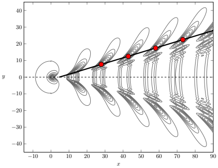
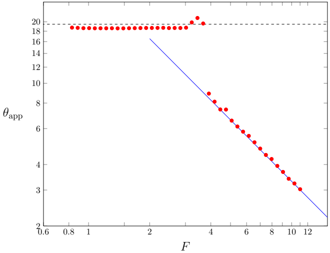
Apparent wake angle for submerged source

what is the angle we actually see?

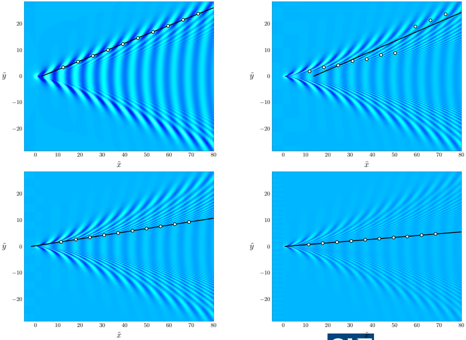


Countour plot $F = 1.5$



$\theta_{\text{app}} \sim 1/(\sqrt{3}F)$ for $F \gg 1$

Recall $F = U/\sqrt{gH}$.



Surface elevation for $F = 1.5,\ 3.5,\ 4.5$ and 8.5.

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Linearised flow past submerged doublet

a prototype problem

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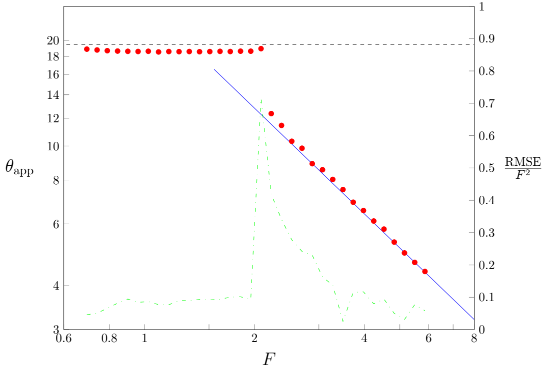
where $g(k,\theta) = F^2 k\sin(k\cos\theta) + \cos\theta\cos(k\cos\theta)$, $\xi(\lambda) = \sqrt{\lambda^2+1}/F^2$

Apparent wake angle for submerged doublet

what is the angle we actually see?



$\theta_{\text{app}} \sim 1/(\sqrt{5}F)$ for $F \gg 1$

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Key points so far

very rough summary

- Rabaud & Moisy (2013; *Phys. Rev. Lett.*), Darmon *et al.* (2014; *J. Fluid Mech.*), Noblesse et al. (2014; *Euro. J. Mech. B/Fluids*) and many others have revisited the topic of wake angle for ship wave patterns.
- Flow past a submerged point source and submerged point doublet are toy problems that are not directly related to the observations of actual ships, but do give rise to "ship waves".

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- Flow past a submerged point source and submerged point doublet are toy problems that are not directly related to the observations of actual ships, but do give rise to “ship waves”.
- Apparent wake angle for large Froude number appears scales like $1/F$ in a similar way to the models above.
- The decrease in apparent wake angle can be explained using method of stationary phase without resorting to arguments like Rabaud & Moisy (2013).
- As no obvious length scale in direction of flow, the decrease in apparent wake angle is not due to wave interference.
- We have considered a range of other disturbances.

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Wakes for small Froude numbers

dominated by transverse waves

- Note that stationary phase gives $\zeta(r,\theta) \sim \dfrac{\epsilon}{r^{1/2}} H(\theta) \mathrm{e}^{-G(\theta)/F^2} \quad \text{as} \quad r \to \infty.$

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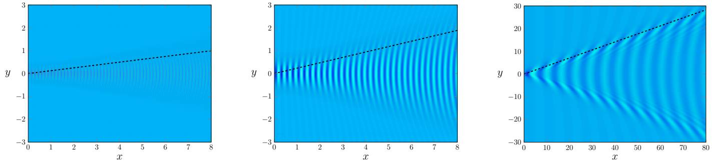
Wakes for small Froude numbers

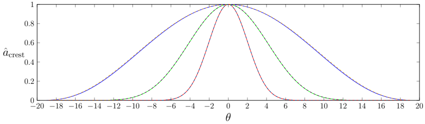
dominated by transverse waves

- Note that stationary phase gives $\zeta(r,\theta) \sim \dfrac{\epsilon}{r^{1/2}} H(\theta) \mathrm{e}^{-G(\theta)/F^2}$ as $r \to \infty$.
- Wake angle seems to decreases with small Froude number ($F = 0.1,\ 0.2,\ 1$):



Here wake angle 20% wave height for (a),(b), but Kelvin's angle for (c).

- Cross section:



Here $F = 0.2$ (blue), $F = 0.1$ (green) and $F = 0.05$ (red).

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SMALL FROUDE NUMBER ASYMPTOTICS

ONE PAGE SUMMARY OF EXPONENTIAL ASYMPTOTICS

Steady ship wave problem:

$$\nabla^2\phi = 0 \quad \text{in fluid}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on boundary}$$

$$\frac{1}{2}F^2\left(|\nabla\phi|^2 - 1\right) + \zeta = 0 \quad \text{on surface}$$

$$\phi \sim x \quad \text{in far-field.}$$

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Small Froude number asymptotics

one page summary of exponential asymptotics

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Asymptotic series:

$$\zeta \sim \sum_{n=0}^{\infty} F^{2n}\zeta_n(\mathbf{x}) \quad \text{as} \quad F \to 0$$

- Compute ζ_1 by hand, perhaps ζ_2 too.
- Series divergent.
- Waves not captured by these terms.

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Small Froude number asymptotics

one page summary of exponential asymptotics

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- Optimally truncate

$$\zeta = \sum_{n=0}^{N-1} F^{2n}\zeta_n(\mathbf{x}) + R_N.$$

- Remainder exponentially small:

$$R_N \sim A(x,y)\mathrm{e}^{-\chi/F^2} \quad \text{as} \quad F \to 0$$

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one page summary of exponential asymptotics

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- Remainder exponentially small:

$$R_N \sim A(x,y)\mathrm{e}^{-\chi/F^2} \quad \text{as} \quad F \to 0$$

- Problem is to calculate A and χ and where waves are 'switched on'.
- Involves WKB and Stokes phenomenon.

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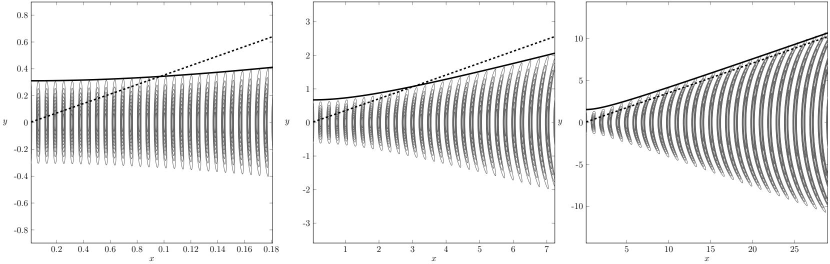
Small Froude number asymptotics

wave envelope



(A) $F = 0.1$ (B) $F = 0.2$ (C) $F = 0.4$

Figure: Contour plots for flow past a submerged source. Solid line is $\mathrm{Re}(\chi) =$ constant. The dashed line is Kelvin's angle.

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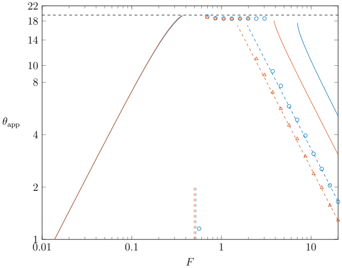
Linear flow past a submerged source

very rough summary for small Froude numbers

- Lots of attention for large Froude number, but little for small Froude number.
- Wake angle decrease like $\theta_{\text{app}} \sim \sqrt{-\ln(\text{percentage})}\, F$:



Source in blue, doublet in purple.

- The small Froude number limit is a problem of exponential asymptotics.

Part II

Nonlinear problem of flow past a submerged source and doublet

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Equations for flow past submerged source

fully nonlinear problem

Velocity potential ϕ:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad \text{for } z < \zeta(x,y)$$

Conditions on free-surface:

$$\phi_x\zeta_x + \phi_y\zeta_y = \phi_z \text{ on } z = \zeta(x,y),$$
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Near submerged source:

$$\phi \sim -\frac{\epsilon}{4\pi\sqrt{x^2+y^2+(z+1)^2}} \text{ as } (x,y,z) \to (0,0,-1).$$

Far-field conditions:

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Numerical solution to nonlinear problem

boundary integral method

- Apply Green's second formula (Forbes 1989; *J. Comp. Phys.*).
- By setting $\phi(x,y) = \Phi(x,y,\zeta(x,y))$, final boundary-integral equation is

$$
\begin{aligned}
2\pi(\phi(x^*,y^*) - x^*) = &- \frac{\epsilon}{\left(x^{*2} + y^{*2} + (\zeta(x^*,y^*)+1)^2\right)^{\frac{1}{2}}} \\
&+ \int_0^\infty \int_{-\infty}^\infty (\phi(x,y) - \phi(x^*,y^*) - x + x^*) K_1(x,y;x^*,y^*) \, \mathrm{d}x \, \mathrm{d}y \\
&+ \int_0^\infty \int_{-\infty}^\infty \zeta_x(x,y) K_2(x,y;x^*,y^*) \, \mathrm{d}x \, \mathrm{d}y, \qquad (1)
\end{aligned}
$$

which holds for any point (x^*,y^*) in the (x,y)-plane.
- Here K_1 and K_2 are the kernel functions

$$
\begin{aligned}
K_1(x,y;x^*,y^*) = &\frac{\zeta(x,y) - \zeta(x^*,y^*) - (x-x^*)\zeta_x - (y-y^*)\zeta_y}{\left((x-x^*)^2 + (y-y^*)^2 + (\zeta(x,y)-\zeta(x^*,y^*))^2\right)^{\frac{3}{2}}} \\
&+ \frac{\zeta(x,y) - \zeta(x^*,y^*) - (x-x^*)\zeta_x - (y+y^*)\zeta_y}{\left((x-x^*)^2 + (y-y^*)^2 + (\zeta(x,y)-\zeta(x^*,y^*))^2\right)^{\frac{3}{2}}}, \\
K_2(x,y;x^*,y^*) = &\frac{1}{\sqrt{(x-x^*)^2 + (y-y^*)^2 + (\zeta(x,y)-\zeta(x^*,y^*))^2}} \\
&+ \frac{1}{\sqrt{\cdots}}.
\end{aligned}
$$

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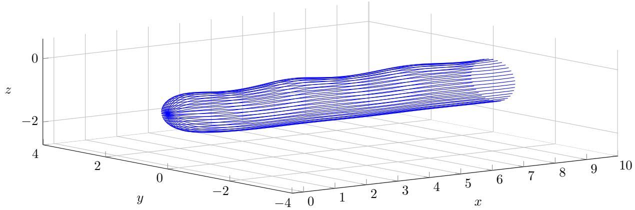
Flow past a submerged source

nonlinearity distorts the Rankine body



- Linear solution $(\epsilon \ll 1)$ corresponds to semi-infinite Rankine body with radius $\sqrt{\epsilon/\pi}$.
- Nonlinearity distorts the particle paths (here $\epsilon = 1.5$).

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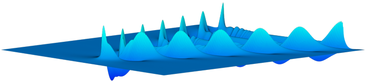
Flow past a submerged source

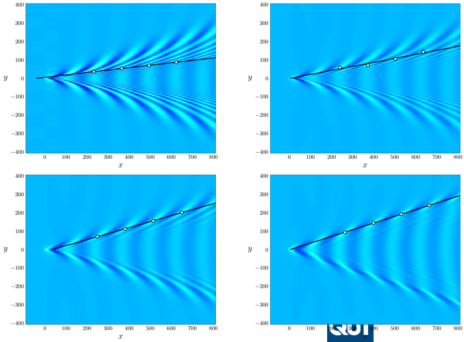
nonlinearity increases apparent wake angle

Profiles for $F = 4.5$ (a fixed speed).

linear $(\epsilon \ll 1)$



nonlinear $(\epsilon = 106)$



Linear $(\epsilon \ll 1)$, $\epsilon = 12$, 55 and 106.

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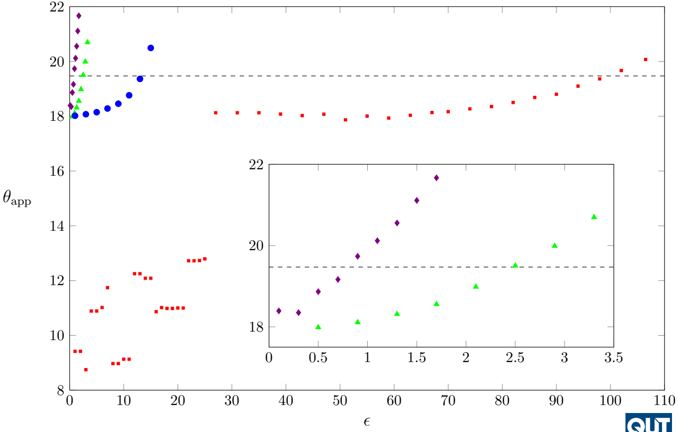
Apparent wake angle for submerged source

effect of nonlinearity



$F = 0.9$ (violet diamonds), 1.4 (green triangles), 2.5 (blue circles), 4.5 (red squares).

Equations for flow past submerged doublet

fully nonlinear problem

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Near submerged doublet:

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Far-field conditions:

$$(\phi_x,\phi_y,\phi_z) \to (1,0,0), \quad \zeta \to 0 \quad \text{as} \quad x \to -\infty,$$
$$(\phi_x,\phi_y,\phi_z) \to (1,0,0), \quad \text{as} \quad z \to -\infty.$$

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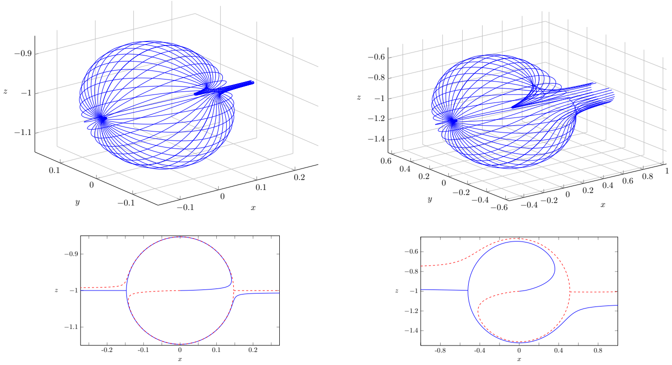
Flow past a submerged doublet

nonlinearity distorts the spherical body



- Linear solution $(\mu \ll 1)$ corresponds to spherical body with radius $(\mu/2\pi)^{1/3}$.
- Nonlinearity distorts the particle paths (here $\mu = 0.02$ and 0.88).

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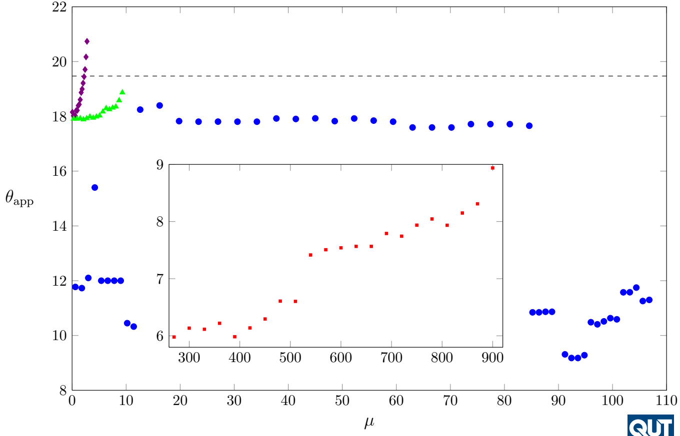
APPARENT WAKE ANGLE FOR SUBMERGED SOURCE

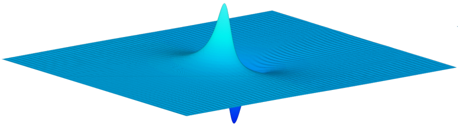
EFFECT OF NONLINEARITY



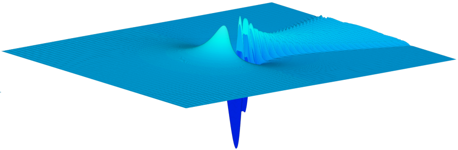
$F = 0.9$ (violet diamonds), 1.4 (green triangles), 2.5 (blue circles), 4.5 (red squares).

Numerical solutions for small Froude number

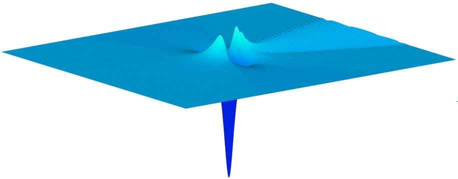
highly nonlinear waves



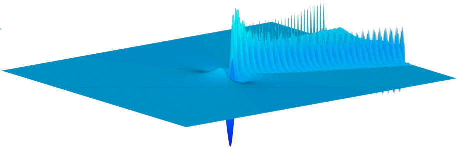
(A) $\epsilon = 0.1$



(B) $\epsilon = 8.5$



(C) $\epsilon = 0.1$



(D) $\epsilon = 1.58$

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Surface profiles for flow past (A,B) point source and (C,D) doublet for $F = 0.3$.

A final summary

and discussion

- Linear problem of flow past a submerged point source is very closely related to the Green's function. Physically represents flow past a Rankine body.
- Linear problem of flow past a submerged point doublet represents flow past a submerged sphere.
- Interesting toy models for "ship waves" as there is no length scale in direction of flow.
- Apparent wake angle has interesting behaviour such as

$$\theta_{\mathrm{app}} \sim \text{constant}\, F^{-1} \quad \text{for} \quad F \gg 1$$

and

$$\theta_{\mathrm{app}} \sim \sqrt{-\ln(\text{percentage})}\, F \quad \text{for} \quad F \ll 1.$$

- Nonlinearity provides interesting mathematical and computational challenges.

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Results documented in:

- R. Pethiyagoda, S.W. McCue, T.J. Moroney & J. M. Back (2014) Jacobian-free Newton-Krylov methods with GPU acceleration for computing nonlinear ship wave patterns, *Journal of Computational Physics* 269, 297–313.
- R. Pethiyagoda, S.W. McCue & T.J. Moroney (2014) What is the apparent angle of a Kelvin ship wave pattern? *Journal of Fluid Mechanics* 758, 468–485.
- R. Pethiyagoda, T.J. Moroney, C.J. Lustri & S.W. McCue (2018) Kelvin wake pattern at small Froude numbers, in preparation.

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