Nonlinear waves in rotating fluids

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Outline

- Rotaitonal Boussinesq eqs. and the resulting rKdV eq.
- rKdV properties, generalizations, and integrals
- “Antisoliton theorem,” terminal damping, and long-time evolution
- Laboratory and field experiments and calculations
- Dynamics of a soliton and a pair of interacting solitons on a long wave
- Solitons and multisolitons at negative dispersion
- What else?

Boussinesq-type equations for rotating fluids

Surface waves (*Shrira, 1981*):

$$\eta_t + \nabla\left[\left(H+\eta\right)\mathbf{u}\right] = 0,$$

$$\mathbf{u}_t + \left(\mathbf{u}\cdot\nabla\right)\mathbf{u} + g\nabla\eta + \frac{\beta}{3H}\nabla\eta_{tt} + \left[\mathbf{f}\times\mathbf{u}\right] = 0$$

H is a depth, $\beta = H^2 - \frac{3\sigma}{\rho g}$, σ is a surface tension, $\mathbf{f}$ is a Coriolis parameter

Internal waves, a single mode (*Ostrovsky, 1978*):

$$\eta_t + H\nabla\mathbf{u} + \frac{1}{2}s\nabla(\eta\mathbf{u}) = 0,$$

$$\mathbf{u}_t + \frac{c^2}{H}\nabla\eta + s\left[\left(\mathbf{u}\cdot\nabla\right)\mathbf{u} - \frac{1}{2H}\left(\eta\mathbf{u}\right)_t\right] + DH\nabla\eta_{tt} + \left[\mathbf{f}\times\mathbf{u}\right] = 0$$

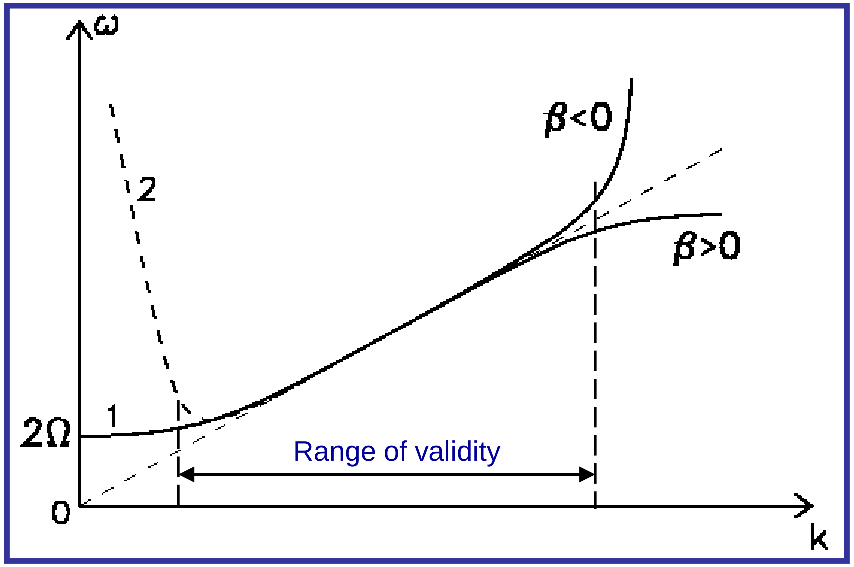
$$D = \left\langle W^2\right\rangle / H^2\left\langle W_z^2\right\rangle,\ s = H\left\langle W_z^3\right\rangle / \left\langle W_z^2\right\rangle$$

$W(z)$ is for the vertical velocity/displacement mode, $\left\langle \ldots \right\rangle = \int_0^H dz$

$\eta(x,y,t)$ is vert. displacement, $\mathbf{u}(x,y,t)$ is horiz. velocity

Dispersion equation for a linear harmonic wave:

$$u, v, \eta \propto \exp[i(kx - \omega t)] \;\Rightarrow\; \boxed{\omega^2 = \frac{f^2 + c_0^2 k^2}{1 + \beta k^2 / 3}}$$



In the <u>intermediate</u> range of wavenumbers the dispersion equation is

$$\boxed{\omega \approx c_0 k - \frac{\beta c_0}{6} k^3 + \frac{f^2}{2 c_0 k}}$$

Rotation modified KdV (rKdV) equation (*Ostrovsky, 1978*)

(about 10^5 items in Google for "Ostrovsky equation:" Thanks to R.Grimshaw!)

$$\boxed{\left(\eta_t + c_0\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx}\right)_x = \gamma\eta}$$

This equation is probably non-integrable.

It is integrable in the limits of $\gamma = 0$ (KdV) and $\beta = 0$ (reduced rKdV).

For water waves: η is a displacement, $\gamma = f^2/2c_0$, and f is the Coriolis frequency

$$c_0 = \sqrt{gh},\quad \alpha = 3c_0/2h,\quad \beta = c_0 h^2/6.$$

For surface waves:

For internal waves in a two-layer fluid:

$$c_0 = \sqrt{\frac{g\delta\rho}{\rho}\frac{h_1 h_2}{h_1+h_2}},\quad \alpha = \frac{3c_0\left(h_1-h_2\right)}{h_1 h_2},\quad \beta = \frac{3c_0 h_1 h_2}{6}.$$

rKdV is well applicable to internal waves with the periods of the order of

10^3 s (15 min) $< T < 3\cdot 10^4$ s (8 h).

Generalizations and reduced versions of rKdV equation

Similar equations were derived later for numerous kinds of waves both with the quadratic and cubic nonlinearites:

- Ultra-short waves in dielectric media (*Kozlov & Sazonov, 1997*);
- Waves in plasmas (*Obregon & Stepanyants, 1998; Litvak et al., 2005*);
- Waves in a particle chain on an elastic substrate (*Yagi & Kawahara, 2001*);

The reduced rKdV with quadratic nonlinearity: $\boxed{\left(\eta_t + c_0\eta_x + \alpha\eta\eta_x\right)_x = \gamma\eta}$

- Waves in a relaxing media (*Vakhnenko, 1991*);
- Sound waves in a bubbly liquid (*Tan & Hunter, 1991*);
- Waves in quark-gluon plasma (*Fogaca et al., 2018*).

And with the cubic nonlinearity: $\boxed{\left(\eta_t + c_0\eta_x + \alpha\eta^2\eta_x\right)_x = \gamma\eta}$

- Ultra-short optical waves in fibers (*Schäfer & Wayne, 2004*);
- Waves in a string resting on an elastic-inertial foundation (*Erofeev et al., 2018*);

There are also various generalizations of rKdV, such as:

"rKP" equation for 2-D waves (*Grimshaw & Melville, 1989, 2015*)

$$\boxed{\left(\eta_t + c_0\eta_x + \alpha\,\eta\eta_x + \beta\,\eta_{xxx}\right)_x = \gamma\eta - \frac{c_0}{2}\eta_{yy}}$$

rGardner" equation with a cubic term (*Holloway et al., 1999*):

$$\boxed{\left(\eta_t + c_0\eta_x + \alpha\,\eta\eta_x + \alpha_1\eta^2\eta^2 + \beta\,\eta_{xxx}\right)_x = \gamma\eta}$$

"rBO" equation for a rotating deep ocean (*Grimshaw, 1985*):

$$\boxed{\left(\eta_t + c_0\eta_x + \alpha\eta\eta_x + \frac{\beta}{\pi}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{+\infty}\frac{\eta(x',t)\,dx'}{x-x'}\right)_x = \gamma\eta - \frac{c_0}{2}\eta_{yy}}$$

Integral invariants for periodic or localized waves

Taken over the entire *x* axis for localized waves and over the period for spatially periodic waves

$$I_1 \equiv \int \eta(x,t)\; dx = 0$$

– zero mass condition (*Ostrovsky 1978*)

$$I_2 \equiv \frac{1}{2}\int \eta^2(x,t)\; dx = \mathrm{const}$$

– wave “energy” or “wave action”

$$I_3 \equiv \frac{1}{2}\int \left[\frac{\beta c_0}{6}(\eta_x)^2 - \frac{c_0}{2h}\eta^3 - \frac{2\Omega^2}{c_0}(\phi_x)^2\right] dx = \mathrm{const},$$

$$\eta = \phi_{xx}$$

– the Hamiltonian

$$J_1 \equiv \int x\eta(x,t)\,dx = 0$$

$$J_2 \equiv \int \left[x^2\eta(x,t) + \frac{\eta^2(x,t)}{2}\right] dx = 0$$

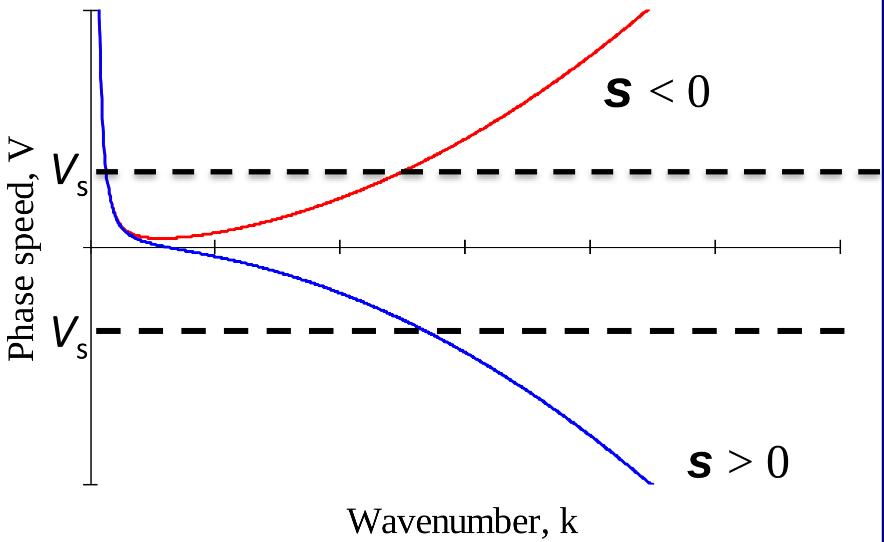
more integrals (*Benilov, 1992*)

“Antisoliton theorem”

(*Leonov, 1981; Galkin & Stepanyants, 1991*)

There are no solitary solutions in the rKdV equation if $s = \beta\gamma > 0$.

Physical interpretation:



Dashed line – soliton speed.

If $s > 0$, a soliton is “fast” ($V_s > 0$); it generates synchronous (resonant) wave and attenuates due to radiation.

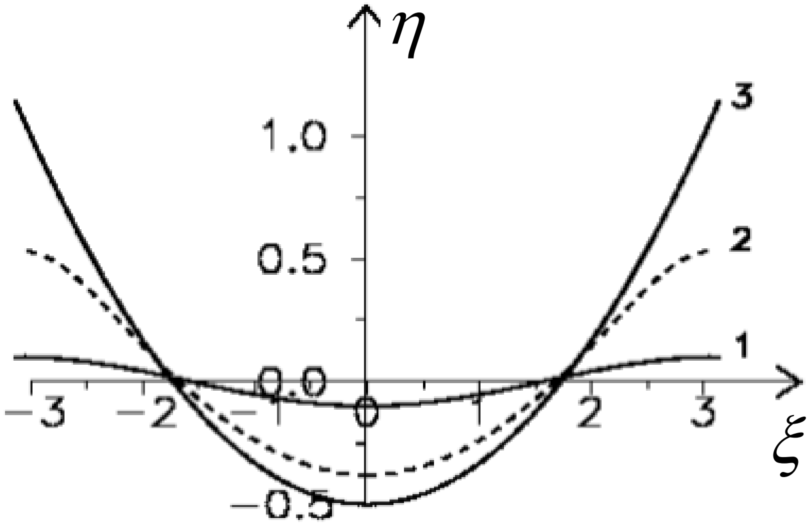
If $s < 0$, “slow” solitons ($V_s < 0$) are possible and they do not radiate.

For surface and internal oceanic waves usually $s > 0$ but in the presence of surface tension and/or shear flows, *s* can be negative for internal waves (*Alias et al., 2013; 2014*).

Reduced rKdV equation ($\beta = 0$)

$$\boxed{\left(\eta_t + c_0\eta_x + \alpha\eta\eta_x\right)_x = \gamma\eta}$$

Stationary periodic waves



Acceleration: $d^2\eta/dt^2 = \mathrm{const}$.

Limiting wave – a parabola (3):

$$\eta = \frac{\gamma}{6\alpha}\left(\xi^2 - \frac{L^2}{12}\right),$$

$$\xi = x - Vt, \quad V = \frac{\gamma L^2}{36}.$$

$$\eta_{\max} = \frac{\gamma}{36\alpha}; \qquad \eta_{\min} = -\frac{\eta_{\max}}{2}$$

Terminal decay of solitons in the rKdV equation

Solution of rKdV equation with small rotation effect can be thought in the form of the asymptotic series (*Grimshaw et al., 1998*):

$$\eta = \eta^{(0)}(\theta, T) + \varepsilon\eta^{(1)}(\theta, T) + \ldots, \quad T = \varepsilon t;$$

$$\theta = x - \int_0^t c(t')dt', \qquad \eta^{(0)} = A(T)\,\mathrm{sech}^2\left(\sqrt{A}\,\theta\right)$$

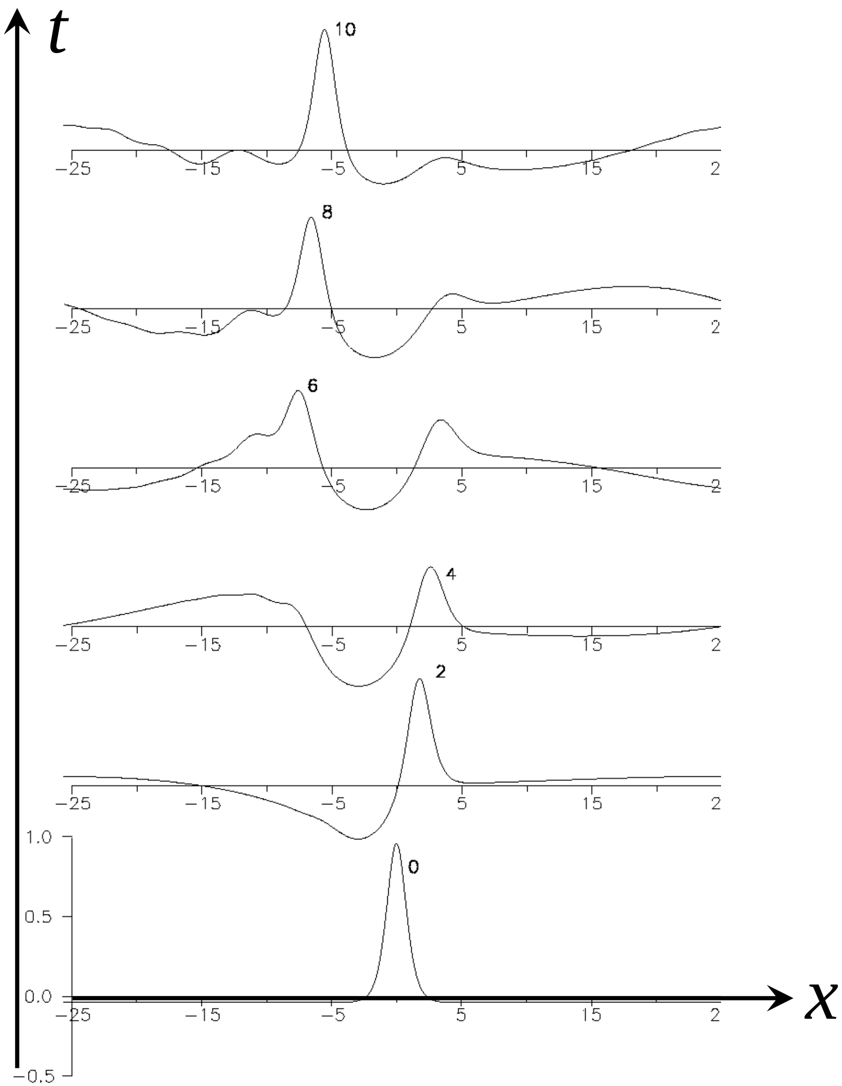
This yields the energy balance equation:

$$\boxed{\frac{d}{dt}\int_{-\infty}^{+\infty}\left(\eta^{(0)}\right)^2 d\theta = -\gamma\left(\int_{-\infty}^{+\infty}\eta^{(0)}d\theta\right)^2} \quad \Rightarrow \quad \boxed{\frac{dA}{dt} = -2\gamma\sqrt{\frac{12\beta}{\alpha}}}$$

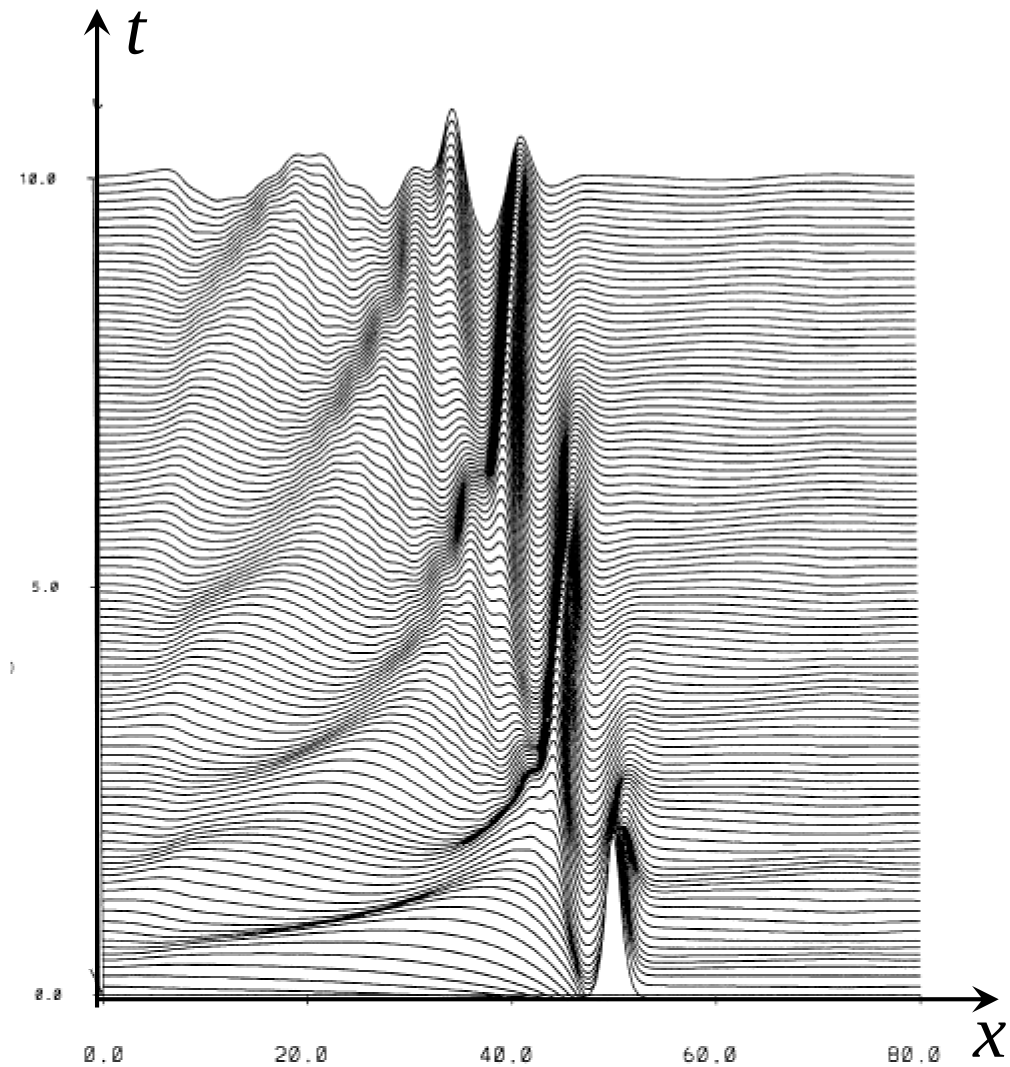
$$\boxed{A(t) = A_0\left(1 - \frac{t}{T_e}\right), \quad T_e = \frac{1}{\gamma}\sqrt{\frac{\alpha A_0}{12\beta}}.}$$

The extinction time for oceanic internal solitons is $T_e \sim 1$ day.

Numerical modelling. Early stage of soliton evolution



(*Ostrovsky & Stepanyants, 1990; Gilman et al., 1996*)

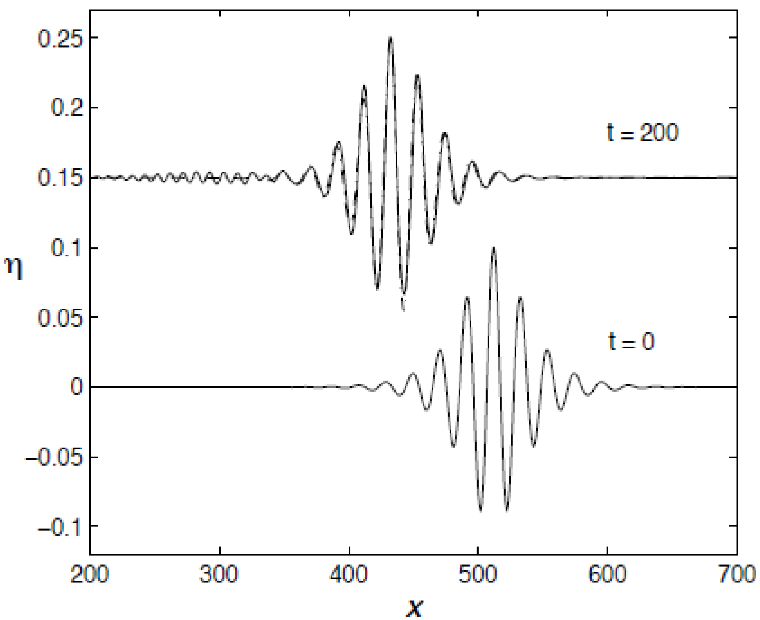


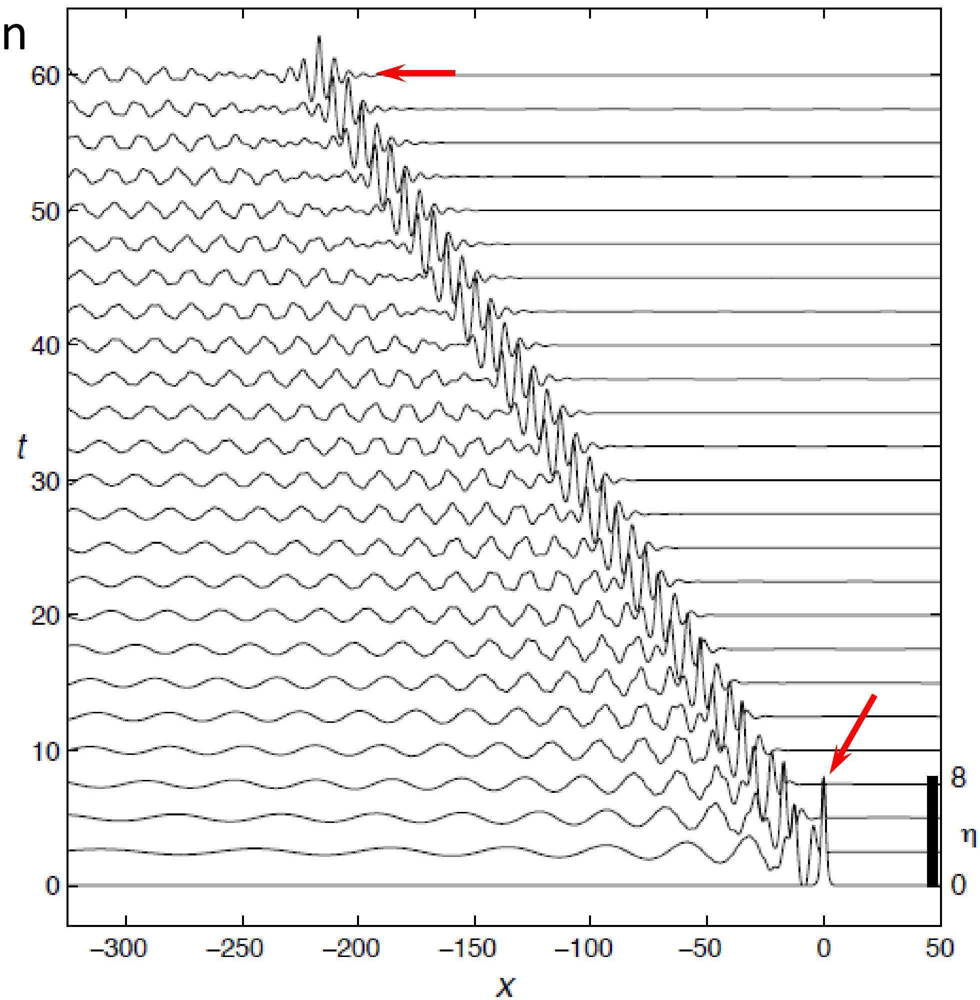
(*Grimshaw et al., 1998*)

Long-term evolution of a KdV soliton

The long-term evolution of a KdV soliton ends up with the formation of the NLS envelope soliton

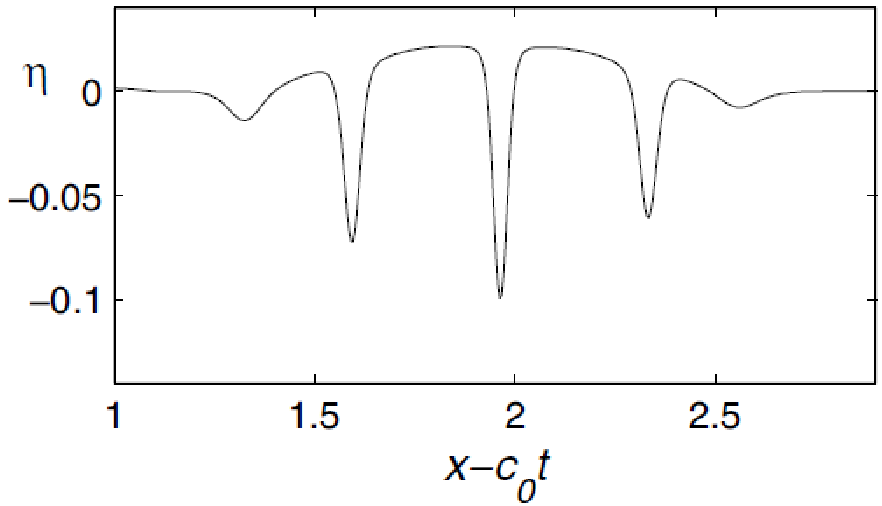
(*Grimshaw & Helfrich, 2008*)



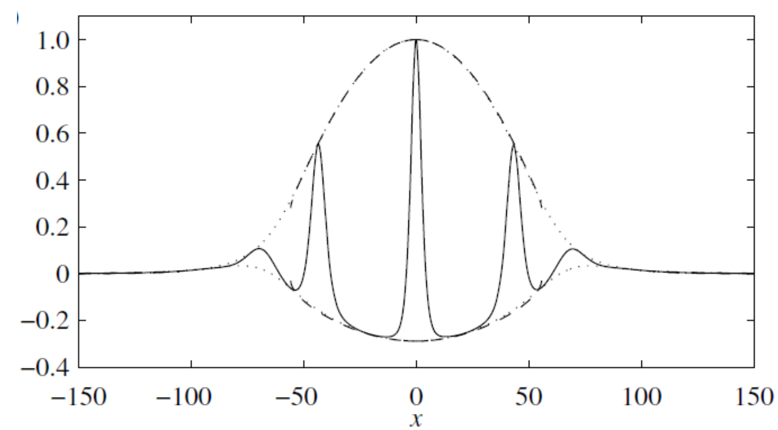


Influence of initial soliton amplitude on the resultant wave train

As has been shown trough the numerical modelling (Helfrich, 2007), the result of evolution of initial solitary wave of small and moderate amplitude is very similar in the fully nonlinear rotation modified Miyata–Choi–Camassa (MCC-f) model, MCC-f model with the linear Boussinesq dispersion, and rKdV model.

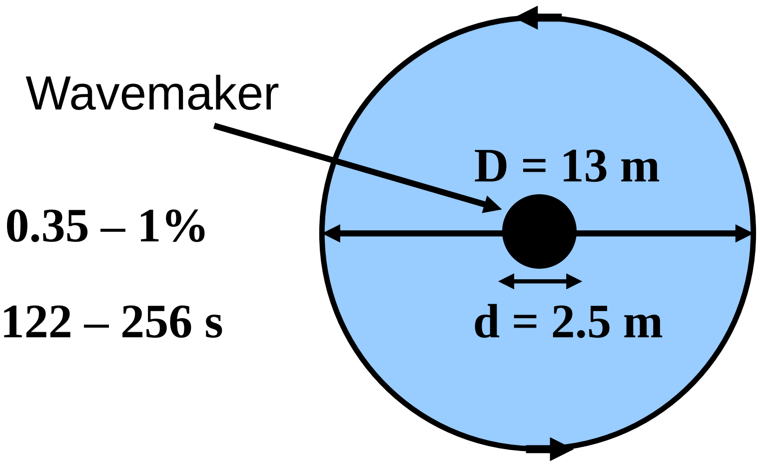


The long-term evolution of a KdV soliton of a large amplitude in two-layer fluid within the fully nonlinear MCC-f model (*Helfrich, 2007*).



The long-term evolution of a KdV soliton of a large amplitude with the rKdV equation (*Whitfield & Johnson, 2016*).

Experimental study of nonlinear waves in a rotating tank



$\Delta\rho/\rho$ = 0.35 – 1%

T_r = 122 – 256 s

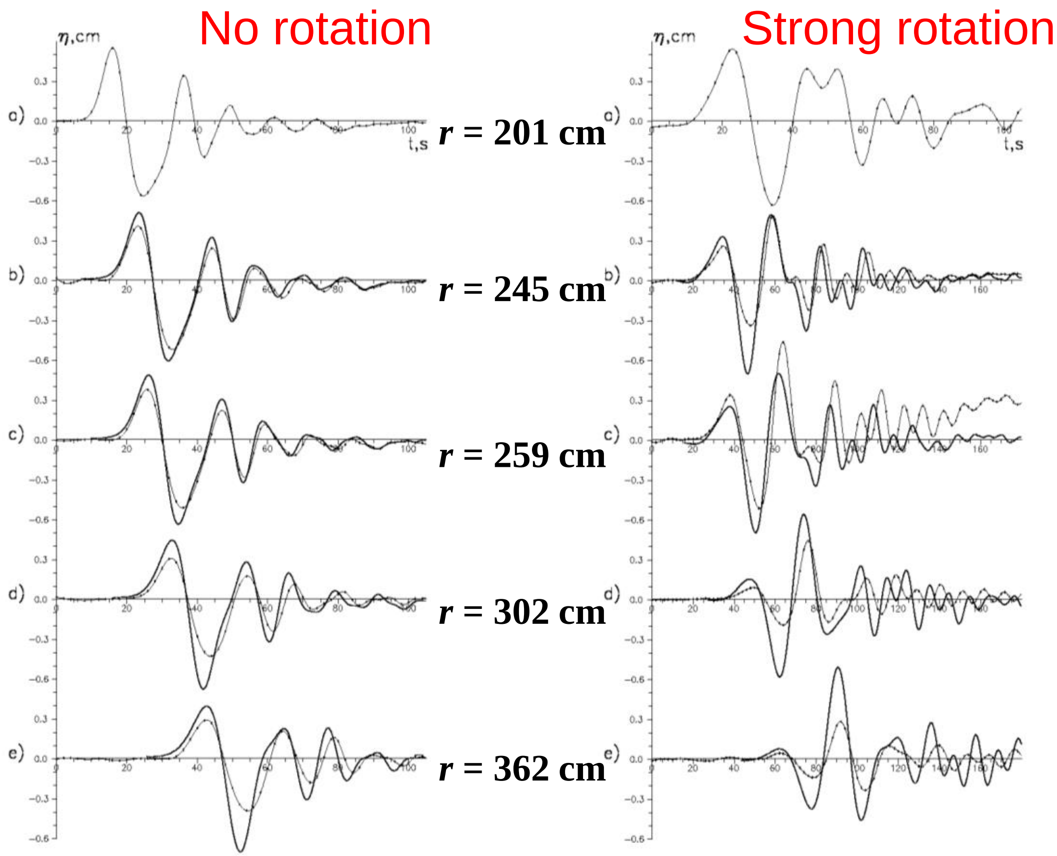
Depth, h = 0.25 m

Grenoble rotating platform “Coriolis”
Internal waves were generated in a 2-layer fluid
(*Ramirez et al., 2002*)

Comparison with experiments

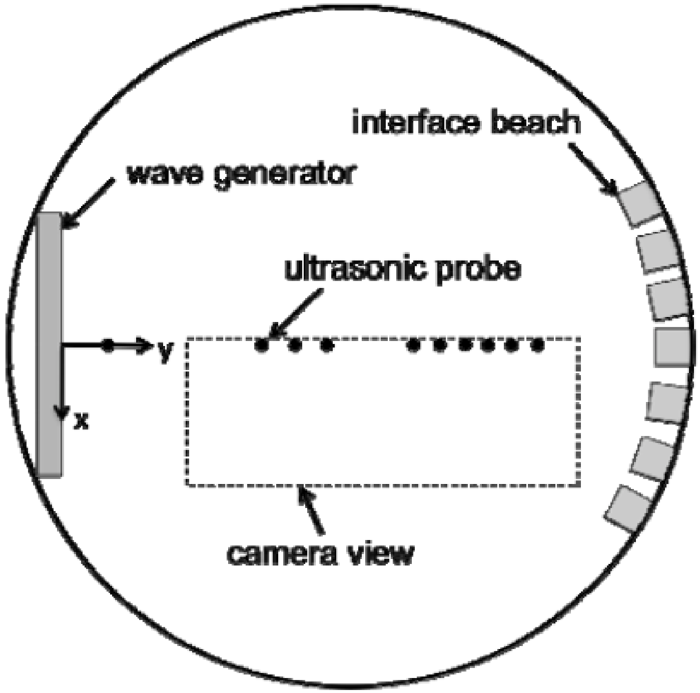
$$\boxed{\left(\eta_r + \frac{1}{c_0}\eta_t - \alpha\eta\eta_t - \beta\eta_{ttt} + \frac{\eta}{2r}\right)_t = \frac{f^2}{2c_0}\eta}$$

– cylindrical rKdV



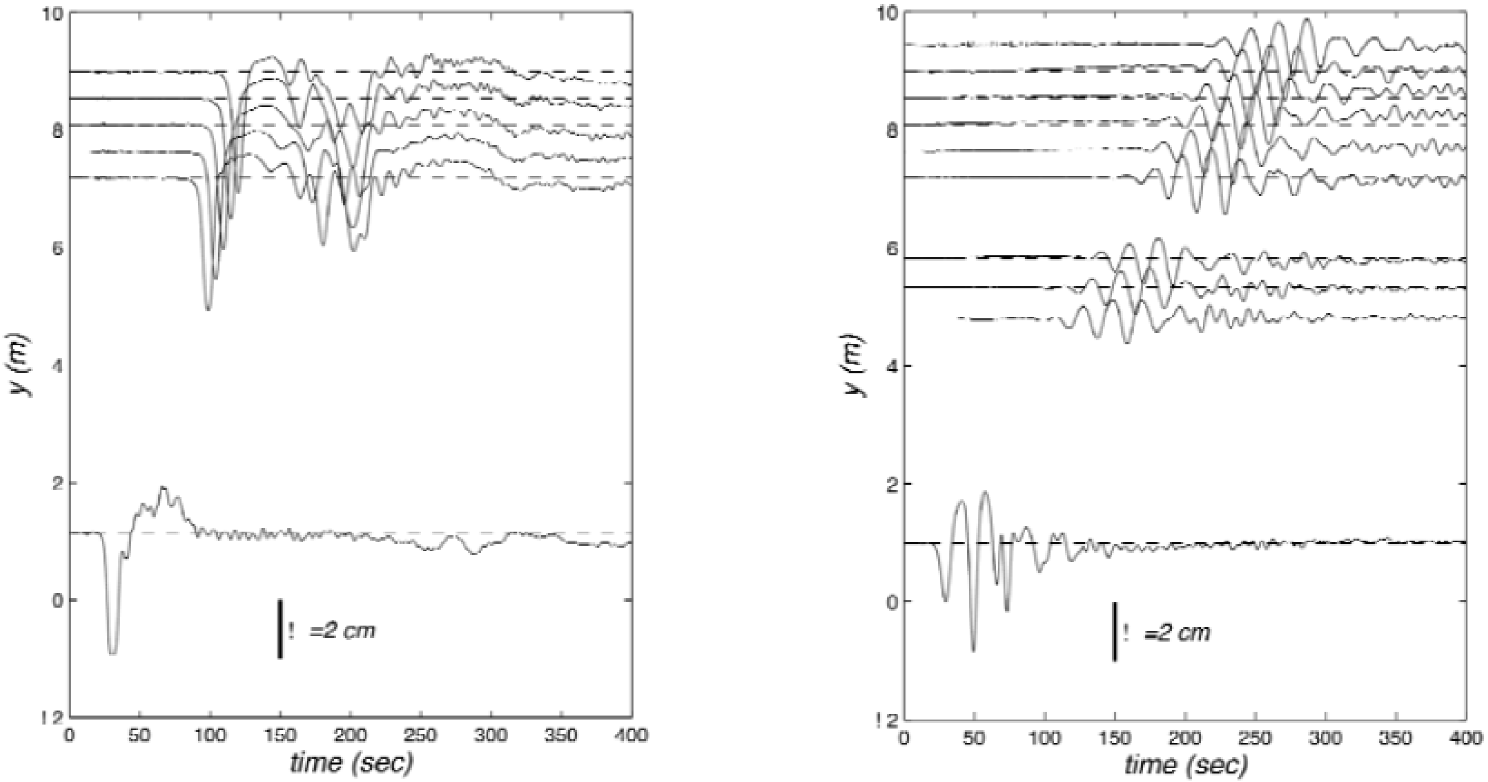
Upper plots-initial waveform at r = 201 cm.
Gray lines-experiment
Black lines- calculations from cylindrical rKdV

Laboratory experiment in the "Coriolis" platform, Grenoble, France (*Grimshaw et al., 2010*)



Plan view sketch of the experimental setup.

Laboratory experiment in the “Coriolis” platform, Grenoble, France (*Grimshaw & Johnson 2013*)

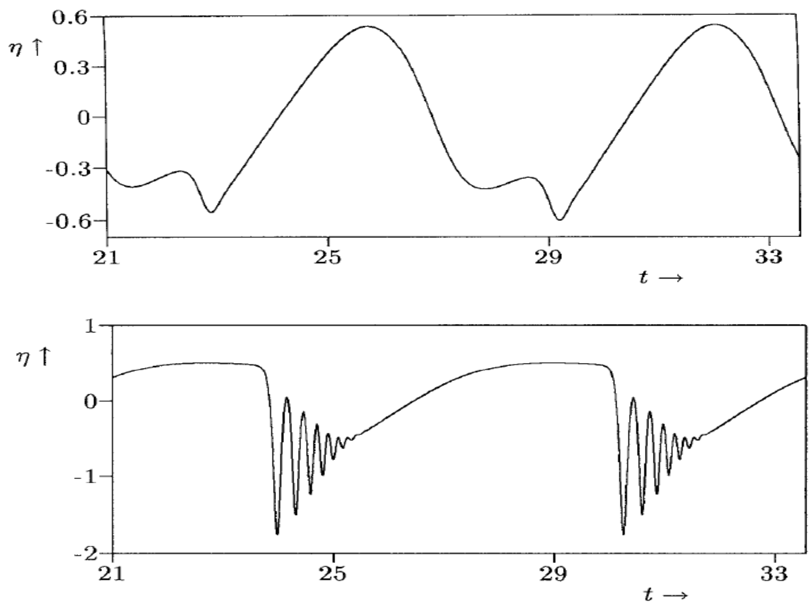


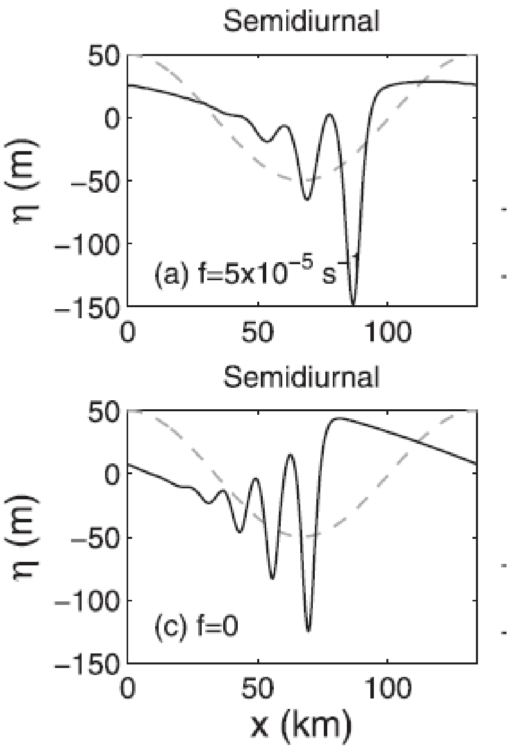
Left: solitary wave evolution in a nonrotating case.

Right: Formation of an envelope soliton in the rotating case.

Modeling for real ocean

Theo Gerkema in his PhD Thesis (1994) has shown that even a small rotation reduces the number of evolving quasi-solitons.



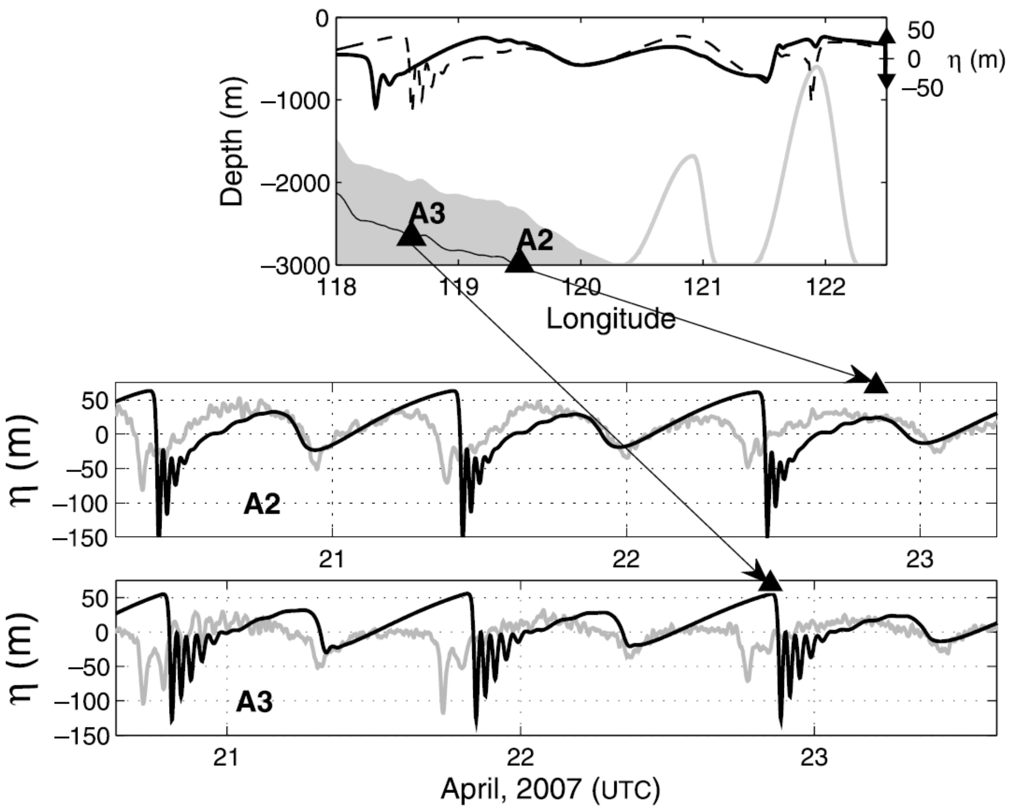


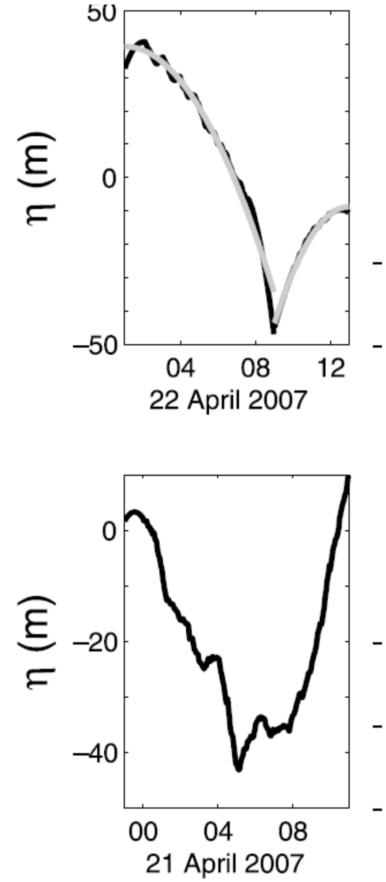
Calculation of pycnocline displacement during two tidal periods corresponding to the observational data in Celtic Sea (*Pingree and Mardell, 1985*) Top - with rotation. Bottom - without rotation (*Gerkema, 1996*).

Rotation effect for a harmonic initial condition, 50 m amplitude (dashed lines). Solid lines: shapes after 30 h. Top - with rotation. Bottom - without rotation (*Li & Farmer, 2011*). Rotation decreases the number of quasi-solitons. South China Sea (SCS)

Comparison with the South China Sea experiment

(*Farmer et al., 2009*)





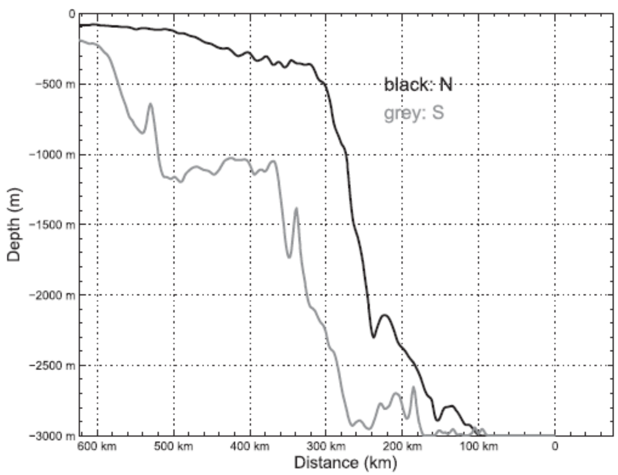
Top: Modeled displacement with (solid) and without (dashed) rotation. Lower panels: Observed (grey) and modeled without rotation (black) time series.

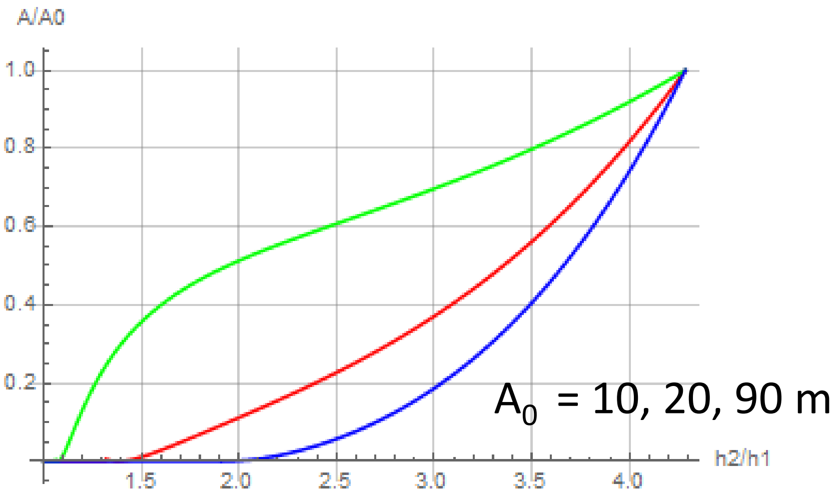
Top: Parabolic function (grey) fitted To A3 time series (black).
Bottom: Corresponding data at A1
Propagation: from A1 to A3

$$O_s = \frac{24\pi c_0 \alpha A}{f^2 L^2} = 1.87$$

Horizontally variable environment

(Grimshaw et al., 2014; Ostrovsky & Stepanyants, 2018)





2-Layer model

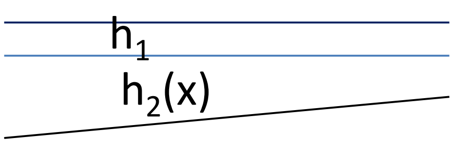
$$A = A_0 \left[\frac{(s-1)s_0^2}{(s_0-1)s^2} \right]^{1/3} \left[1 - q_1 \int_{s_0}^{s} \left(\frac{1+s}{s} \right) \left(\frac{s^2}{s-1} \right)^{2/3} ds/(ds/dx) \right]^2, \; s = h_2/h_1 \geq 1$$

$$q_1 \sim f^2 > 0$$

The integral *I* is expressed via hypergeometric functions

Flat sloping bottom:

$$A = A_0 \left[\frac{s_0^2(s-1)}{s^2(s_0-1)} \right]^{1/3} \left[1 - |qI| \right]^2, \; q \sim f^2 > 0, \; I < 0$$

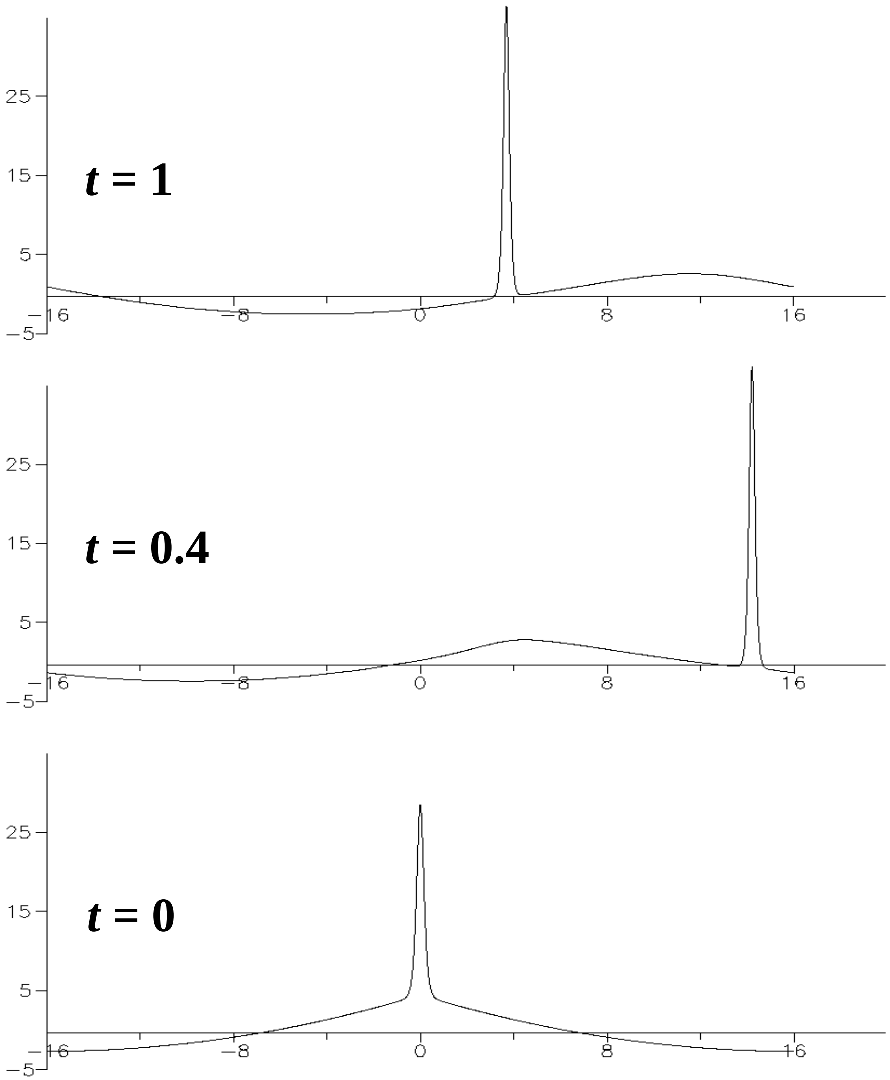


Three factors play here: 1) terminal damping, 2) zeroing quadratic nonlinearity, and 3) reaching the beach. For this case terminal damping always occurs first.

Soliton on a long wave. Numerical result

(*Gilman et al., 1996*)

Solitary waves can exist in the rKdV equation, if they are supported by a background wave due to the balance between the energy "pumping" and radiation losses.



Soliton on a long wave. Asymptotic theory

(Ostrovsky & Stepanyants, 2016)

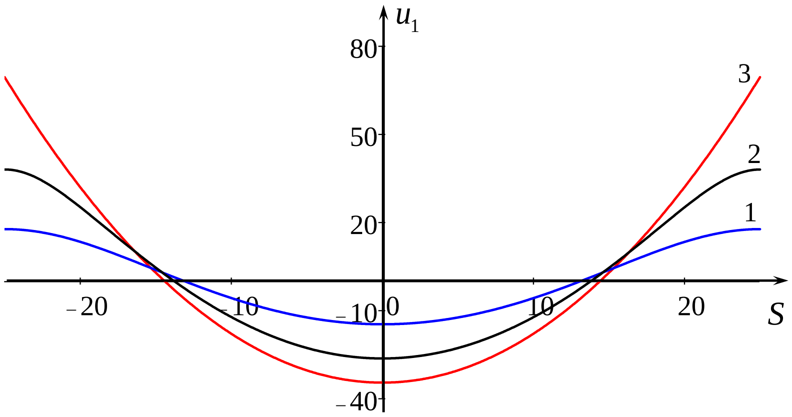
$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)=u}$$

$$u = u_1(S) + u_2(t,x),\ S = x - ct$$

u_1 is a given background long wave:

$$\frac{d^2}{dS^2}\left(\frac{1}{2}u_1^2 - c_1 u_1\right)=u_1 \Rightarrow$$

A family of long background waves:



u_2 is a KdV soliton + perturbation:

$$\boxed{u_2 = A\,\mathrm{sech}^2\frac{\zeta - S}{\Delta} - p}$$

$$\zeta = x - \int_0^t V\,dt,\quad V = \frac{A}{3} - p + u_1(S),\quad \Delta = \sqrt{\frac{12}{A}},\quad p = \frac{4\sqrt{3A}}{\Lambda}$$

(p secures zero mass)

Dynamics of a soliton as a particle

$$\frac{\partial u_2}{\partial t}+u_2\frac{\partial u_2}{\partial x}+\frac{\partial^3 u_2}{\partial x^3}=-\frac{\partial\left(u_1 u_2\right)}{\partial x}$$

Multiplying this by u_2 and integrating over x, one obtains:

$$\boxed{\frac{dA}{dt}=-2A\frac{du_1\left(S\right)}{dS}}$$

with the kinematic equation

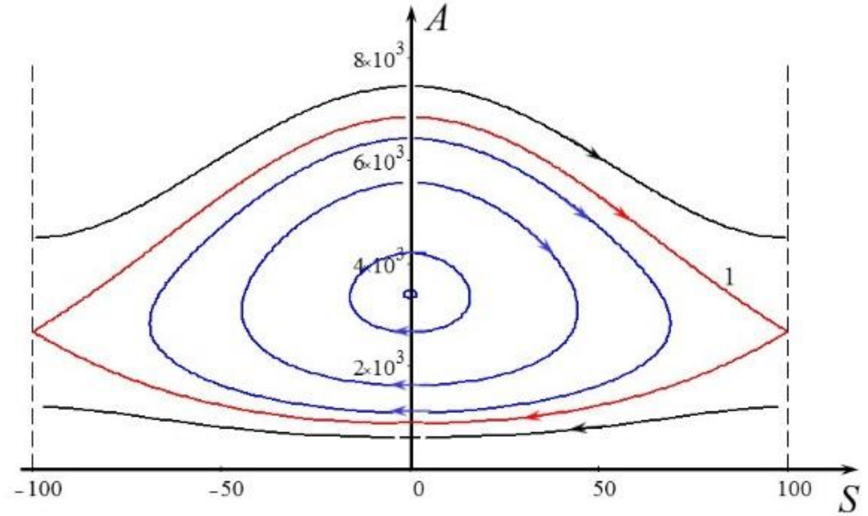
$$\boxed{\frac{dS}{dt}=\frac{A}{3}+u_1\left(S\right)-c_0}$$

This is a dynamical system for a soliton as a particle localized in the point $S(t)$ on the background wave u_1. c_0 is the long wave velocity.

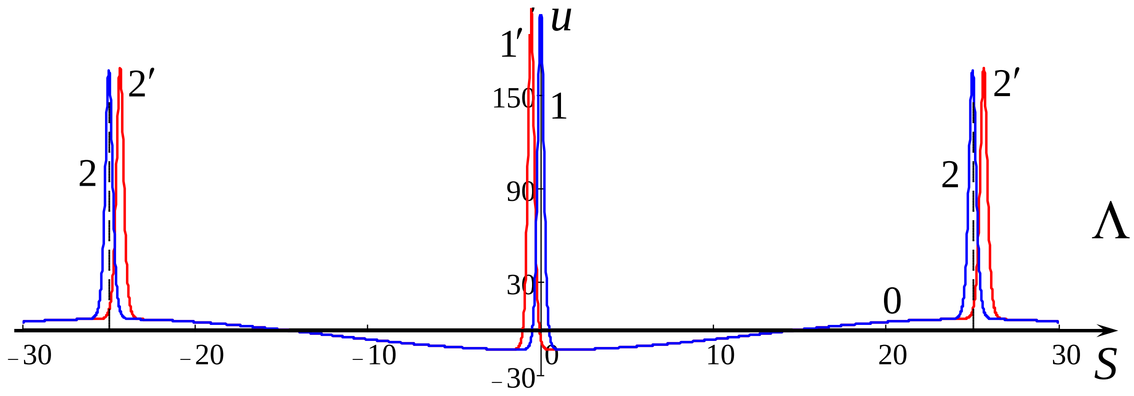
Soliton on a sinusoidal wave

$$\frac{dS}{dt} = \frac{A}{3} - U_0 \cos kS - \frac{1}{k^2}$$

$$\frac{dA}{dt} = -\frac{4}{3} A U_0 k \sin kS$$



Phase portrait on the (A,S) plane for
$\Lambda = 2\pi / k = 200$ and $U_0 = 120$.

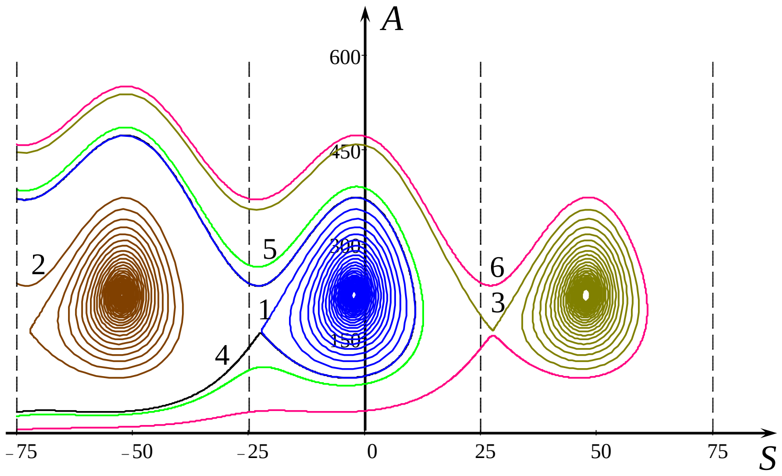


$\Lambda = 50; \quad U_0 = 10$

Stationary solitons on the sinusoidal background
Red-same with radiation losses (see next slide)

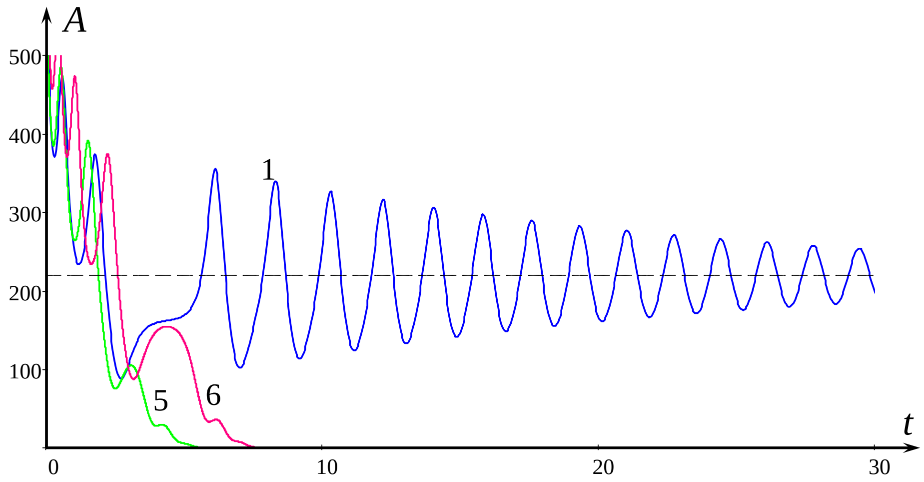
The same system, but with the radiative losses

$$\frac{dS}{dt} = \frac{A}{3} - U_0 \cos kS - \frac{1}{k^2} \qquad \frac{dA}{dt} = -2AU_0 k \sin kS - 4\sqrt{3A}$$



Phase portrait on the (A, S) plane with the radiation losses.

$\Lambda = 50$ and $U_0 = 10$.

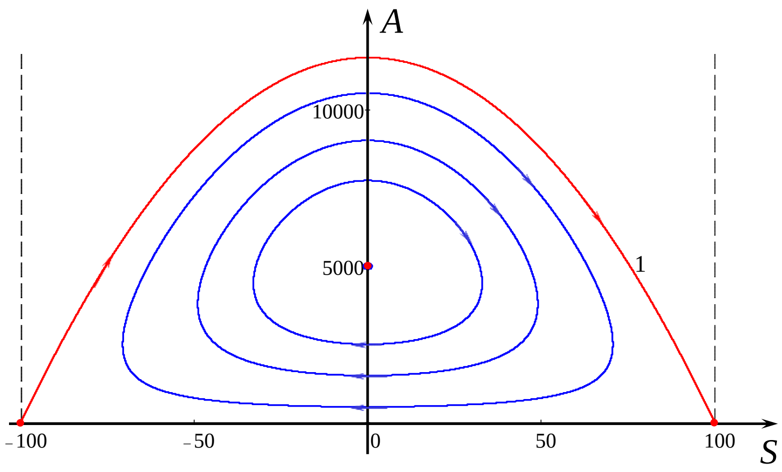


Variation of soliton amplitude for selected trajectories on the phase plane. The numbers correspond to those on the left.

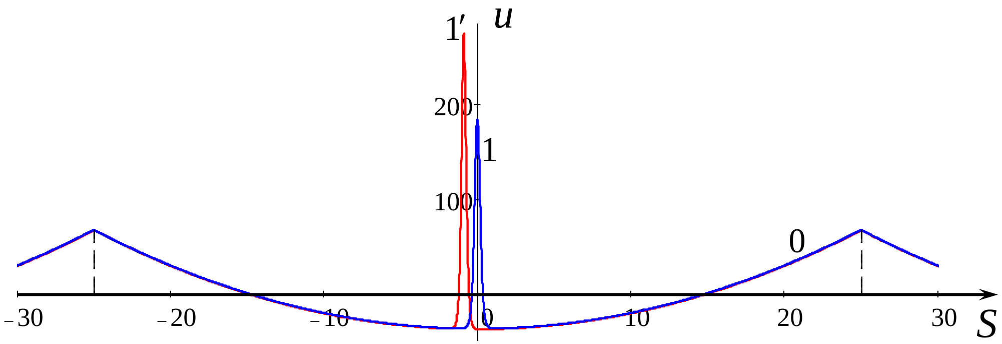
Soliton on a parabola

$$u_1(S)=\frac{1}{6}\left(S^2-\frac{\Lambda^2}{12}\right),\qquad -\frac{\Lambda}{2}\le S\le\frac{\Lambda}{2},\qquad U_{\max}=\frac{\Lambda^2}{36},\; c=\frac{\Lambda^2}{4\pi^2}$$

$$\frac{dS}{dt}=\frac{A}{3}+\frac{S^2}{6}-\frac{\Lambda^2}{24}\qquad\qquad \frac{dA}{dt}=-\frac{4}{9}AS$$



Phase portrait for a parabolic background without radiation losses.



Equilibrium soliton on a parabola.
Blue line – without losses,
red line - with radiation losses.

Two solitons on a parabola

$$\frac{dS_1}{dt} = \frac{A_1}{3} + \frac{1}{6}\left(S_1^2 - \frac{\Lambda^2}{12}\right) + A_2 \exp\left(-\sqrt{\frac{A_2}{3}}\left|S_2 - S_1\right|\right) - \frac{\Lambda^2}{36};$$

$$\frac{dS_2}{dt} = \frac{A_2}{3} + \frac{1}{6}\left(S_2^2 - \frac{\Lambda^2}{12}\right) + A_1 \exp\left(-\sqrt{\frac{A_1}{3}}\left|S_2 - S_1\right|\right) - \frac{\Lambda^2}{36};$$

$$\frac{dA_1}{dt} = -4\sqrt{3A_1} - \frac{4}{9}A_1 S_1 - \frac{16}{3}A_1 A_2 \sqrt{\frac{A_2}{3}} \exp\left(-\sqrt{\frac{A_2}{3}}\left|S_2 - S_1\right|\right),$$

$$\frac{dA_2}{dt} = -4\sqrt{3A_2} - \frac{4}{9}A_2 S_2 + \frac{16}{3}A_1 A_2 \sqrt{\frac{A_1}{3}} \exp\left(-\sqrt{\frac{A_2}{3}}\left|S_2 - S_1\right|\right)$$

In the forthcoming examples, $\Lambda = 100$

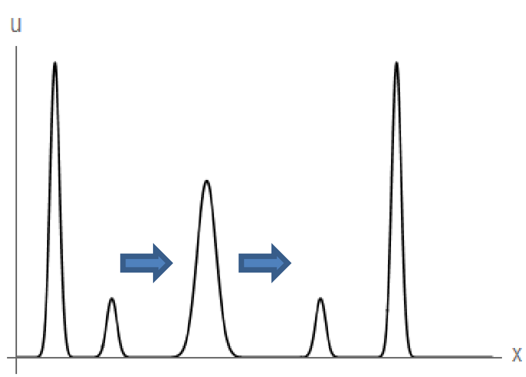
Two solitons without rotation

$$\frac{dS_1}{dt} = \frac{A_1}{3} + A_2 \exp\left(-\sqrt{\frac{A_2}{3}}\,|s|\right);$$

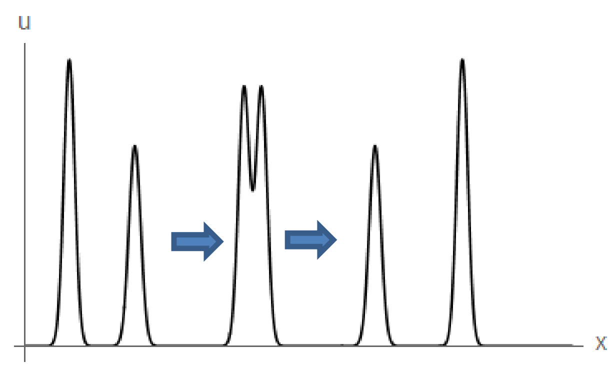
$$\frac{dS_2}{dt} = \frac{A_2}{3} + A_1 \exp\left(-\sqrt{\frac{A_1}{3}}\,|s|\right);$$

$$\frac{dA_1}{dt} = -\frac{16}{3} A_1 A_2 \sqrt{\frac{A_2}{3}} \exp\left(-\sqrt{\frac{A_2}{3}}\,|s|\right);$$

$$\frac{dA_2}{dt} = \frac{16}{3} A_1 A_2 \sqrt{\frac{A_1}{3}} \exp\left(-\sqrt{\frac{A_2}{3}}\,|s|\right)$$



Overlapping interaction

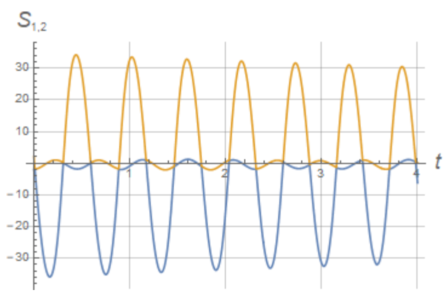


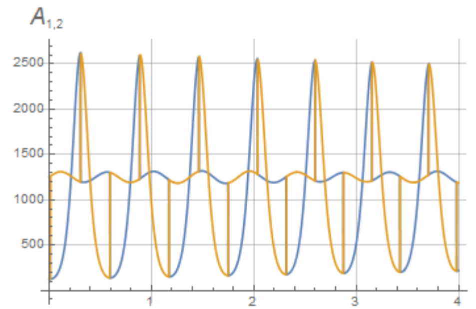
Exchange interaction

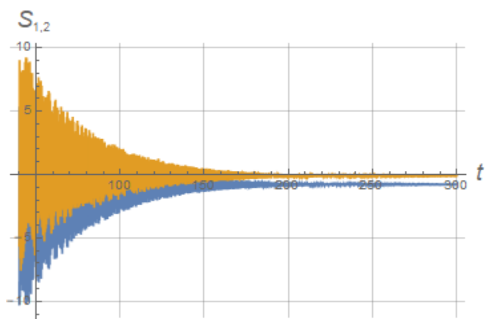
The rear soliton always provides energy to the frontal one

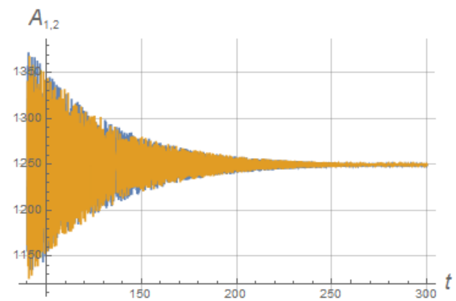
Two solitons on a background. Overlapping interaction.

$S_1(0) = -2, \quad S_2(0) = 2, \quad A_1(0) = 1250, \quad A_2(0) = 125$

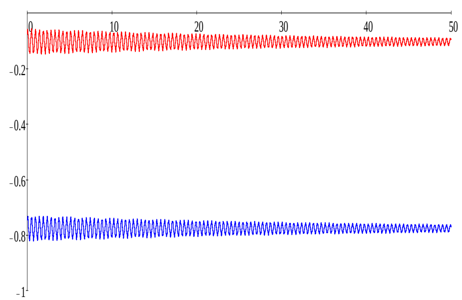


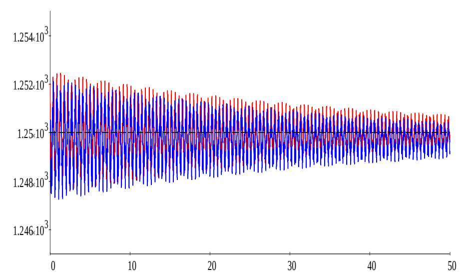






Asymptotics is two solitons of equal amplitudes at different (not symmetric!) positions

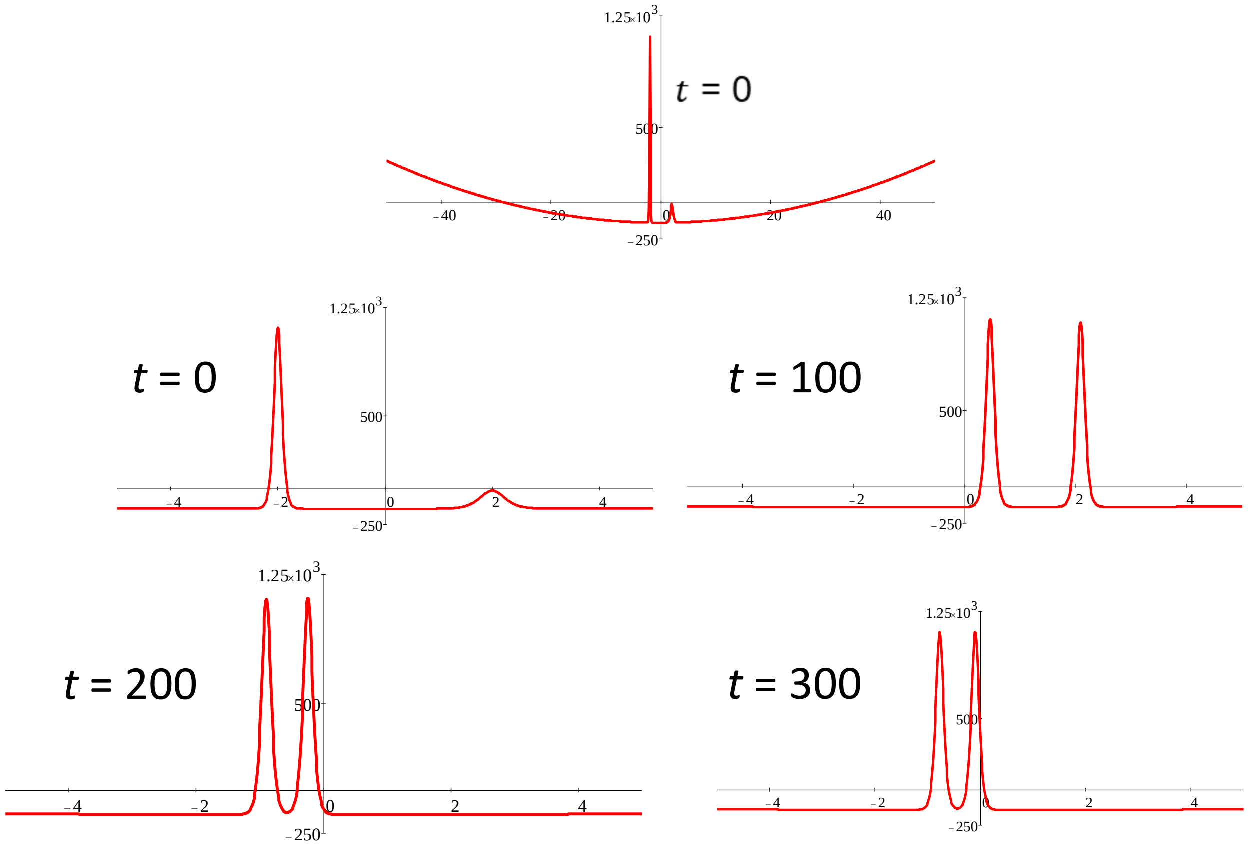




Long-time stage of evolution

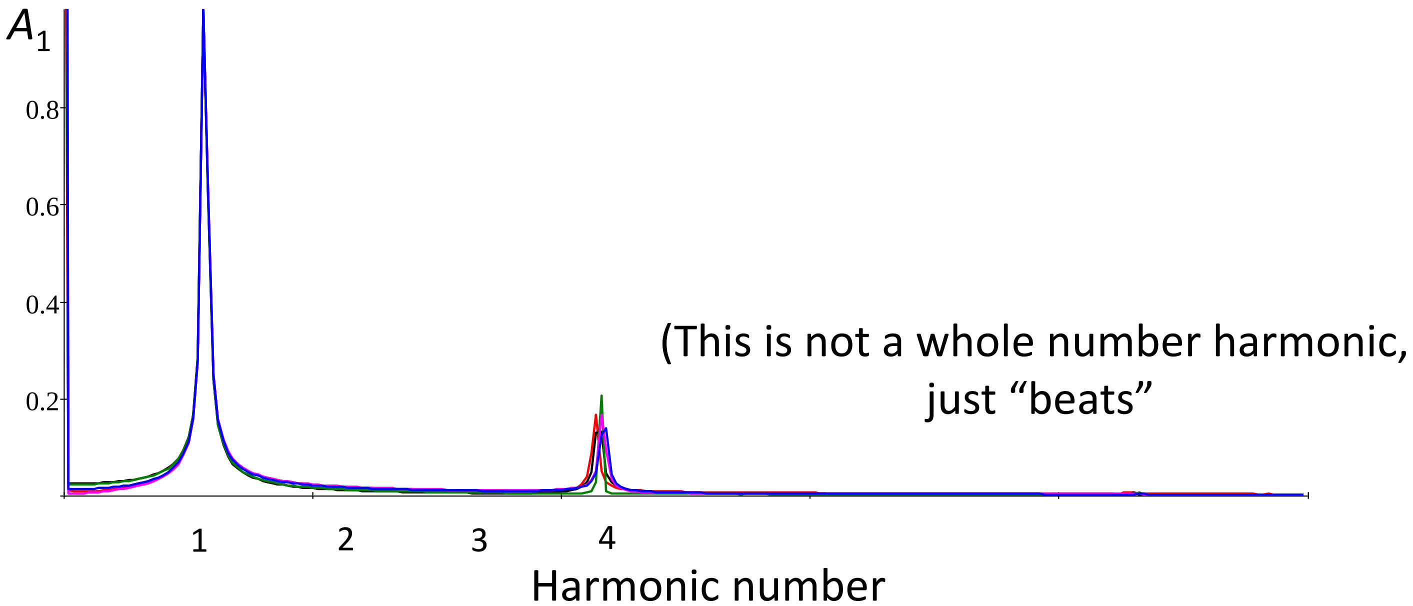
Space-time dynamics.

Overlapping interactions becomes exchange interactions



Solitons’ amplitudes quickly become close to each other

Spectra of overlapping soliton interactions



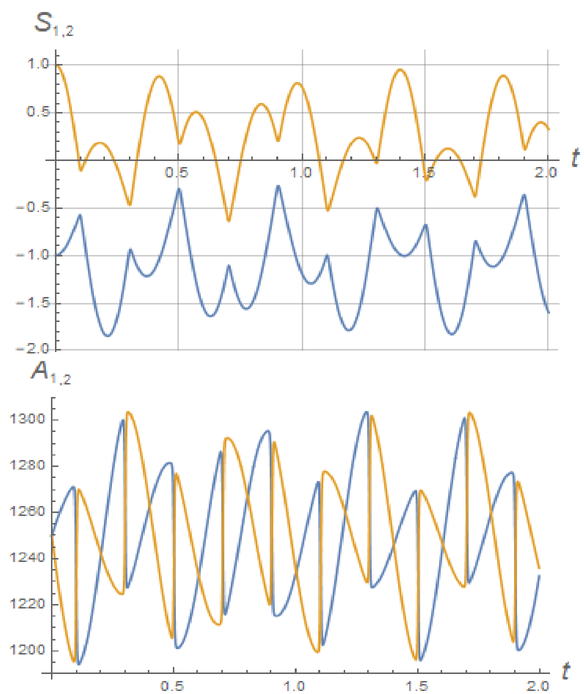
Normalized spectra of oscillations of soliton amplitude A_1:
1 - for $0 < t < 10$; 2 - for $10 < t < 20$; 3 – for $20 < t < 30$; 4 - for $30 < t < 40$;
5 - for $40 < t < 50$.

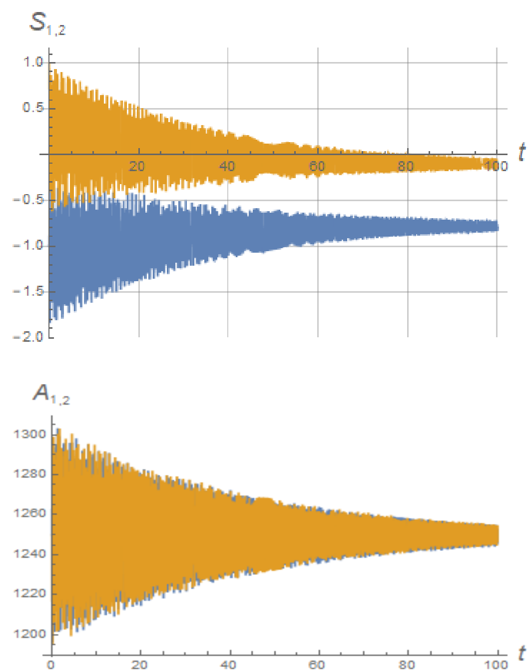
The basic oscillations are due to the "pump".

The faster oscillations are due to the interaction.

Two solitons on the background.
Exchange interaction.

$S_1(0) = -1, \quad S_2(0) = 1, \quad A_1(0) = 1250, \quad A_2(0) = 1250$



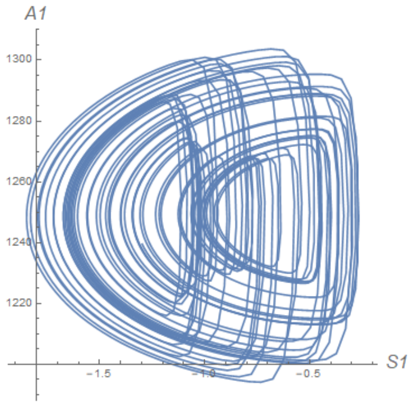


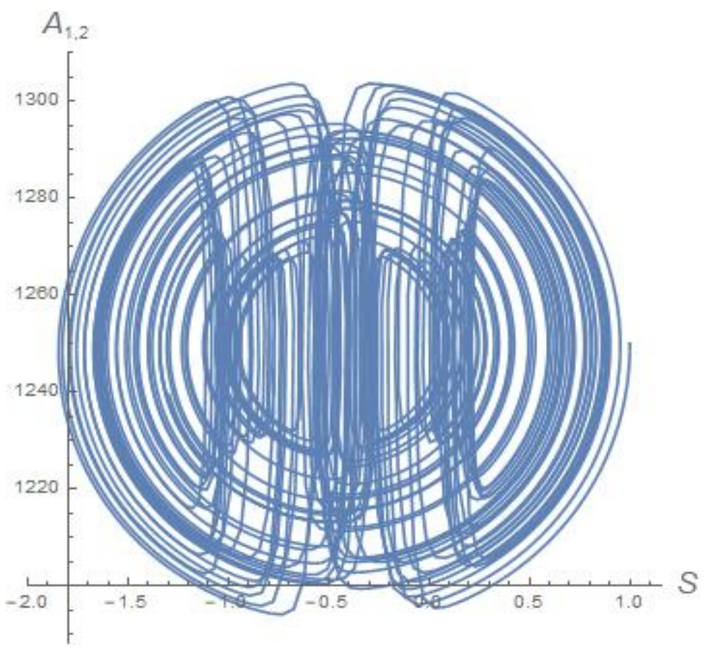
Again, asymptotics is two solitons of equal amplitudes at different (not symmetric!) positions.

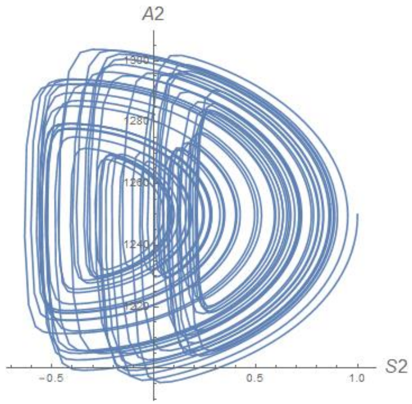
In this case solitons never overlap

Projections of the 4D phase space

$0 < t < 10$

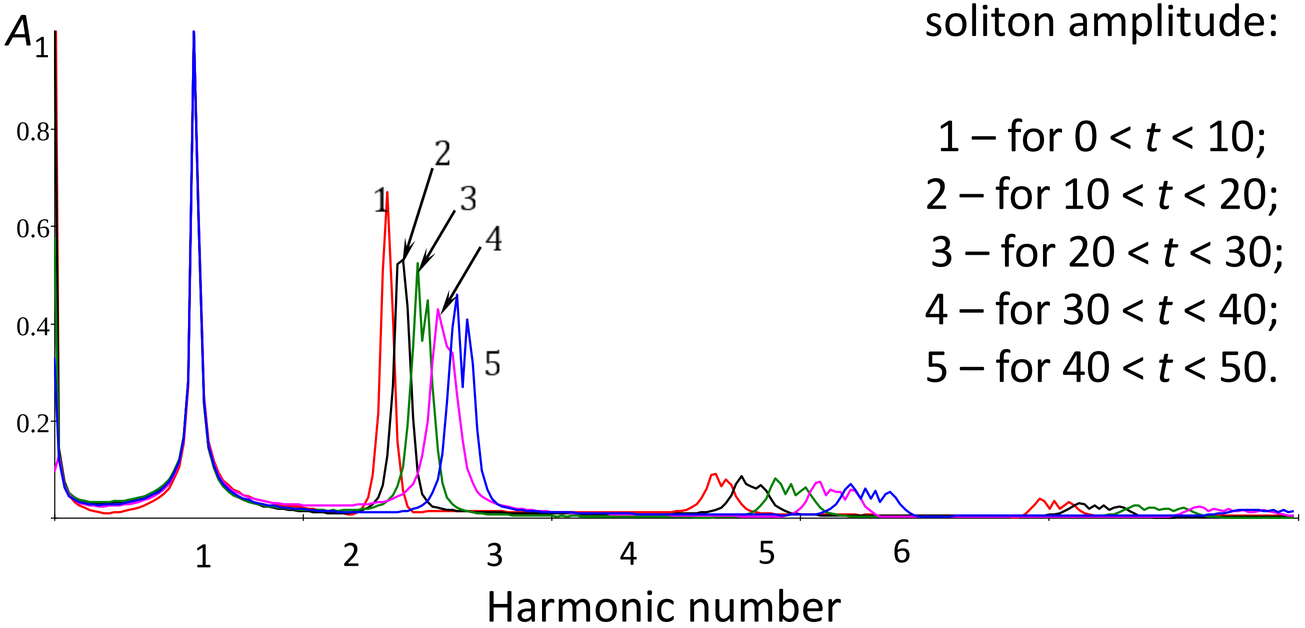






Again, there is a shift to the left.

Spectra of exchange soliton interactions



Normalized spectra of oscillations of soliton amplitude:

1 – for $0 < t < 10$;
2 – for $10 < t < 20$;
3 – for $20 < t < 30$;
4 – for $30 < t < 40$;
5 – for $40 < t < 50$.

Again, the basic oscillations are due to the “pump”.
The faster oscillations are due to the interaction
The exchange becomes faster in time.

Solitary waves in the case of $s = \beta\gamma < 0$

The dimensionless form of rKdV equation in this case is

$$(u_t + uu_x - u_{xxx})_x = u$$

For stationary solutions of the form $u = u(x + V_s t)$

the equation reduces to the ODE:

$$(u'' - V_s u - u^2/2)'' + u = 0$$

Linearization at the tails:

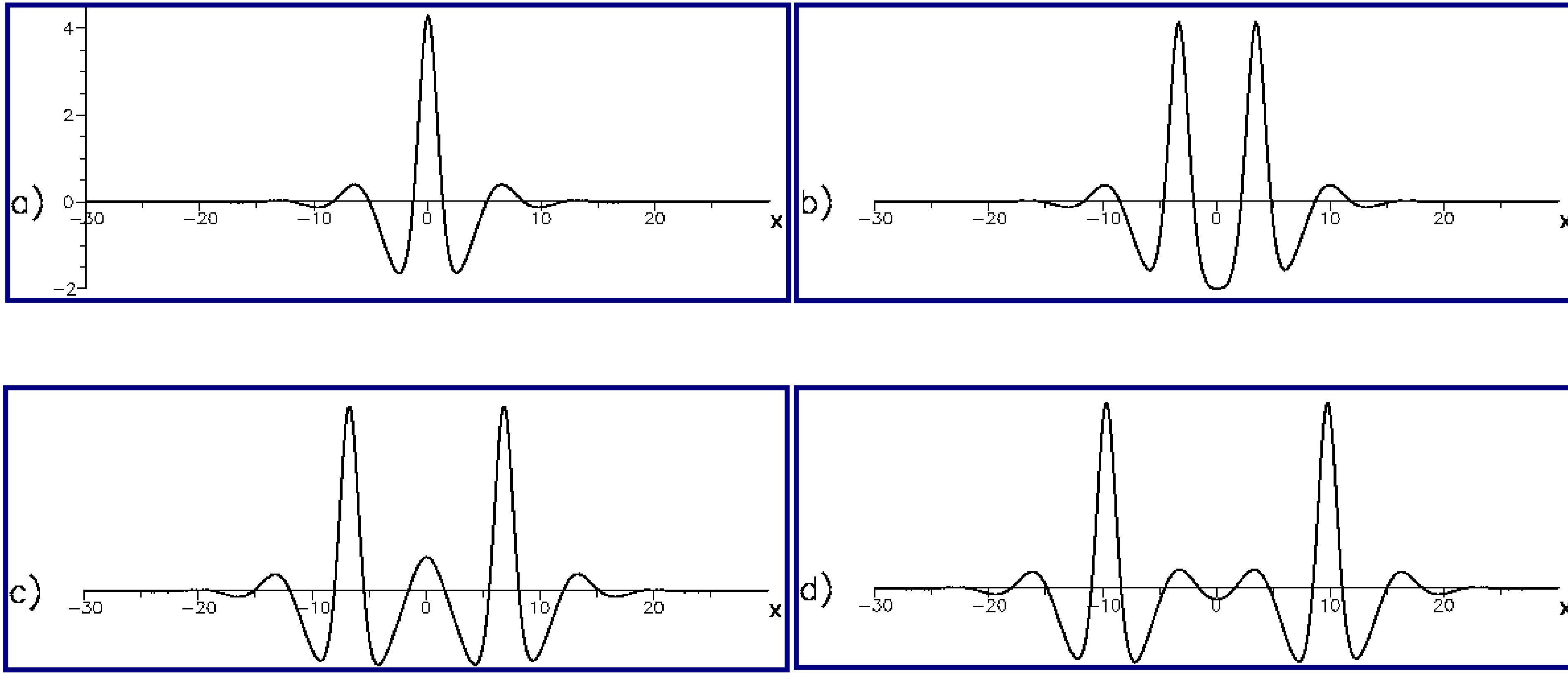
$$(u'' - V_s u)'' + u = 0$$

for the solution in the form $u \sim e^{\mu x}$ gives the characteristic equation:

$$\mu^4 - V_s \mu^2 + 1 = 0$$

Solitons and multi-solitons, computed

(*Obregon & Stepanyants, 1998*)



From single soliton through the bi-solitons to multisolitons,

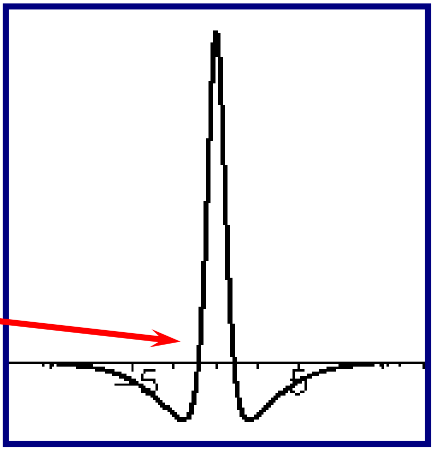
then, to the random chains of solitons and "frozen turbulence"

(*Gorshkov et al., 1979*)

Possible soliton asymptotics and real profiles

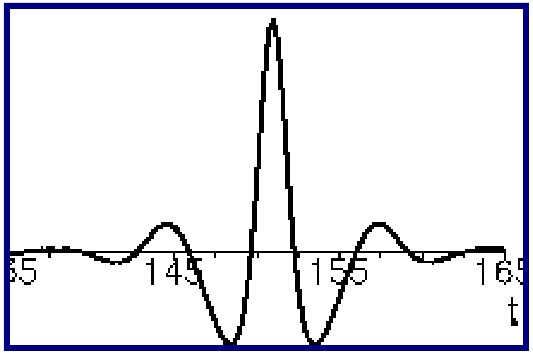
The roots of the characteristic equation are: $\mu = \pm\sqrt{\dfrac{V_s}{2} \pm \sqrt{\left(\dfrac{V_s}{2}\right)^2 - 1}}$

a) $V_s \le -2$ – the roots are purely imaginary;
no solitary waves, $u \sim \cos(\mu\,\xi)$

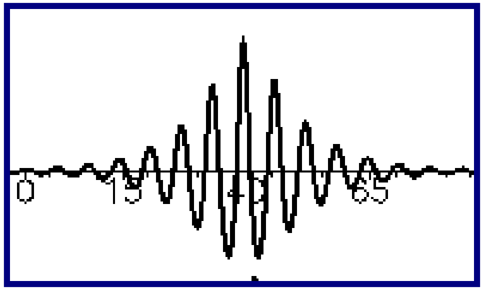


b) $V_s \ge 2$ – the root are purely real, $u \sim e^{\pm\mu\,\xi}$

c) $-2 < V_s < 2$ – the roots are complex, $u \sim \mathrm{e}^{\pm\alpha\,\xi}\cos(\beta\xi - \varphi)$

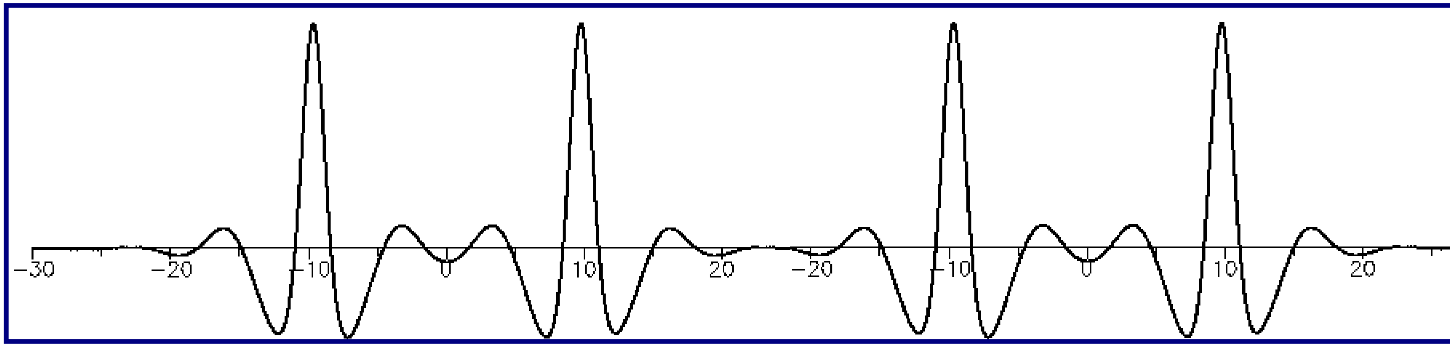


$$\int_{-\infty}^{+\infty} u(x,t)\,dx = 0$$



Dynamics of solitons as classical particles

(*Gorshkov et al., 1976; Kawahara & Takaoka, 1988; Obregon & Stepanyants, 1993*)



$$m\ddot{s}_n = f\left(s_n - s_{n-1}\right) - f\left(s_{n+1} - s_n\right)$$

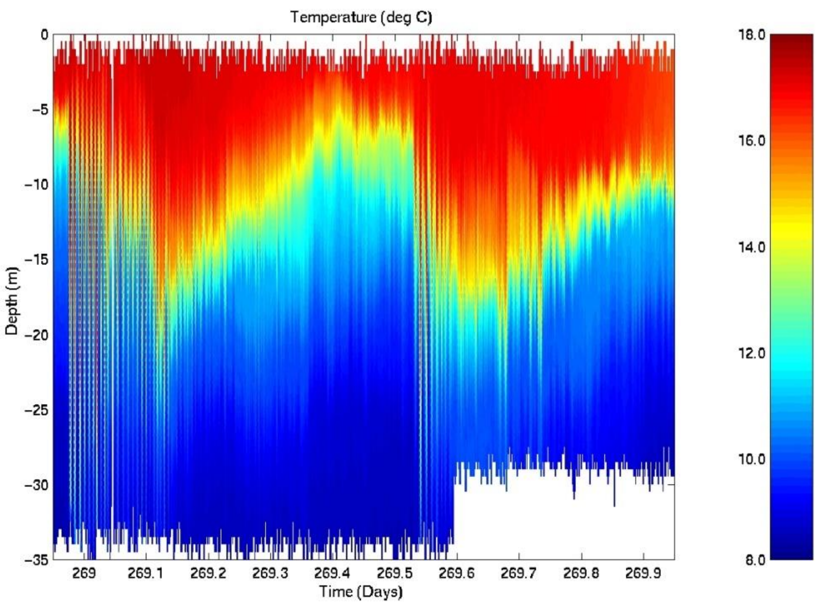
$$f(x) = Ae^{-\alpha x}\cos\left(\beta x - \varphi\right)$$

What is worth doing in the perspective

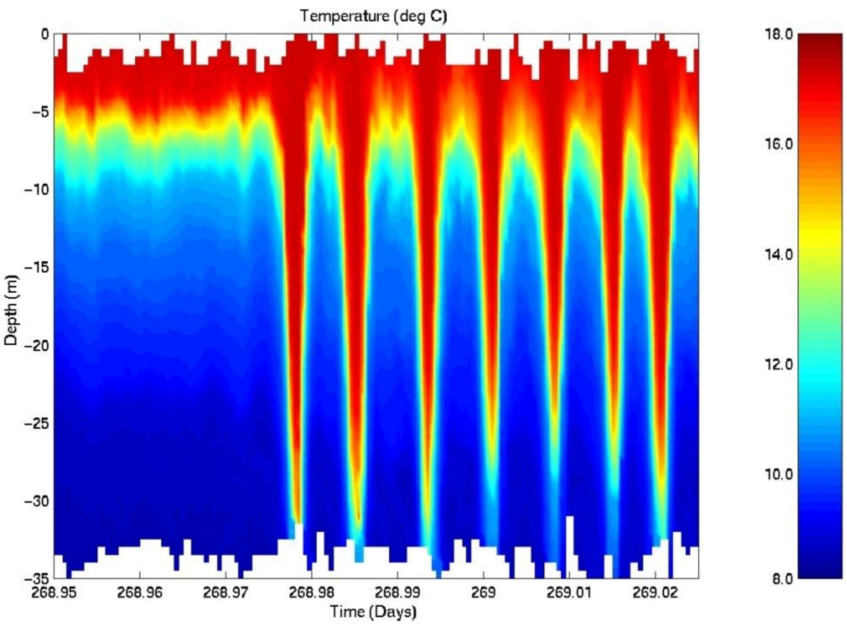
(to name a few)

- Could the non-integrability of the rKdV equation be proven rigorously?
- 2D in a horizontal direction (geometrical theory of waves with the rotation and nonlinearity)
- Really strong nonlinearity plus rotation (*Re: K. Helfrich*)
- Analysis of oceanic data for evaluating the role of Coriolis force (too few examples exist)
- How does rotation affect nonlinear magnetic sound in the ionosphere? This includes negative dispersion (solitons)

Really strong internal solitons



A



B

Groups of extremely strong internal solitary waves generated by tide on the North-West Pacific shelf off the US. The color bar shows temperatures in centigrade.
(A) Two tidal periods, (B) The first 1.7 hours of the first tidal period. Nonlinearity is so strong that the Korteweg–de Vries equation is absolutely inapplicable.
From Stanton & Ostrovsky, Geophys. Res. Lett., v. 25, 2695-2698, 1998.

Conclusions

- Earth rotation brings the additional, large-scale dispersion
- The model rKdV equation was derived for nonlinear waves in rotating media.
- Rotation prohibits existence of stationary solitons in the case of “normal dispersion”
- Initial KdV solitons undergo a terminal decay due to rotation.
- In the long-time asymptotic a Shrödinger-type wave packets can emerge.
- However, solitons can exist on the background of a long wave.
- Two solitons on a long wave can reveal a complex dynamics.
- Laboratory experiments and oceanic data corroborate some of these results.
- In the case of “anomalous dispersion” there exist solitons, bounded solitons, and multisolitons.

HAPPY BIRTHDAY ROGER!