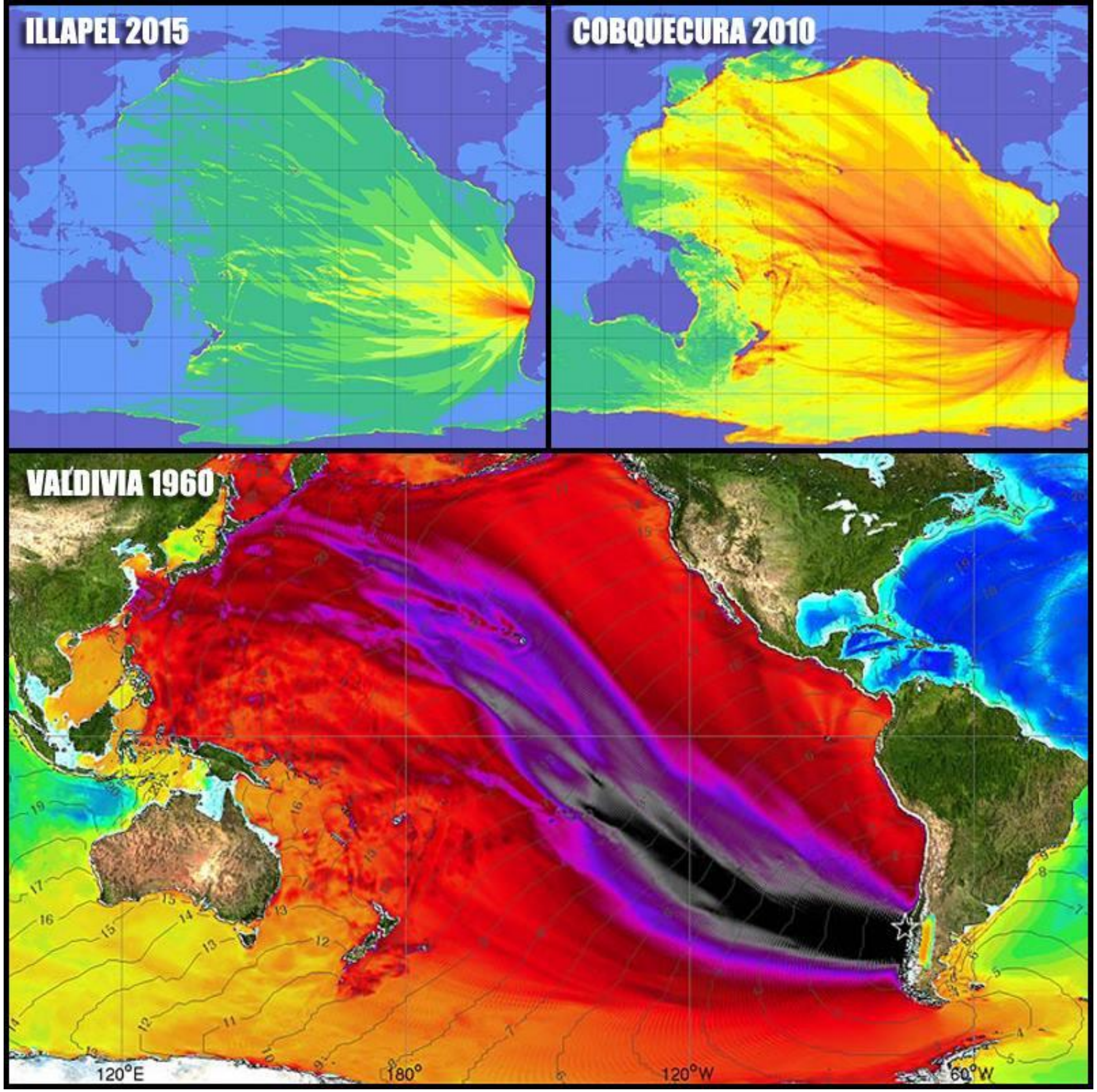
Long traveling waves in a fluid of variable depth

Efim Pelinovsky

Department of Nonlinear Geophysical Processes,
Institute of Applied Physics,
State Technical University,
Nizhny Novgorod, RUSSIA

Cooperation with I. Didenkulova, T. Talipova, R. Grimshaw and others

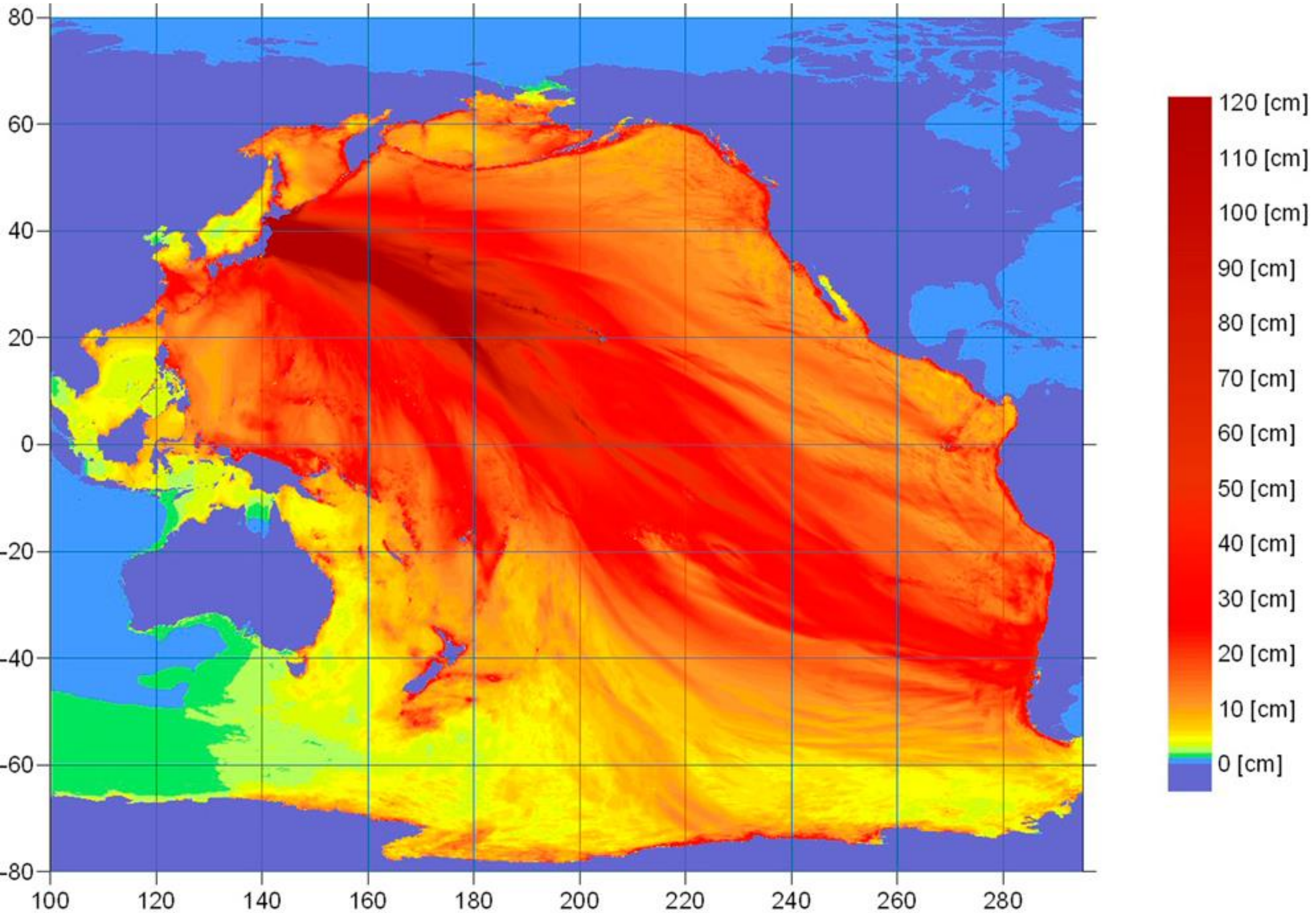
Workshop on Nonlinear Waves, Toowoomba, Australia



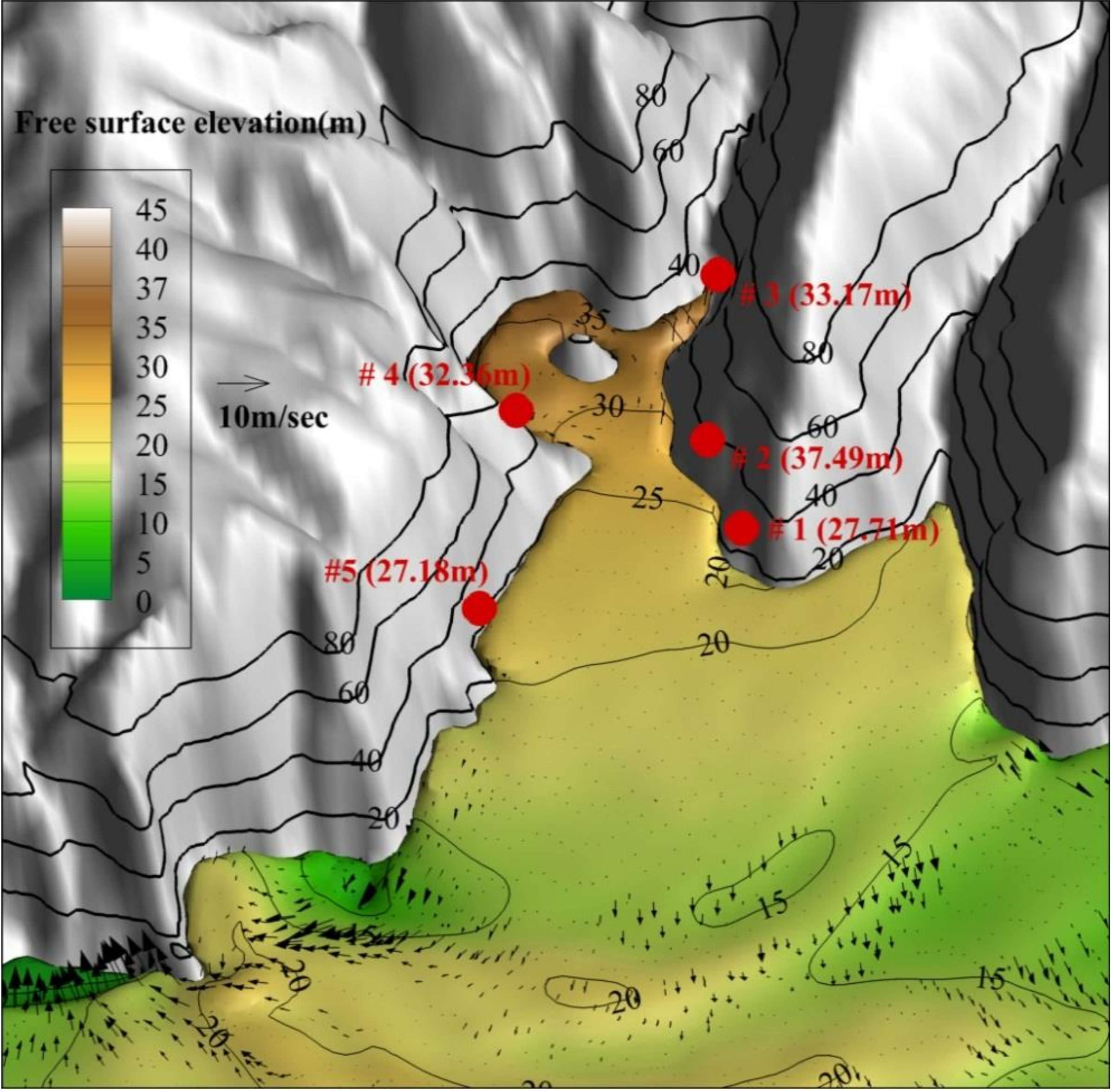
Chilean Tsunamis 1960, 2010, 2015

Wave Height Distribution in Pacific

Japan, 11 March 2011 M = 9.0



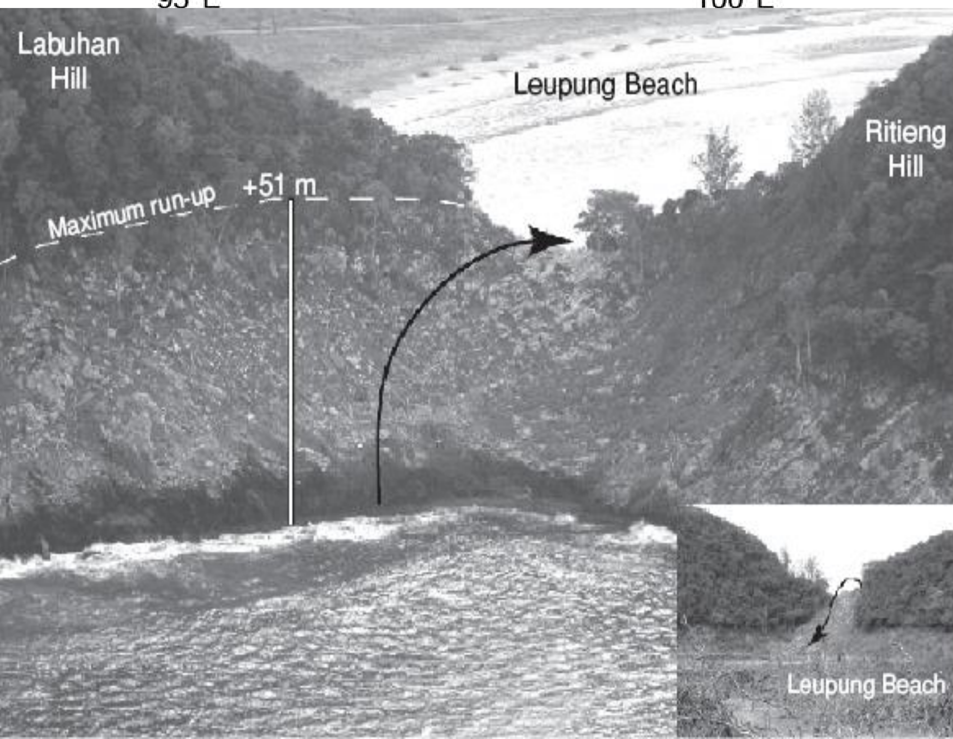
Wave Height Distribution

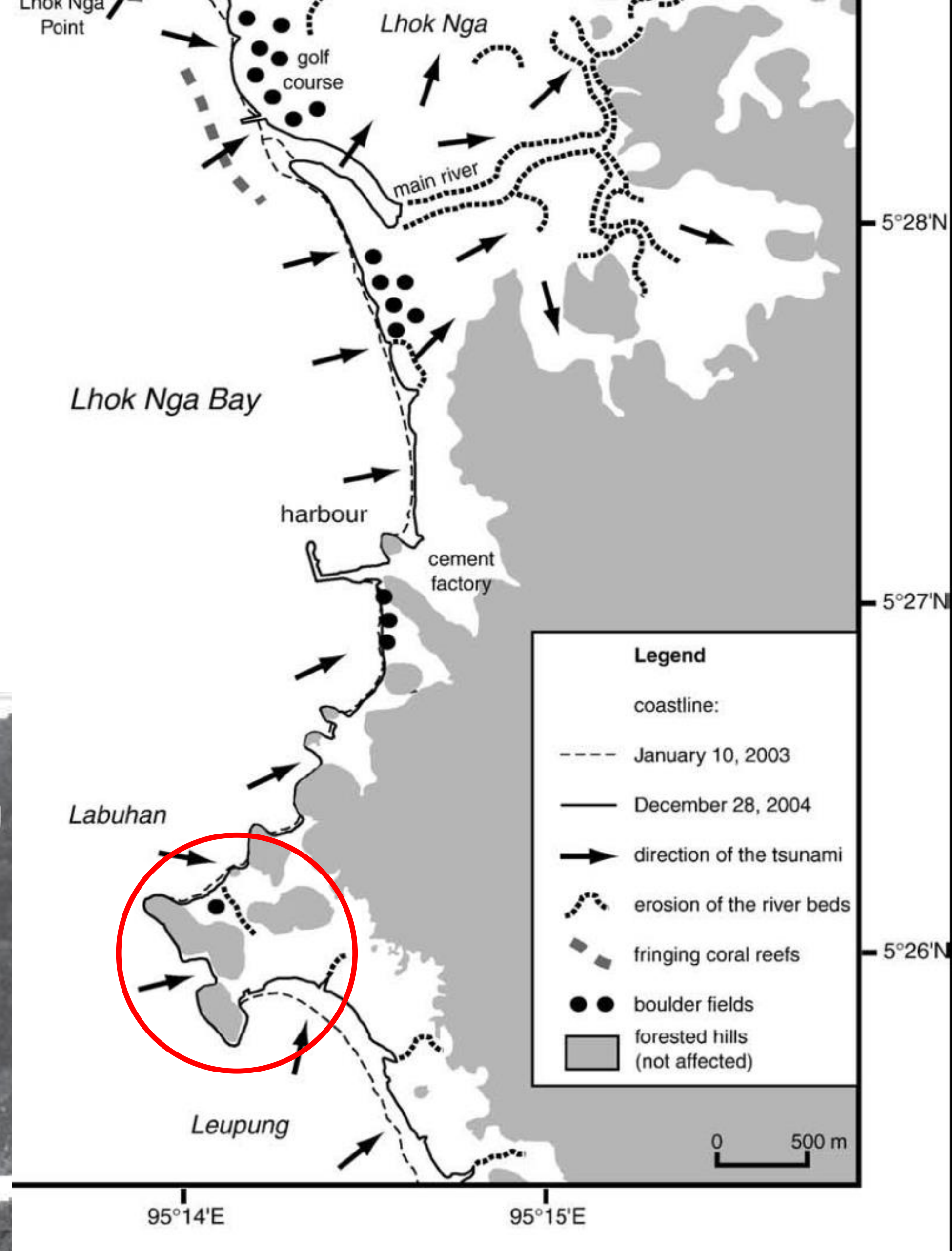


2011

Koborinai Cetl. Maximum of tsunami runup 37.5 m

Kim D.C., Kim K.O., Pelinovsky E., Didenkulova I., and Choi B.H. Three-dimensional tsunami runup simulation at the Koborinai port, Sanriku coast, Japan. ***Journal of Coastal Research***, 2013, vol.65, 266-271

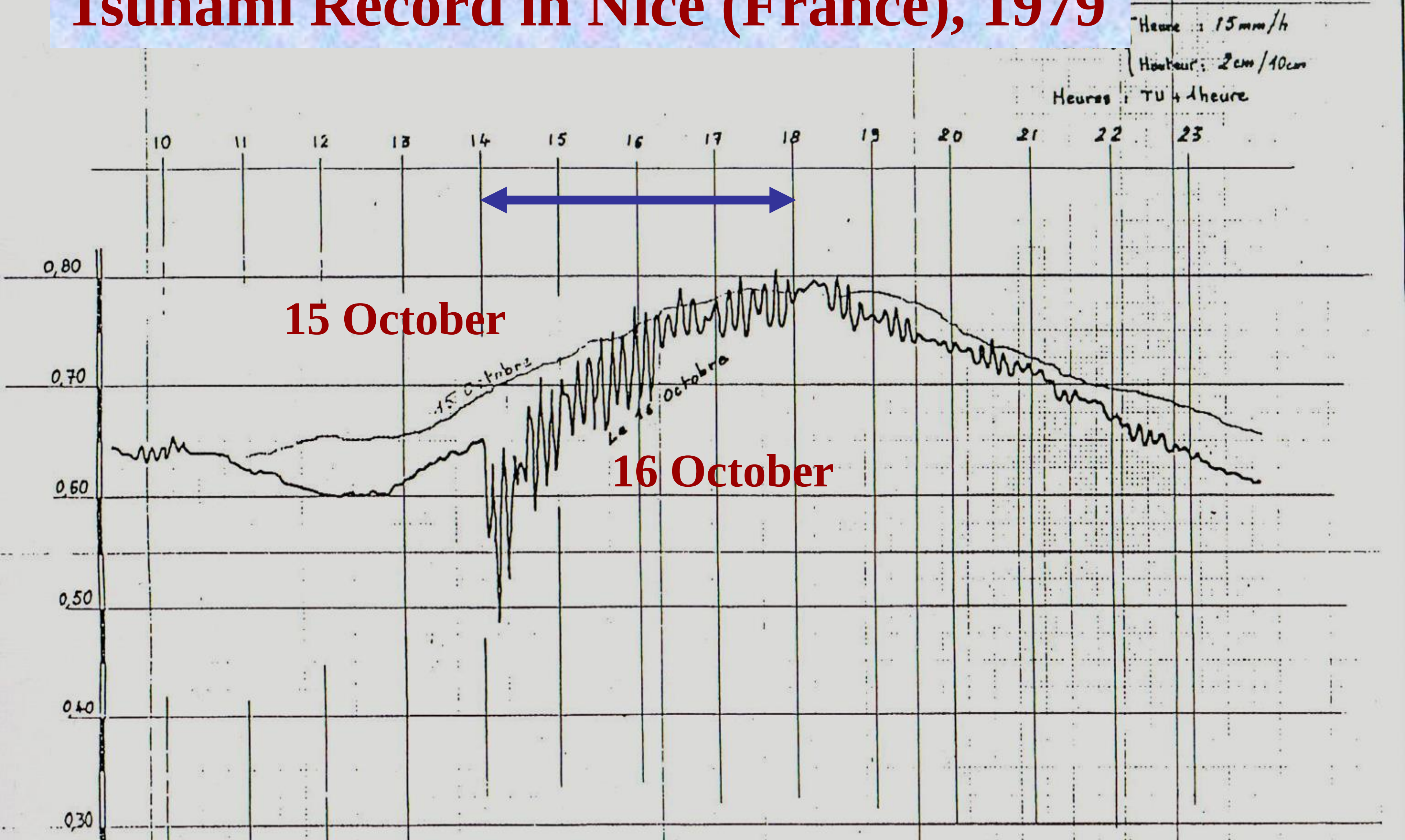




Indonesia, 2004

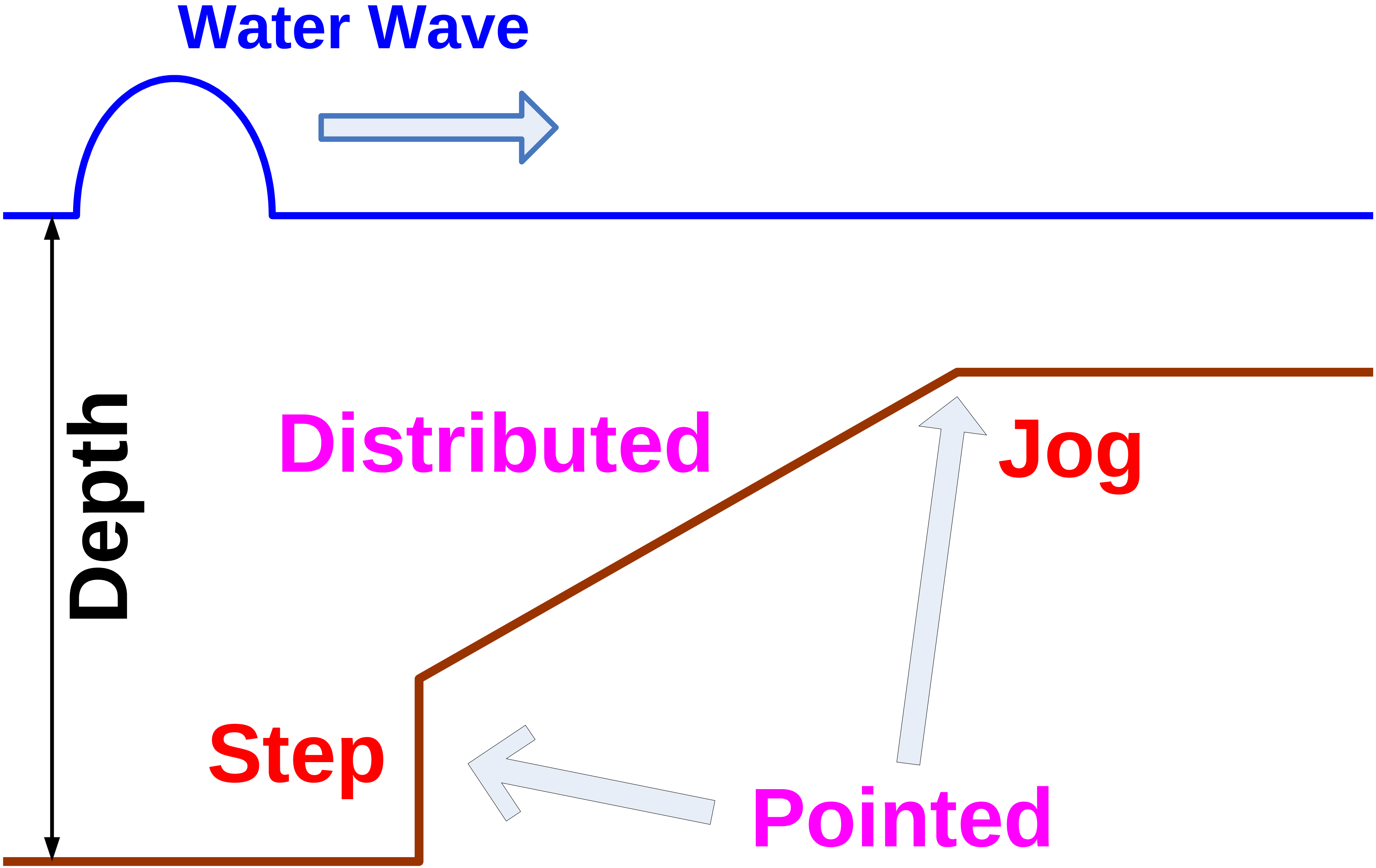
Maximum Runup Height 51 m

Tsunami Record in Nice (France), 1979



Wave Height 10 cm, meanwhile on coast – 1 m

Motivation: Role of Each Factor



Simplified Linear Theory of 1D Shallow Water Waves

$$\boxed{\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left[c^2(x) \frac{\partial \eta}{\partial x} \right] = 0}$$

$$c(x) = \sqrt{g h(x)}$$ - **Wave Speed**

η(x,t) – Water Displacement

h(x) – Water Depth

"Non-Reflected" Beach with BIG Amplification

Seek Solution of Wave Equation

$$\boxed{\eta(x,t) = A(x)\exp\left[i\left\{\omega t - \Psi(x)\right\}\right]}$$

Two unknown Functions: A and Ψ

Exact Separation

$$\boxed{\left[\frac{\omega^2}{gh(x)} - k^2(x)\right]A + \left[\frac{d^2A}{dx^2} + \frac{1}{h}\frac{dh}{dx}\frac{dA}{dx}\right] = 0}$$

$$\boxed{2k\frac{dA}{dx} + A\frac{dk}{dx} + \frac{1}{h}\frac{dh}{dx}kA = 0}$$

where $k(x) = \frac{d\Psi}{dx}$ ***- wavenumber***

One equation is integrated exactly

$$\boxed{2k\frac{dA}{dx} + A\frac{dk}{dx} + \frac{1}{h}\frac{dh}{dx}kA = 0}$$

$$\Downarrow$$

$$\boxed{A^2(x)k(x)h(x) = \mathrm{const}}$$

Energy Flux Conservation

Second Equation can not be integrated generally

$$\boxed{\left[\frac{\omega^2}{gh(x)} - k^2(x)\right]A + \left[\frac{d^2A}{dx^2} + \frac{1}{h}\frac{dh}{dx}\frac{dA}{dx}\right] = 0}$$

It is a Variable-Coefficient 2d Order Equation

No simple than Initial Wave Equation

If Depth varies smoothly – WKB Approach

$$\boxed{\left[\frac{\omega^2}{gh(x)} - k^2(x)\right]A + \cancel{\left[\frac{d^2A}{dx^2} + \frac{1}{h}\frac{dh}{dx}\frac{dA}{dx}\right]} = 0}$$

$$\boxed{k(x) = \frac{\omega}{\sqrt{gh(x)}}}$$

eikonal

together

$$\boxed{A^2(x)k(x)h(x) = \text{const}}$$

Asymptotic Solution for h(εx)

$$\boxed{\eta(x,t) = A(\varepsilon x)\exp\left[i\left\{\omega t - \Psi(x)\right\}\right] + \varepsilon\ldots}$$

$$k(\varepsilon x) = \frac{d\Psi}{dx}$$

Described slowly varied propagated wave

Reflection – beyond asymptotic method
As exp(-1/ε)

Mathematics: Theory of catastrophes, caustics, Maslov operator, ray approach….

Arnold, Maslov, Berry, Dobrokhotov, ……

New Journal of Physics
The open-access journal for physics

Tsunami asymptotics

M V Berry[1]
H H Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, UK

New Journal of Physics 7 (2005) 129
Received 4 April 2005
Published 23 May 2005
Online at http://www.njp.org/
doi:10.1088/1367-2630/7/1/129

'There was an awful rainbow once in heaven' (John Keats, 1820)

Abstract. By applying the technique of uniform asymptotic approximation to the oscillatory integrals representing tsunami wave profiles, the form of the travelling wave far from the source is calculated for arbitrary initial disturbances. The approximations reproduce the entire profiles very accurately, from the front to the tail, and their numerical computation is much faster than that of the oscillatory integrals. For one-dimensional propagation, the uniform asymptotics involve Airy functions and their derivatives; for two-dimensional propagation, the uniform asymptotics involve products of these functions. Separate analyses are required when the initial disturbance is specified as surface elevation or surface velocity as functions of position, and when these functions are even or odd.

ISSN 1061-9208, Russian Journal of Mathematical Physics, Vol. 14, No. 1, 2007, pp. 28–56. © *Pleiades Publishing, Ltd., 2007.*

Asymptotics of Localized Solutions of the One-Dimensional Wave Equation with Variable Velocity. I. The Cauchy Problem

S. Yu. Dobrokhotov*, S. O. Sinitsyn, and B. Tirozzi*****

**Institute for Problems in Mechanics RAS*
E-mail: `dobr@ipmnet.ru`
***Moscow State Institute of Electronics and Mathematics*
E-mail: `serg.sinitsyn@gmail.com`
****Department of Physics, University "La Sapienza," Rome*
E-mail: `brunello.tirozzi@roma1.infn.it`

Received December 25, 2006

Abstract. We present a systematic study of the construction of localized asymptotic solutions of the one-dimensional wave equation with variable velocity. In part I, we discuss the solution of the Cauchy problem with localized initial data and zero right-hand side in detail. Our aim is to give a description of various representations of the solution, their geometric interpretation, computer visualization, and illustration of various general approaches (such as the WKB and Whitham methods) concerning asymptotic expansions. We discuss ideas that can be used in more complicated cases (and will be considered in subsequent parts of this paper) such as inhomogeneous wave equations, the linear surge problem, the small dispersion case, etc. and can eventually be generalized to the 2- (and n-) dimensional cases.

Try to keep Features of Travelling Wave

$$\boxed{\left[\frac{\omega^2}{gh(x)} - k^2(x)\right]A + \left[\frac{d^2A}{dx^2} + \frac{1}{h}\frac{dh}{dx}\frac{dA}{dx}\right] = 0}$$

$$\boxed{\left[\frac{\omega^2}{gh(x)} - k^2(x)\right] = 0}$$

Overdetermined System

$$\boxed{\left[\frac{d^2A}{dx^2} + \frac{1}{h}\frac{dh}{dx}\frac{dA}{dx}\right] = 0}$$

“Non-Reflected” Beach

$$\boxed{\left[\frac{d^2A}{dx^2}+\frac{1}{h}\frac{dh}{dx}\frac{dA}{dx}\right]=0}$$

$$\boxed{h(x)\frac{dA}{dx}=\text{const}}$$

together with

$$\boxed{A^2(x)k(x)h(x)=\text{const}}$$

$$\boxed{k(x)=\frac{\omega}{\sqrt{gh(x)}}}$$

gives

$$\boxed{h(x)\sim x^{4/3}}$$

“Non-Reflected” Beach

$$\boxed{\eta(x,t) = A\left[\frac{h_0}{h(x)}\right]^{1/4} f\left[t-\tau(x)\right]}$$

$$\boxed{\tau(x) = \int_{-\infty}^{x} \frac{dx'}{\sqrt{gh(x')}}}$$

Propagated Wave

Impulse posses a shape

But it is a singular solution

at x = 0 (h = 0)

Velocity Field

$$u(x.t) = -g\int_{-\infty}^{t}\frac{\partial\eta}{\partial x}dt' = -g\frac{\partial}{\partial x}\int_{-\infty}^{t}\eta(x,t')dt'$$

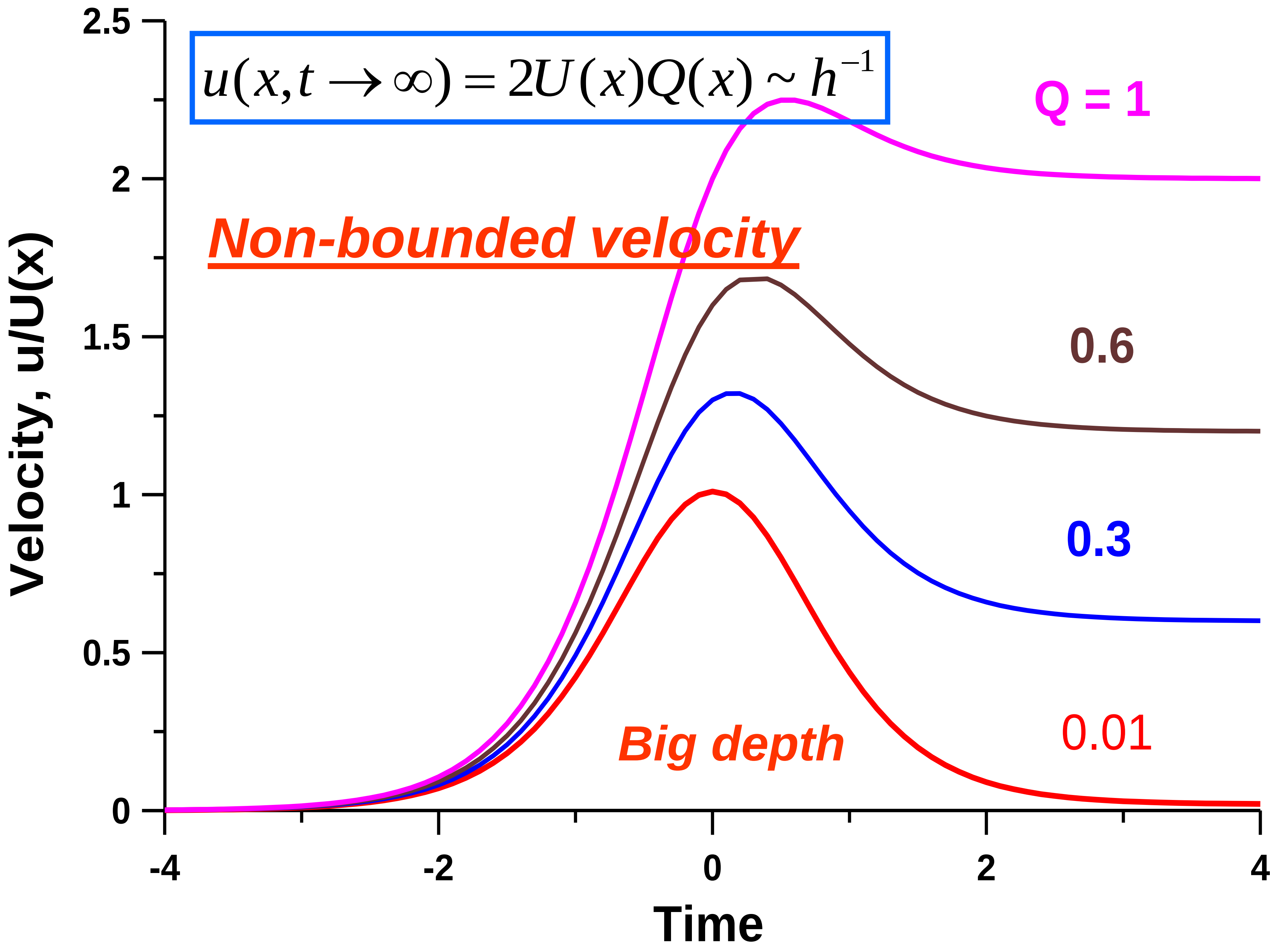
$$\eta(x,t) = A\left[\frac{h_0}{h(x)}\right]^{1/4}\operatorname{sech}^2\{\Omega[t-\tau(x)]\}$$

$$u(x,t) = U(x)\{\operatorname{sech}^2(T) + Q(x)[\tanh(T)+1]\}$$

$$U(x) = A\sqrt{\frac{g}{h(x)}}\left[\frac{h_0}{h(x)}\right]^{1/4} \sim h^{-3/4}$$

WKB amplitude

$$Q(x) = \frac{\sqrt{gh(x)}}{3L\Omega}\left[\frac{h_0}{h(x)}\right]^{3/4} \sim h^{-1/4}$$


$u(x, t \to \infty) = 2U(x)Q(x) \sim h^{-1}$
Q = 1
Non-bounded velocity
0.6
0.3
0.01
Big depth
Velocity, u/U(x)
Time
2.5
2
1.5
1
0.5
0
-4
-2
0
2
4

Physical Solution Vanishing on the Ends

$$\eta(x,t) = A\left[\frac{h_0}{h(x)}\right]^{1/4} f\left[t - \tau(x)\right]$$

$$u(x) = A\sqrt{\frac{g}{h(x)}}\left[\frac{h_0}{h(x)}\right]^{1/4} f\left[t - \tau(x)\right] + Q(x)\int f(\xi)\,d\xi$$

Sign-variable pulse

$$\int_{-\infty}^{+\infty} f(t)\,dt = 0$$

It is a Wave with biggest amplification

Piston Model of Wave Generation

$$\boxed{\eta(x,0) = \eta_0(x) \quad u(x,0) = 0}$$

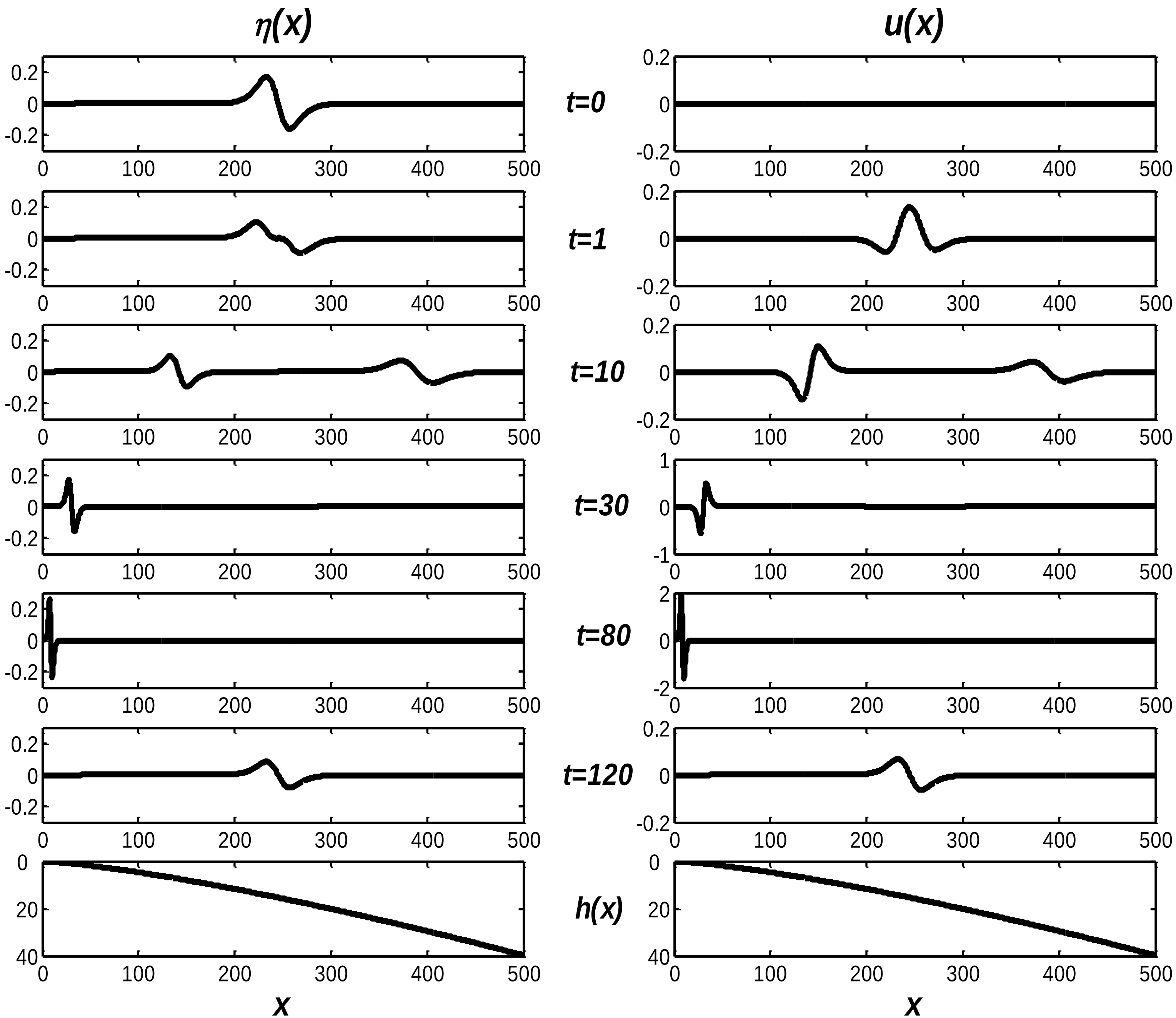
$$\eta(x,t) = \frac{1}{x^{1/3}}\left\{f_0[\tau(x) - t] + f_0[\tau(x) + t] - f_0[-\tau(x) + t]\right\}$$

$$u(x,t) = \sqrt{\frac{g}{p}}\frac{1}{x}\left[f_0(\tau - t) - f_0(\tau + t) - f_0(-\tau + t)\right] -$$

$$-\frac{g}{3x^{4/3}}\left[\Phi_0(\tau - t) - \Phi_0(\tau + t) - \Phi_0(-\tau + t)\right]$$

If initial disturbance is sign-variable

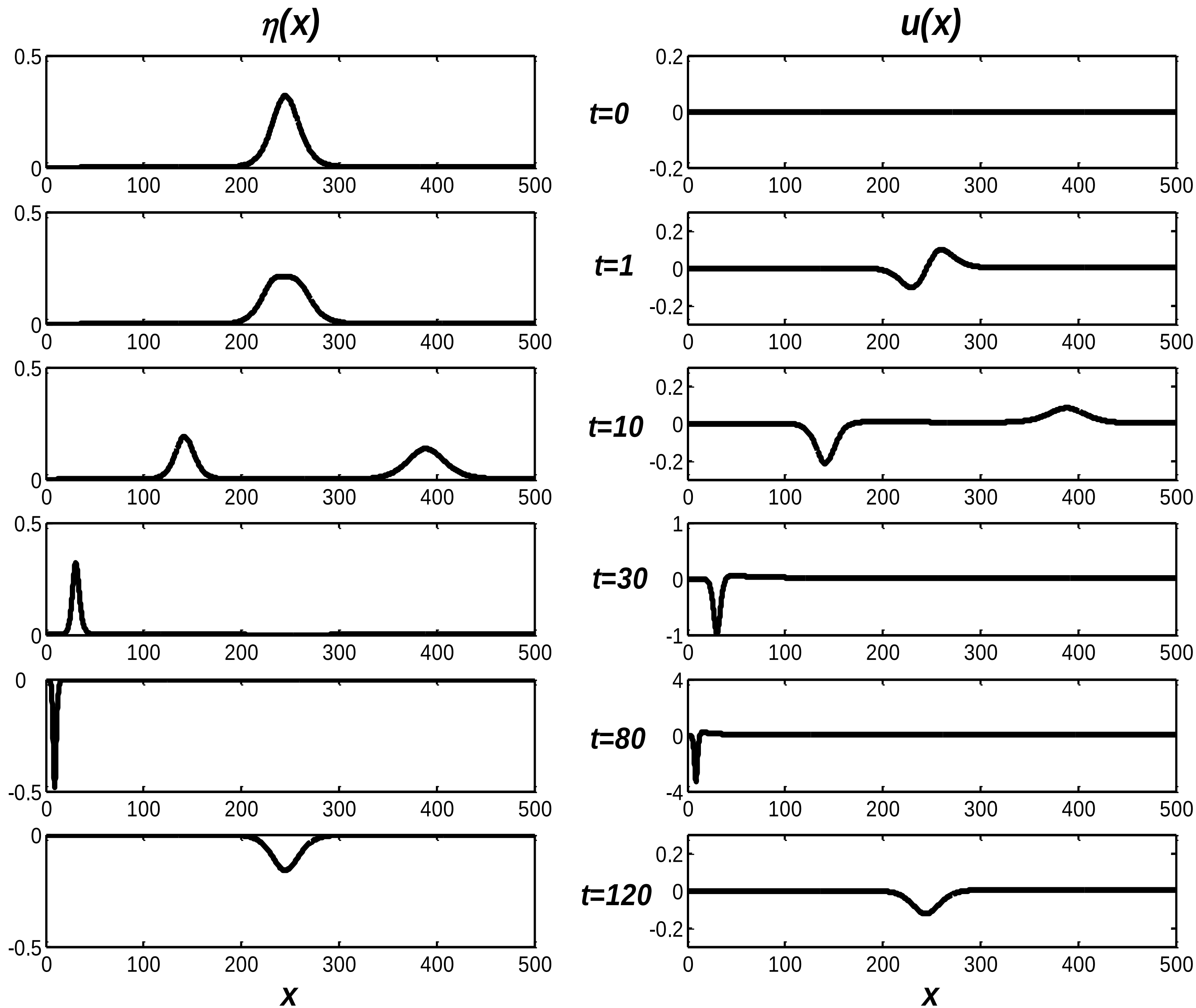
$$f_0(\tau) = -\frac{4}{3}\frac{\tanh[2(\tau-60)/3]}{\cosh^2[2(\tau-60)/3]}$$



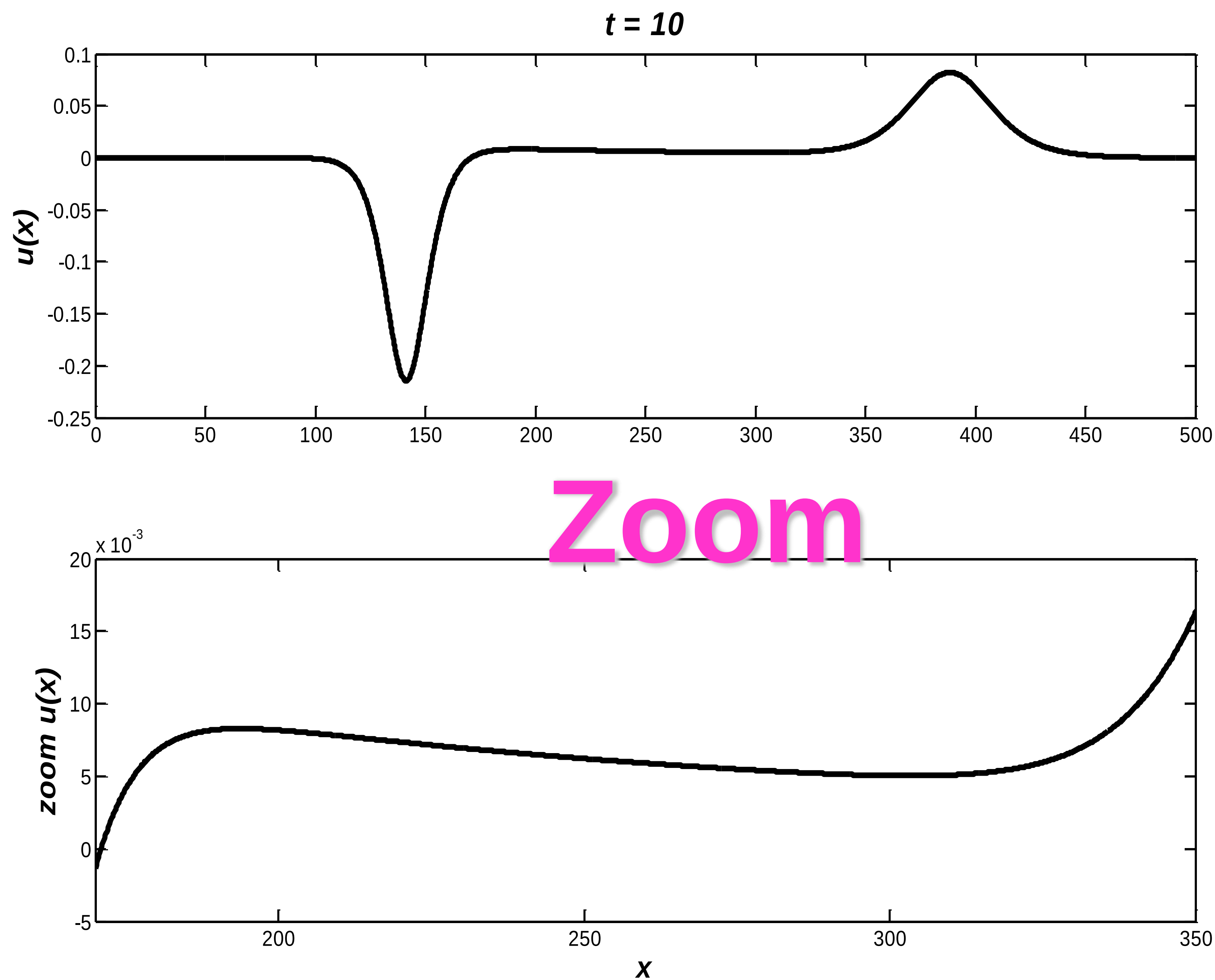
like constant depth

Sign – constant initial disturbance

$$\boxed{f_0(\tau) = \operatorname{sech}^2[2(\tau - 60)/3]}$$



Relict current


t = 10
u(x)
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
0
50
100
150
200
250
300
350
400
450
500
Zoom
x 10^-3
zoom u(x)
20
15
10
5
0
-5
200
250
300
350
x

Runup on beach $x^{4/3}$

$$\boxed{\eta(x,t) = A_0\left[\frac{h_0}{h(x)}\right]^{1/4}\left\{f[t+\tau(x)] - f[t-\tau(x)]\right\}}$$

$$\boxed{\tau(x) = \int_{-L}^{x}\frac{dy}{\sqrt{gh(y)}} = \frac{3L}{\sqrt{gh_0}}\left[\frac{h(x)}{h_0}\right]^{1/4}}$$

Bounded on shore x = 0 (runup)

$$\boxed{R(t) = \eta(x=0,t) = 2\tau_0\frac{df(t+\tau_0)}{dt}}$$

Velocity Field on Shoreline

$$\boxed{u(x \to 0, t) \sim \frac{f(t+\tau_0)}{x}}$$

But discharge

$$\boxed{h(x)u(x,t) \to x^{1/3} f(t+\tau_0) \to 0}$$

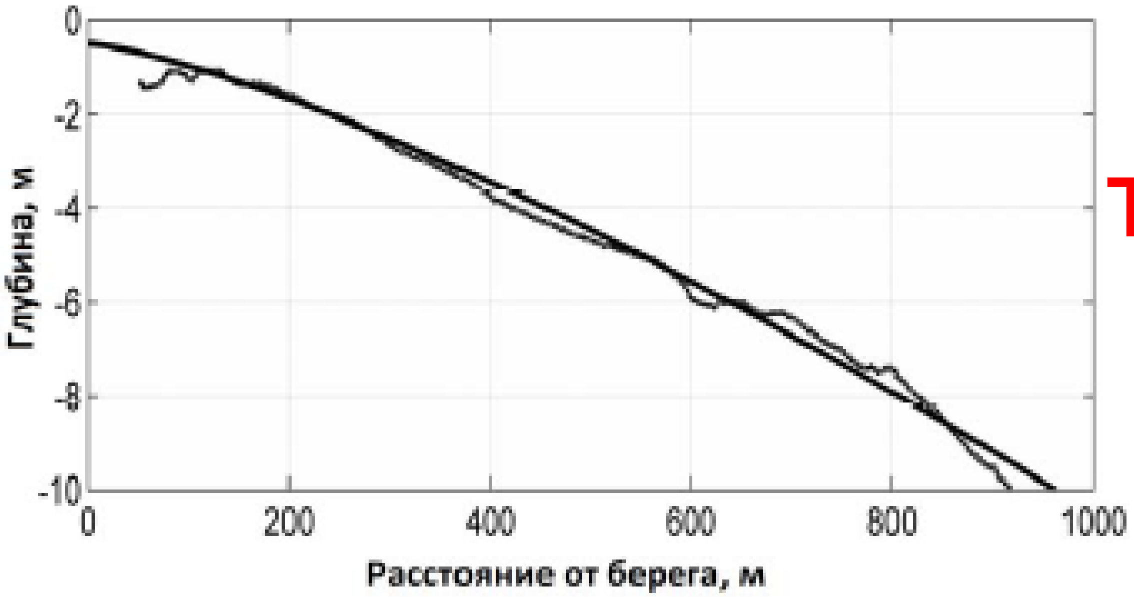
Soliton Runup

$$R_{\max} = 4\frac{A}{\alpha}\sqrt{\frac{A}{h}} \sim A^{3/2}$$

Plane Beach

$$R_{\max} = 2.8312\frac{A}{\sqrt{\alpha}}\left(\frac{A}{h}\right)^{1/4} \sim A^{5/4}$$

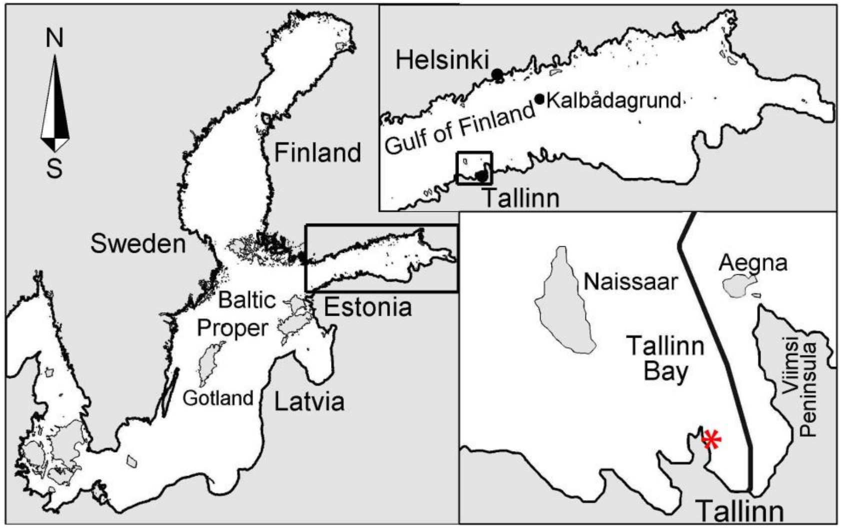
Didenkulova I., Pelinovsky E., Soomere T. Long surface wave dynamics along a convex bottom. *J Geophysical Research - Oceans.* 2009. Vol. 114, C07006.



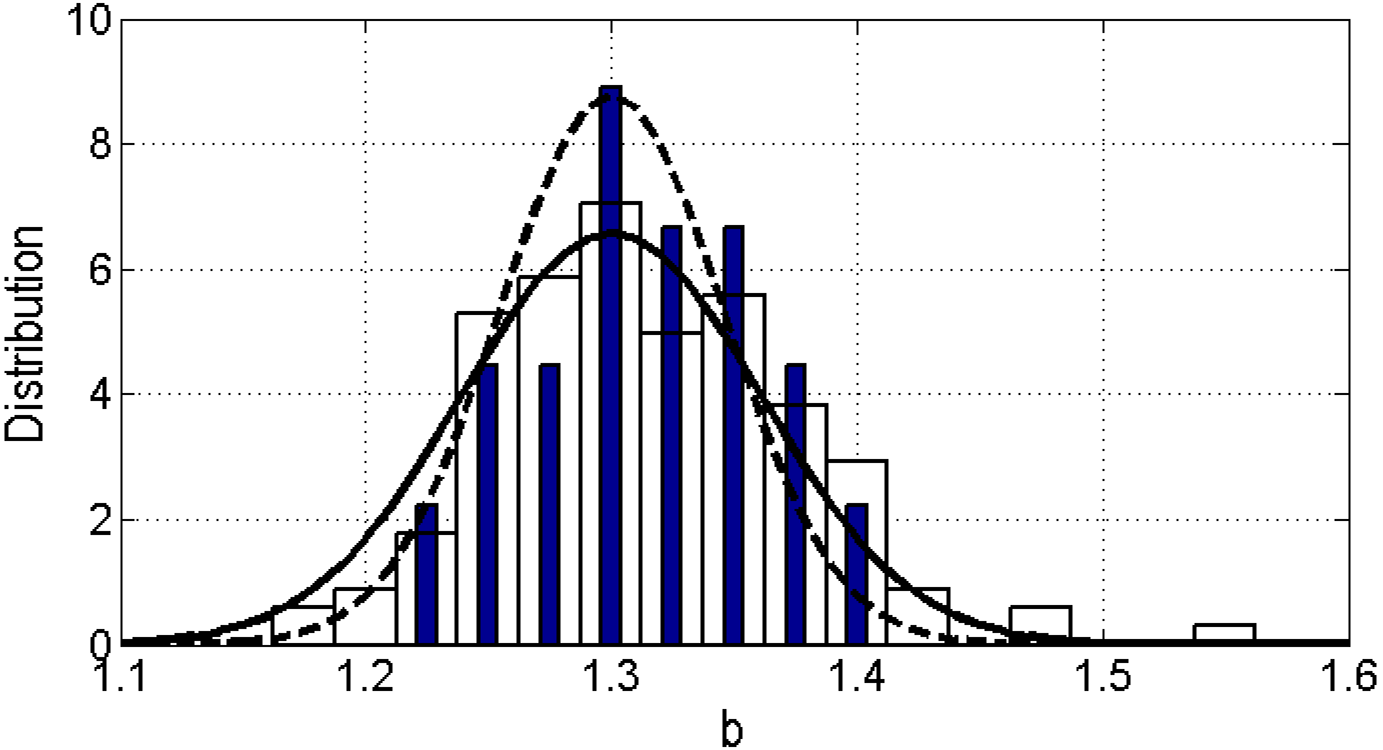
Pirita Beach
Tallinn, Estonia

Approximation
$h(x) \sim x^{4/3}$

Didenkulova I., Soomere T. Formation of two-section cross-shore profile under joint influence of random short waves and group of long waves. Marine Geology, 2011, vol. 289, 29-33



$$\boxed{h \sim x^{b}}$$



How many non-reflected profiles?

General scheme

$$\boxed{\frac{\partial^2 u}{\partial t^2} - c^2(x)\frac{\partial^2 u}{\partial x^2} = 0}$$

Substitution

$$\boxed{u(x,t) = A(x)U[t,\tau(x)]}$$

Three unknown functions

Idea is to get constant-coefficient wave-like equation

$$\boxed{A\left[\frac{\partial^2 U}{\partial t^2}-c^2\left(\frac{d\tau}{dx}\right)^2\frac{\partial^2 U}{\partial \tau^2}\right]=c^2\left[\frac{dA}{dx}\frac{d\tau}{dx}+\frac{d}{dx}\left(A\frac{d\tau}{dx}\right)\right]\frac{\partial U}{\partial \tau}+c^2\frac{d^2 A}{dx^2}U}$$

Should be constant!

$$\boxed{\frac{d\tau}{dx}=\frac{1}{c(x)}}$$

$$\boxed{\tau(x)=\int_{x_0}^{x}\frac{dx}{c(x)}}$$

$$\boxed{A\left[\frac{\partial^2 U}{\partial t^2}-\frac{\partial^2 U}{\partial \tau^2}\right]=c^2\left[\frac{dA}{dx}\frac{d\tau}{dx}+\frac{d}{dx}\left(A\frac{d\tau}{dx}\right)\right]\frac{\partial U}{\partial \tau}+c^2\frac{d^2 A}{dx^2}U}$$

Should be zero!

$$\boxed{A(x)=A_0 c^{1/2}}$$

As in WKB approximation

$$\boxed{A\left[\frac{\partial^2 U}{\partial t^2}-\frac{\partial^2 U}{\partial \tau^2}\right]=c^2\,\frac{d^2 A}{dx^2}U}$$

$$\boxed{\frac{\partial^2 U}{\partial t^2}-\frac{\partial^2 U}{\partial \tau^2}=PU}$$

Klein-Gordon Equation

where

$$\boxed{P(\tau)=c^{3/2}\,\frac{d^2}{dx^2}(c^{1/2})}$$

$$\boxed{P = c^{3/2}\frac{d^2}{dx^2}(c^{1/2}) = \text{const}}$$

Non-reflected profiles

$$\downarrow$$

$$\boxed{\left[\frac{dc^{1/2}}{dx}\right]^2 + \frac{P}{c} = \text{constant}}$$

1) $P = 0$

$$\boxed{c(x) \sim x^2}$$

Variabe-coeff wave equation reduces to constant coeff wave equation

2) $P > 0$

Variable-coeff wave equation reduces to constant coeff Klein – Gordon equation

$$\boxed{c(x) = c_0 + qx^2}$$

$$\boxed{P = c_0 q > 0}$$

"Underwater Hill" ($c_0 > 0$, $q > 0$)

τ interval is bounded!

2) $P < 0$

Variable-coeff wave equation reduces to constant coeff Klein – Gordon equation

i) $$\boxed{c(x) = c_0 + qx^2}$$ $$\boxed{P = c_0 q < 0}$$

- Closed resonator with two "beaches" (q<0)

- Monotonic profile from x* to infinity (q>0)

ii) $$\boxed{c(x) = qx}$$ $$\boxed{P = -q^2/4 < 0}$$

- Monotonic profile from 0 to infinity

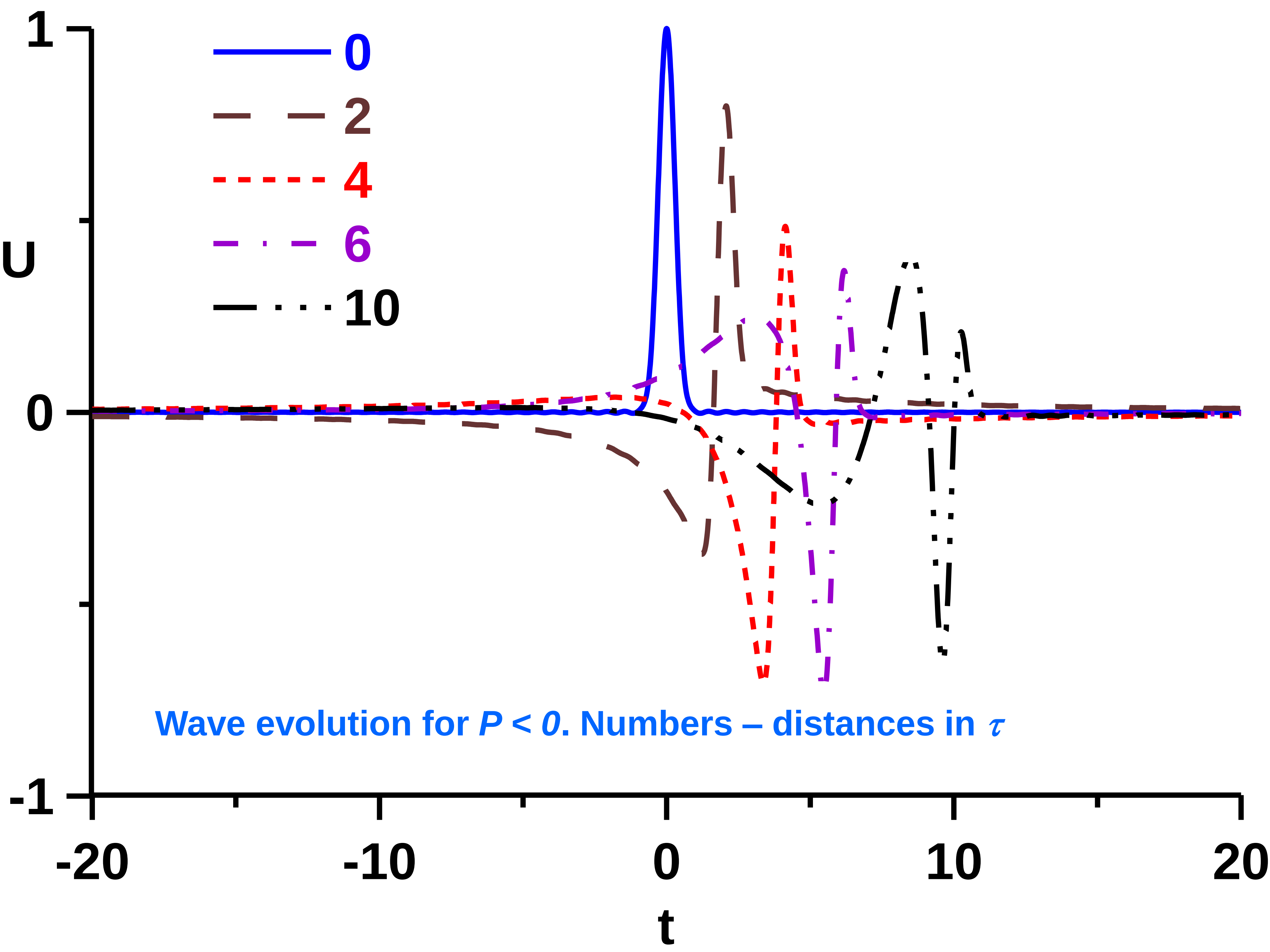
Traveling waves in inhomogeneous medium

For non-zero P
Constant-coefficient
Klein-Gordon equation

$$\boxed{\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial \tau^2} = PU}$$

$$\boxed{U(t,\tau) = \int_{-\infty}^{\infty} B(k)\exp(ik\tau - i\omega(k)t)dk}$$

$$\boxed{\omega^2 = k^2 - P}$$



1
0
-1
U
0
2
4
6
10
Wave evolution for *P* < *0*. Numbers – distances in τ
-20
-10
0
10
20
t

Link to Mathematics

ANALYTIC SOLUTION OF THE LINEARIZED SHALLOW-WATER WAVE EQUATIONS FOR CERTAIN CONTINUOUS DEPTH VARIATIONS

D. L. CLEMENTS and C. ROGERS

(Received 17 July, 1974)

(Revised 26 September, 1974)

Abstract

The linear long-wave equations with (and without) small ground motion are considered. The governing equations are represented in a matrix form and transformations are sought which reduce the system to (for example) a form associated with the conventional wave equation. Integration of the system is then immediate. It is shown that such a reduction may be achieved provided the variation in water depth is specified by certain multi-parameter forms.

Reprinted from
THE JOURNAL OF THE AUSTRALIAN MATHEMATICAL SOCIETY
Volume XIX (Series B) - part 1, p.p 81–94
1975

Reduce to
h = const and h(x) ~ x

Link to Mathematics

SIAM J. APPL. MATH.
Vol. 43, No. 6, December 1983

0036-1399/83/4306-0004 $01.25/0

ON MAPPING LINEAR PARTIAL DIFFERENTIAL EQUATIONS TO CONSTANT COEFFICIENT EQUATIONS*

GEORGE W. BLUMAN†

Abstract. A constructive algorithm is developed to determine whether or not a given linear p.d.e. can be mapped into a linear p.d.e. with constant coefficients. The algorithm is based on analyzing the infinitesimals of the Lie group of transformations leaving invariant the given p.d.e. As consequences, in two dimensions, necessary and sufficient conditions are given for mapping:

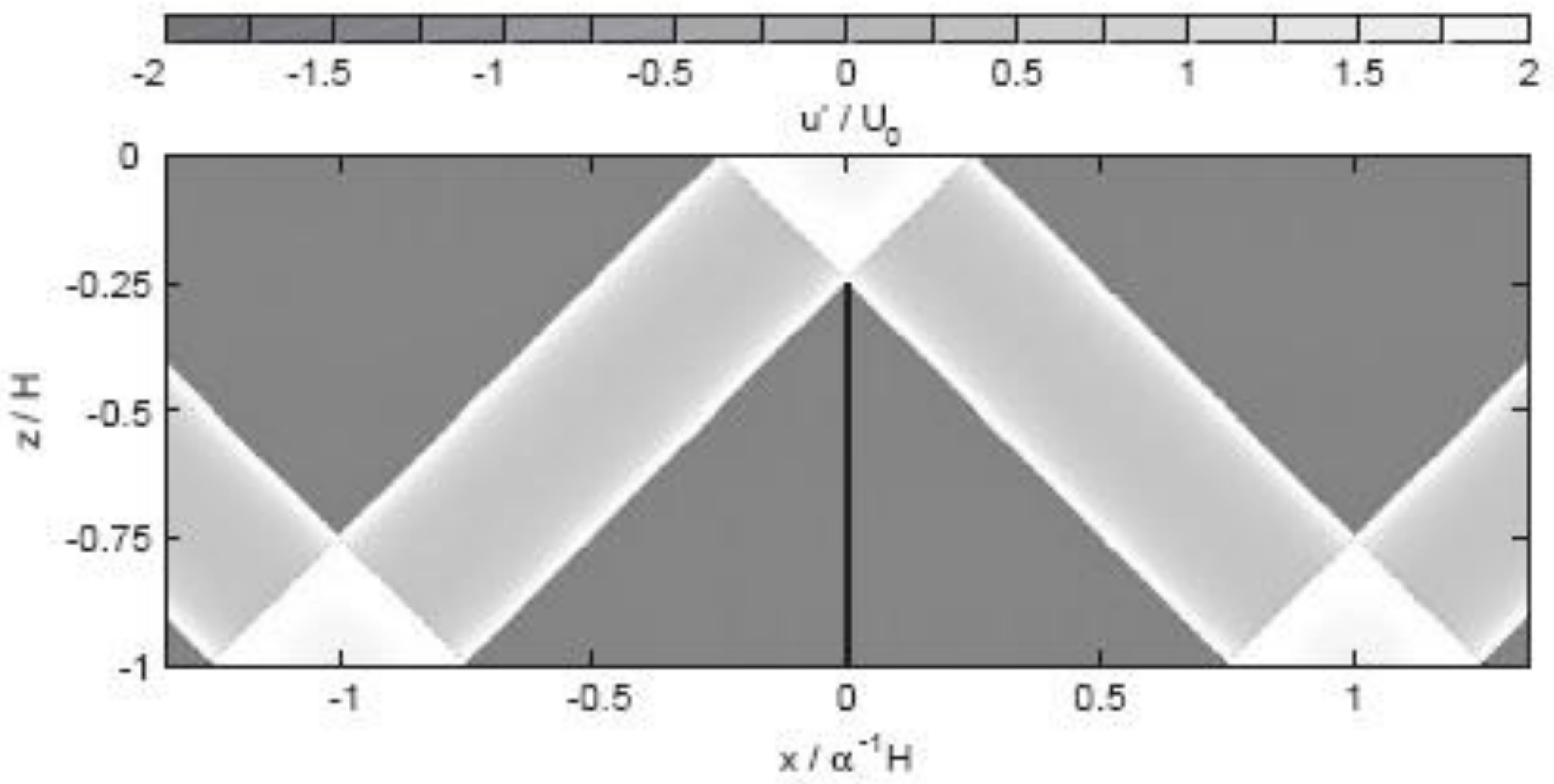
(1) a parabolic p.d.e. into the heat equation;
(2) a hyperbolic p.d.e. into the wave equation;
(3) an elliptic p.d.e. into Laplace's equation or the Helmholtz equation.

The corresponding mappings are given explicitly.

Internal Wave Penetration in the Ocean

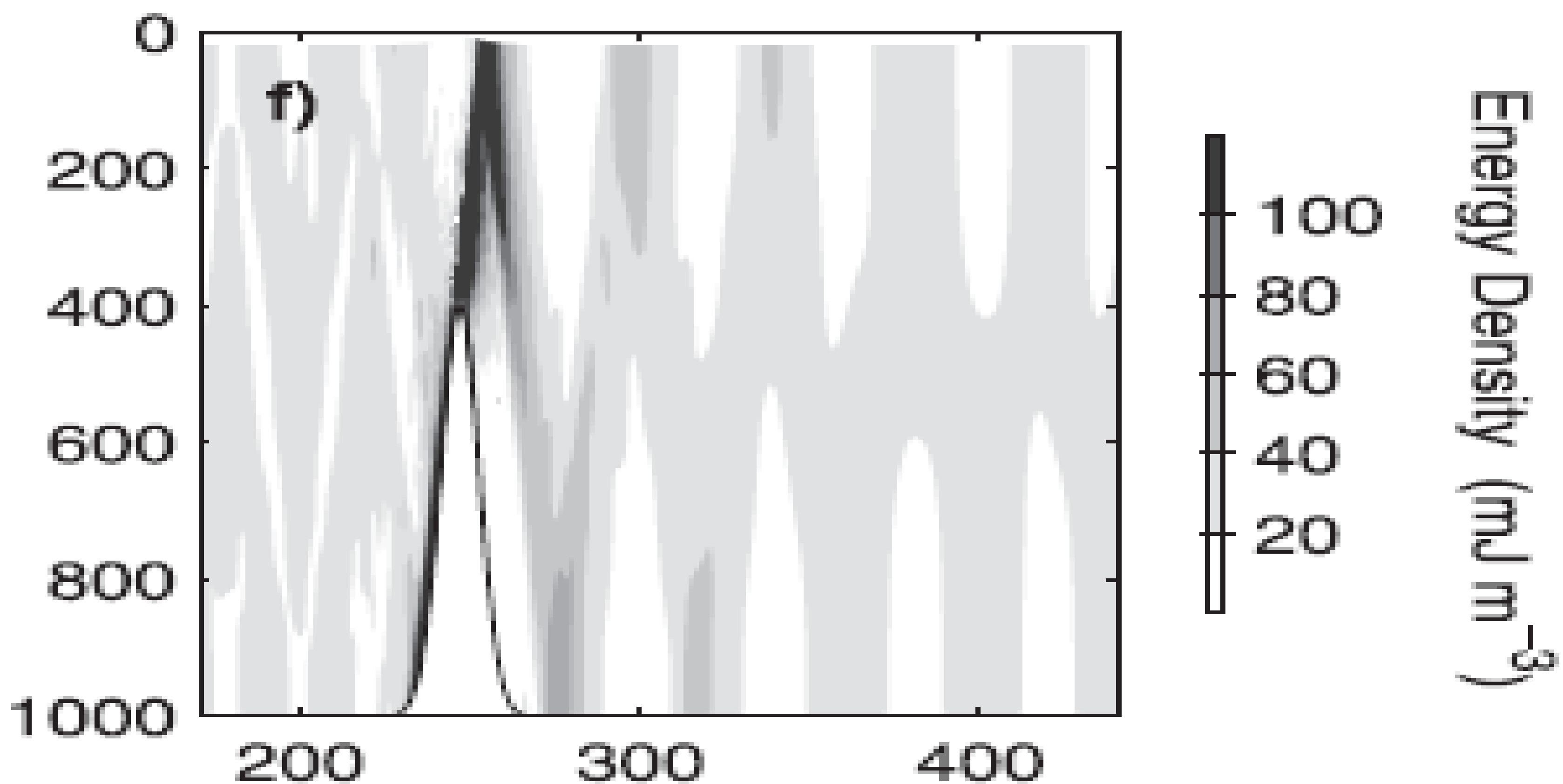
N(z) = constant

L. St. Laurent et al. / Deep-Sea Research I 50 (2003) 987–1003



JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 108, NO. C6, 3180, doi:10.1029/2002JC001528, 2003

Internal tide scattering at seamounts, ridges, and islands



Linear Theory of Internal Waves

$$\boxed{\frac{\partial^2 W}{\partial z^2} - \frac{N^2(z) - \omega^2}{\omega^2 - f^2}\frac{\partial^2 W}{\partial x^2} = 0}$$

Hyperbolic Wave Equation

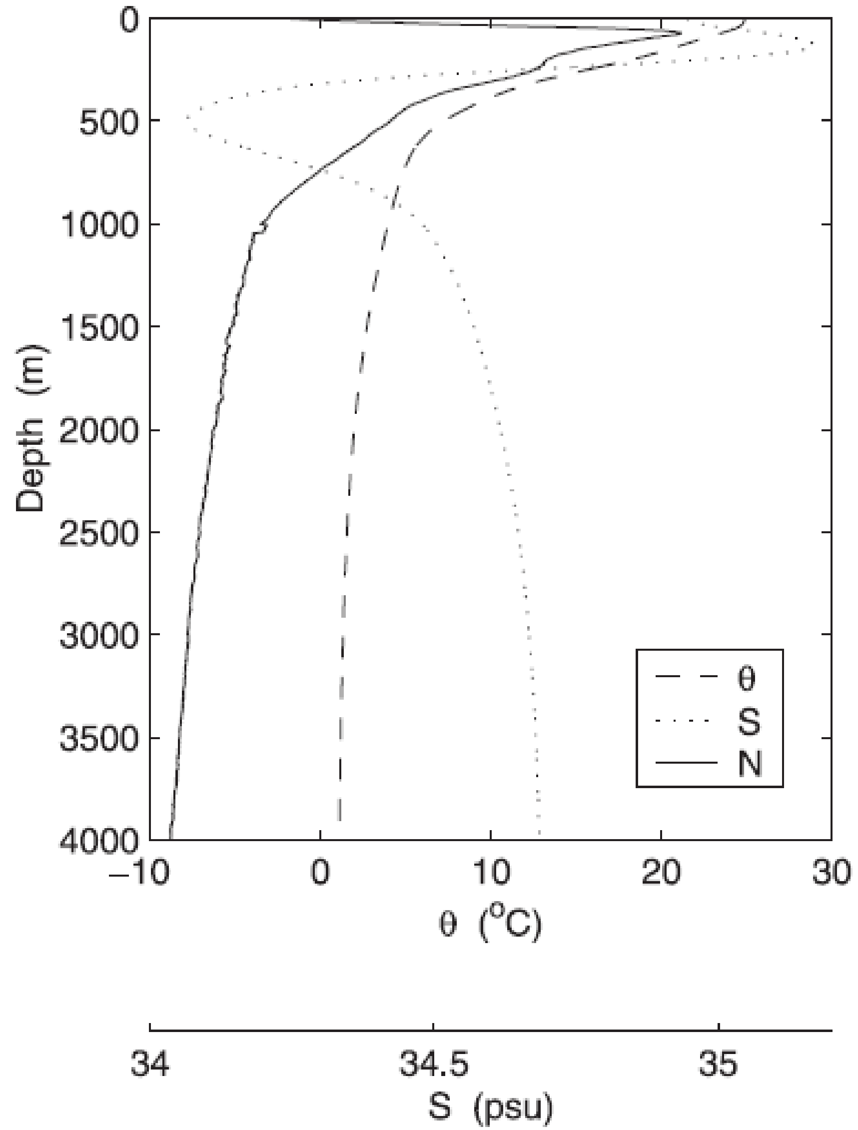
Rays

$$\frac{dz}{dx} = \pm\sqrt{\frac{\omega^2 - f^2}{N^2(z) - \omega^2}}$$

$$N(z) = \frac{N_0}{(1+z/L)^2}$$

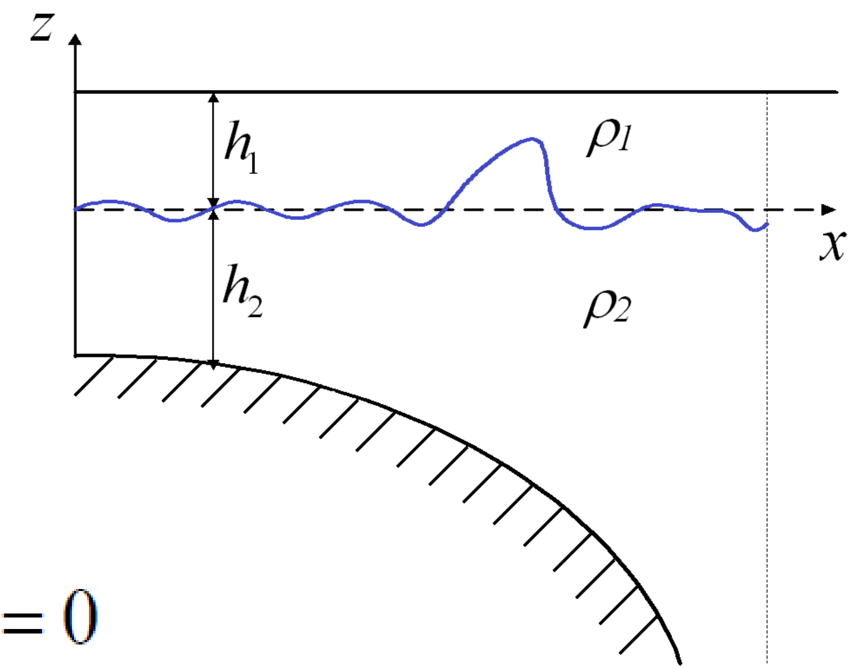
Many real N(z) can be approximated

Perhaps, this explain strong penetration of internal waves in the deep ocean



Grimshaw, R., Pelinovsky, E., and Talipova, T. Non-reflecting internal wave beam propagation in the deep ocean. *J. Phys. Oceanography*, 2010, vol. 40, No. 4, 802-813.

Two-Layer Flow



Nonlinear equations

$$\frac{\partial}{\partial t}\left(h_1-\eta\right)+\frac{\partial}{\partial x}\left[\left(h_1-\eta\right)u_1\right]=0$$

$$\frac{\partial}{\partial t}\left(h_2+\eta\right)+\frac{\partial}{\partial x}\left[\left(h_2+\eta\right)u_2\right]=0$$

$$\frac{\partial}{\partial t}\left(u_1-u_2\right)+u_1\frac{\partial u_1}{\partial x}-u_2\frac{\partial u_2}{\partial x}-g'\frac{\partial \eta}{\partial x}=0$$

$$\boxed{g' = g\,\frac{\rho_2-\rho_1}{\rho_2}}$$

Total discharge = 0 and

$$u_1 = -\frac{h_2 + \eta}{h_1 - \eta} u_2$$

As a result,

$$\frac{\partial}{\partial t}\left(h_2 + \eta\right) + \frac{\partial}{\partial x}\left[\left(h_2 + \eta\right) u_2\right] = 0$$

$$\frac{\partial}{\partial t}\left(\frac{u_2}{h_1 - \eta}\right) + g' \frac{1}{h_1 + h_2} \frac{\partial \eta}{\partial x} + \frac{1}{2(h_1 + h_2)} \frac{\partial}{\partial x}\left[\frac{h_1^2 - h_2^2 - 2(h_1 + h_2)\eta}{\left(h_1 - \eta\right)^2} u_2^2\right] = 0$$

In linear approximation

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[h_2 u_2\right] = 0$$

$$\frac{\partial u_2}{\partial t} + g' \frac{h_1}{h_1 + h_2} \frac{\partial \eta}{\partial x} = 0$$

$$\boxed{\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left[c^2(x) \frac{\partial \eta}{\partial x}\right] = 0}$$

$$c(x) = \sqrt{\frac{g' h_1 h_2(x)}{h_1 + h_2(x)}}$$

As for surface waves where

$$c(x) = \sqrt{g h(x)}$$

The same transformation

$$\eta(t,x) = A(x)\Phi(t,\tau) \qquad A(x) = \frac{const}{\sqrt{c(x)}} \qquad \tau(x) = \int \frac{dx}{c(x)}$$

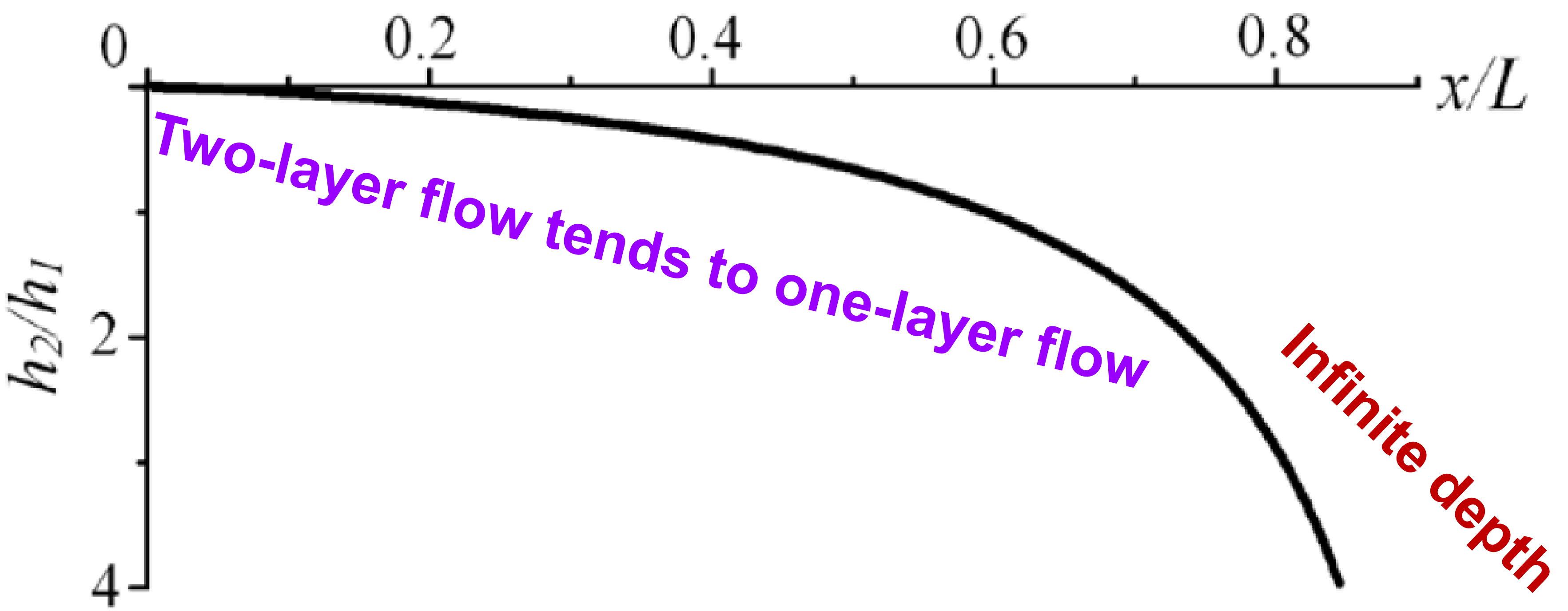
leads to

$$\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial \tau^2} - P\Phi = 0$$

$$\frac{d^2 c^{3/2}}{dx^2} + \frac{3P}{c^{1/2}} = 0$$

"Non-dispersive" profile P = 0

$$c(x) = c_0 \left(\frac{x}{L} \right)^{2/3} \qquad h_2(x) = h_1 \frac{(x/L)^{4/3}}{1-(x/L)^{4/3}}$$



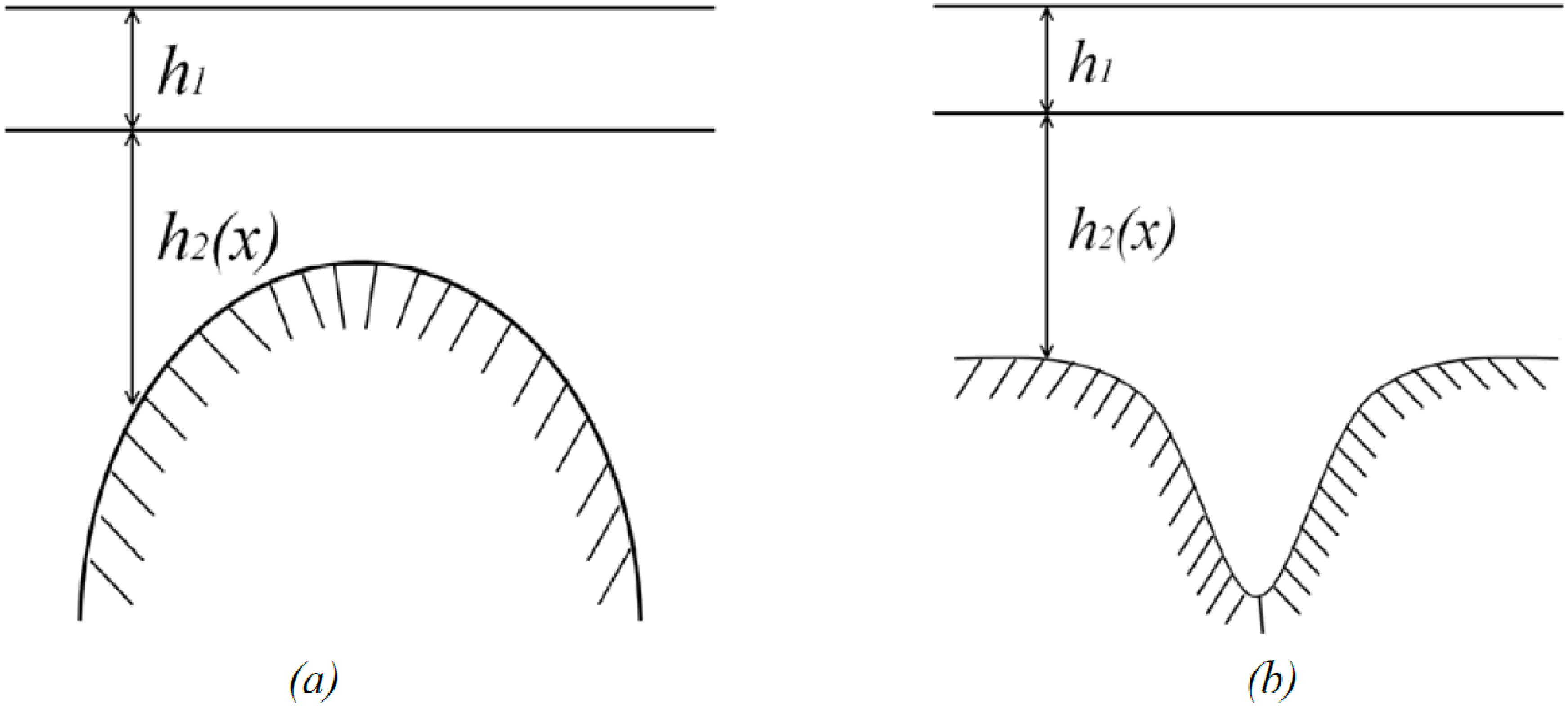
Another example

$$P = -\frac{c_0^2}{4L^2} < 0$$

$$c(x) = c_0\left(\frac{x}{L}\right)$$

$$h_2(x) = h_1 \frac{(x/L)^2}{1-(x/L)^2}$$

with two singular points as above



All profiles have two bifurcation points

Bifurcation of two-layer flow in one-layer flow

1. Internal wave breaks

$$A(x) = \frac{const}{\sqrt{c(x)}}$$

2. Wave fully reflected

Standing waves

$$\eta(x,t) = \frac{Q}{\sqrt{c(x)}} \sin\left[k \int_0^x \frac{dy}{c(y)} \right] \exp(i\omega t)$$

For P = 0

$$\eta(x=0,t) = \frac{3\omega QL}{c_0^{3/2}} \exp(i\omega t) = \frac{\omega \tau_0 Q}{\sqrt{c_0}} \exp(i\omega t)$$

Internal wave “runup”

$$\eta(x=0,t) = 2\tau_0 \frac{d\eta_{inc}(t-\tau_0)}{dt}$$

Nonlinear Internal Wave "Runup"

If $h_2 << h_1$ **Nonlinear equations for two-layer flow reduce to "lower-layer" flow**

$$\frac{\partial}{\partial t}\left(h_2+\eta\right)+\frac{\partial}{\partial x}\left[\left(h_2+\eta\right)u_2\right]=0$$

$$\frac{\partial u_2}{\partial t}+u_2\frac{\partial u_2}{\partial x}+g'\frac{\partial \eta}{\partial x}=0$$

As for surface waves with another gravity acceleration

Carrier-Greenspan (1958) Transformation for constant slope $h_2(x) = \alpha x$

$$\frac{\partial^2 \Phi}{\partial \lambda^2} - \frac{\partial^2 \Phi}{\partial \sigma^2} - \frac{1}{\sigma}\frac{\partial \Phi}{\partial \sigma} = 0$$

All physical variables are expressed by

$$\eta = \frac{1}{2g'}\left(\frac{\partial \Phi}{\partial \lambda} - u_2^2\right) \qquad u_2 = \frac{1}{\sigma}\frac{\partial \Phi}{\partial \sigma}$$

$$x = \frac{1}{2\alpha g'}\left(\frac{\partial \Phi}{\partial \lambda} - u_2^2 + \frac{\sigma^2}{2}\right) \qquad t = \frac{1}{\alpha g'}(\lambda - u_2)$$

$$\sigma = 2\sqrt{gH} = 2\sqrt{g'(\alpha x + \eta)}$$

Common Properties of Runup

Linear Transformation

$$\frac{\partial^2 \Phi}{\partial \lambda^2} - \frac{\partial^2 \Phi}{\partial \sigma^2} - \frac{1}{\sigma}\frac{\partial \Phi}{\partial \sigma} = 0$$

$$\sigma = 2\sqrt{g(h + \cancel{\eta})} \geq 0$$

Implicit form

$$\eta = \frac{1}{2g}\left(\frac{\partial \Phi}{\partial \lambda} - \cancel{u^2}\right)$$

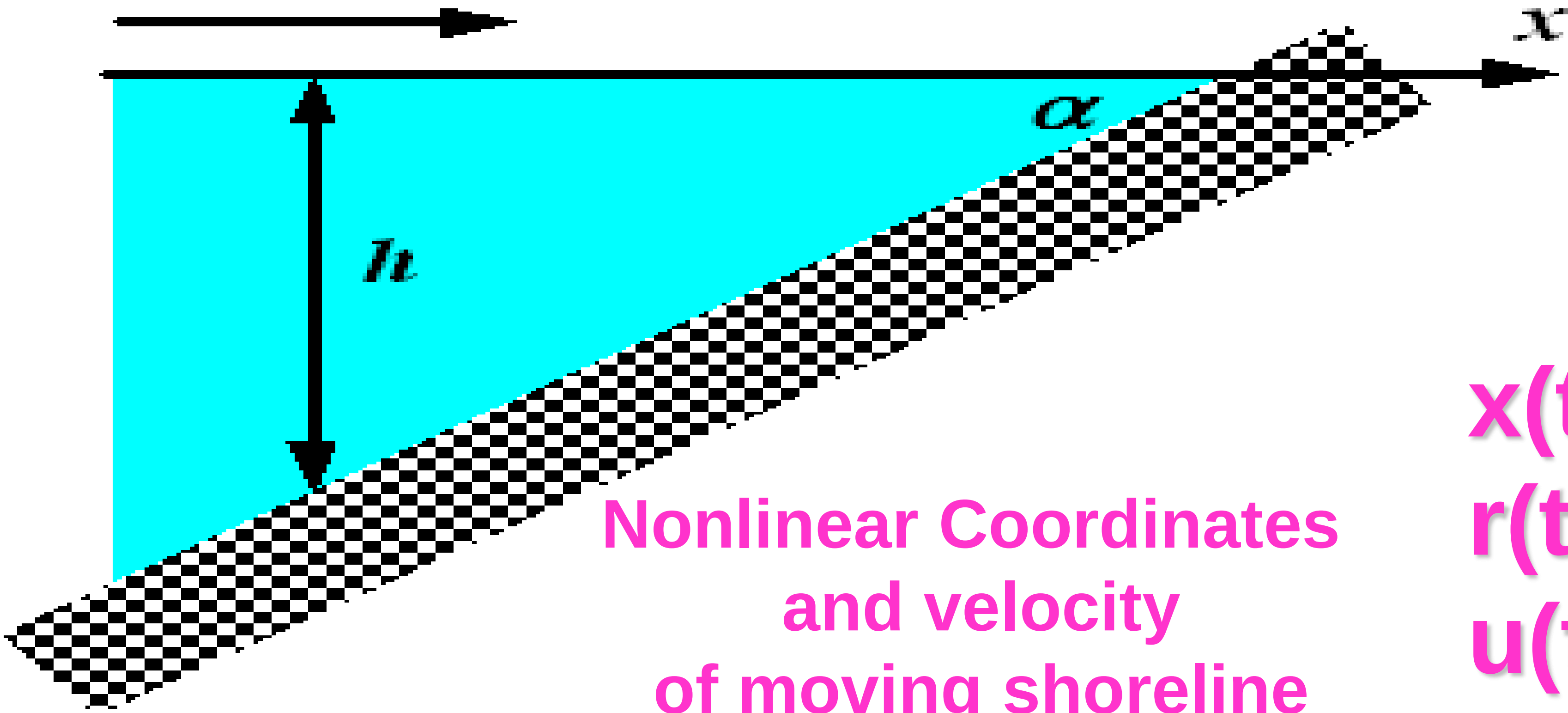
$$u = \frac{1}{\sigma}\frac{\partial \Phi}{\partial \sigma}$$

$$x = \frac{1}{2\alpha g}\left(\cancel{\frac{\partial \Phi}{\partial \lambda}} - \cancel{u^2} - \frac{\sigma^2}{2}\right)$$

$$t = \frac{1}{\alpha g}\left(\lambda - \cancel{\frac{1}{\sigma}\frac{\partial \Phi}{\partial \sigma}}\right)$$

Explicit Solution for Moving Shoreline if Incident Wave is given Far from Shoreline where It is Linear

Maximum Runup is the same in Linear and Nonlinear Theories



Nonlinear Coordinates
and velocity
of moving shoreline

x(t)
r(t)
u(t)

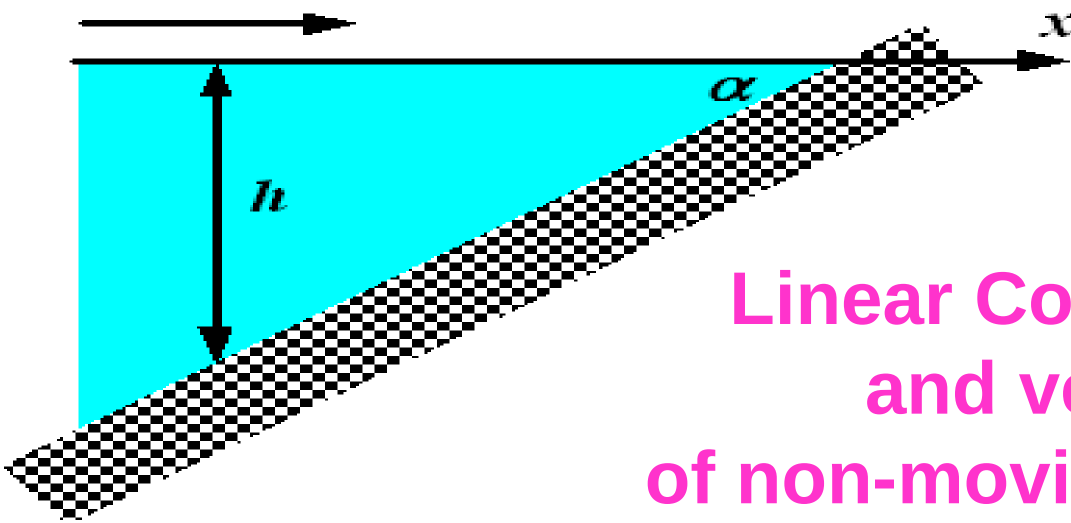
First Step – Solution of Linear Equations For Wave Transformation on Beach

Incident Wave

$$\eta(t) = \sum_{n=1}^{\infty} A_n \sin(n\omega t + \varphi_n)$$

$$R(t) = 2\pi\sqrt{\frac{2L}{\lambda}}\sum_{n=1}^{\infty}\sqrt{n}A_n \sin\left(n\omega t + \varphi_n + \frac{\pi}{4}\right)$$

$$U(t) = \frac{1}{\alpha}\frac{dR}{dt}$$



Linear Coordinates
and velocity
of non-moving shoreline

x=0
R(t)
U(t)

Hodograph Transformation

$$\boxed{\frac{\partial^2\Phi}{\partial\lambda^2} - \frac{\partial^2\Phi}{\partial\sigma^2} - \frac{1}{\sigma}\frac{\partial\Phi}{\partial\sigma} = 0}$$

$$\boxed{\sigma = 2\sqrt{g(h+\eta)} \geq 0}$$

$$\boxed{u = \frac{1}{\sigma}\frac{\partial\Phi}{\partial\sigma}}$$

$$\boxed{t = \frac{1}{\alpha g}\left(\lambda - \frac{1}{\sigma}\frac{\partial\Phi}{\partial\sigma}\right)}$$

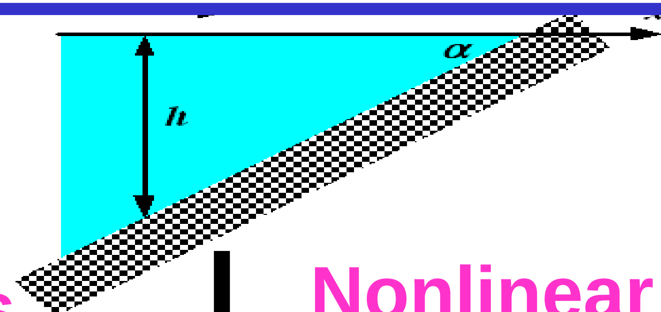
At shoreline

$$\sigma = 0$$

$$\boxed{t = \frac{1}{\alpha g}\left[\lambda - u(0,\lambda)\right]}$$

Second Step – “Nonlinear” Moving Shoreline

$$u(t) = U\left(t + \frac{u}{\alpha g}\right) \qquad r(t) = \alpha \int u(t)\,dt$$



Linear Coordinates
and velocity
of non-moving shoreline

x=0, R(t), U(t)

Nonlinear Coordinates
and velocity
of moving shoreline

x(t), r(t), u(t)

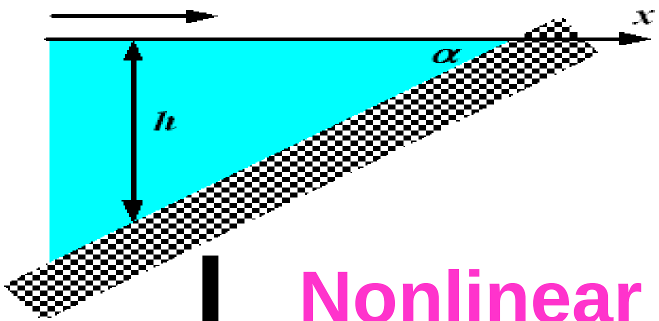
First Result: Linear Theory predicts Maxima

$$u_{\max} = U_{\max}$$

$$r_{\max} = R_{\max}$$

for sine wave

$$\frac{R}{H_0} = 2\pi\sqrt{\frac{2L}{\lambda_0}}$$



Linear Coordinates and velocity of non-moving shoreline

x=0, R(t), U(t)

Nonlinear Coordinates and velocity of moving shoreline

x(t), r(t), u(t)

Second Result: Linear Theory "predicts" Wave Breaking

$$u(t) = U\left(t + \frac{u}{\alpha g}\right)$$

$$\downarrow$$

$$\frac{\partial u}{\partial t} = \frac{U'}{1 - \dfrac{U'}{\alpha g}}$$

$$\boxed{Br = \frac{1}{g\alpha^2}\max\left(\frac{d^2R}{dt^2}\right) = 1}$$

The same as Jacobian Transformation

$$\frac{\partial(t, x)}{\partial(I_+, I_-)} = 0$$

"Linear" Vertical Acceleration = Gravity Acceleration

Breaking Runup Height

$$\boxed{R_{breaking} = \frac{g\alpha^2}{\omega^2}}$$

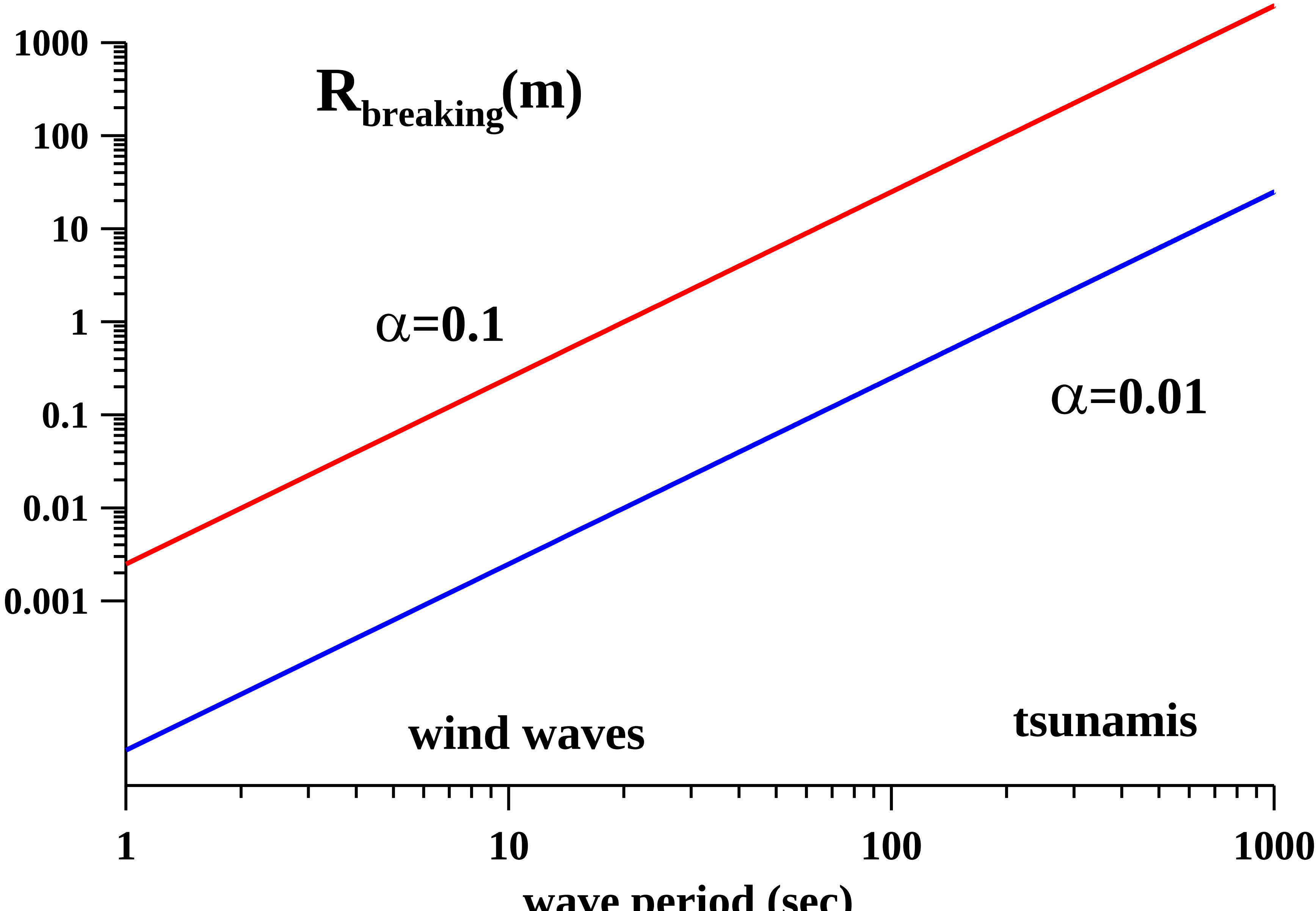
Function only of:
1. Faceslope
2. Wave frequency

For any channel geometry!!

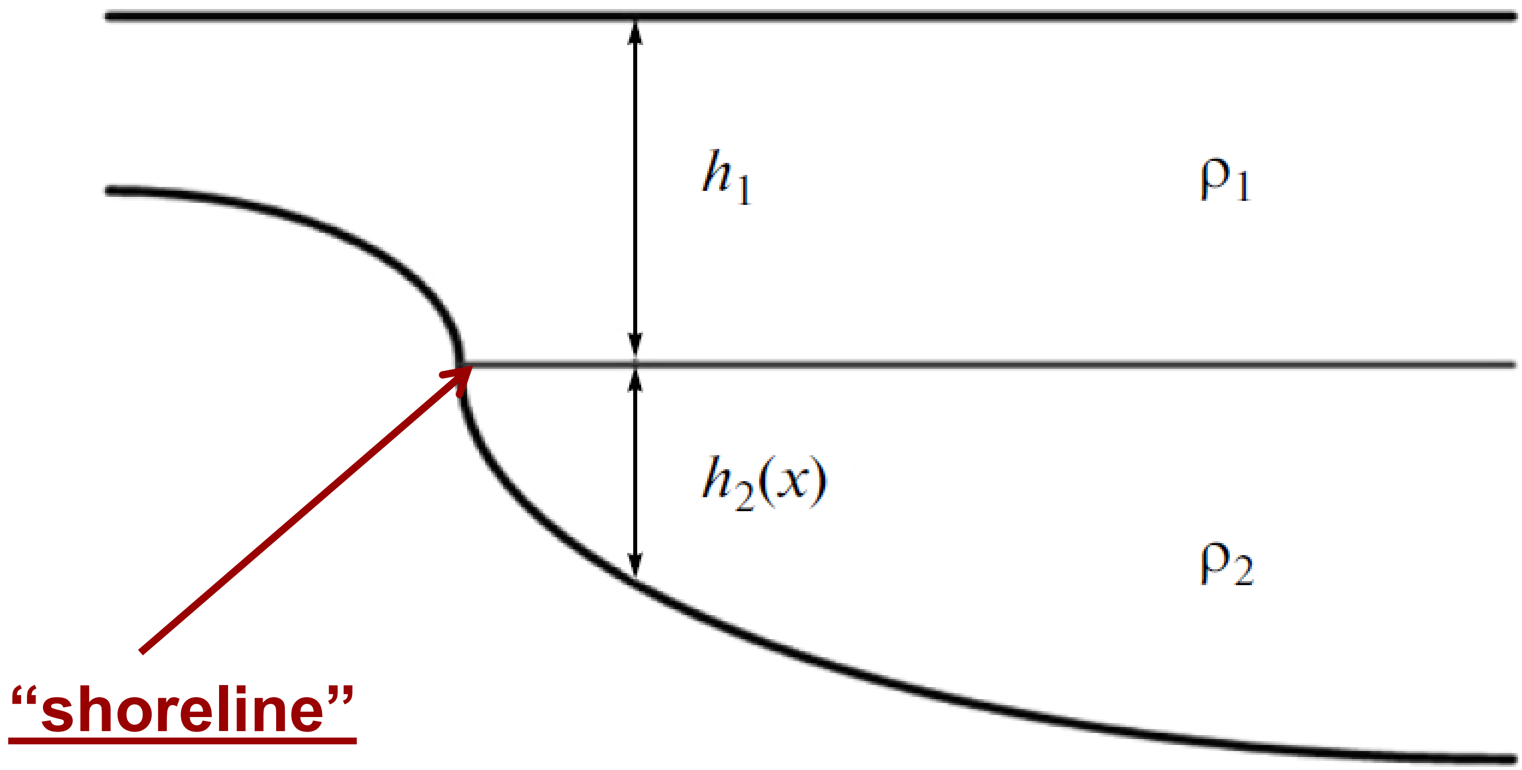
But runup ratio, R/A depends on channel geometry, and wave frequency!!

75% of tsunamis are NOT breaking waves

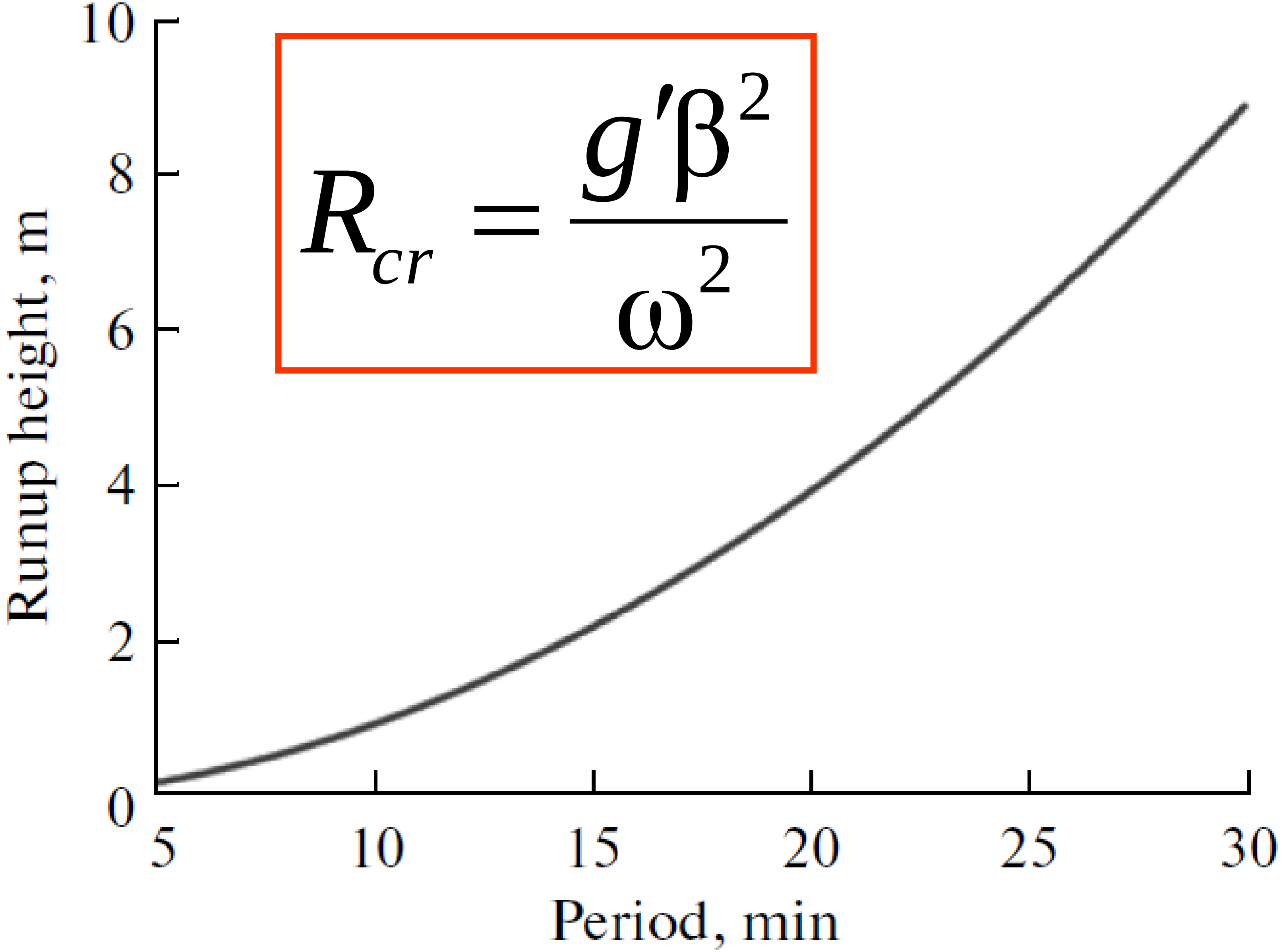
Mazova, Pelinovsky and Soloviev, 1982



“Runup” of Interfacial Waves



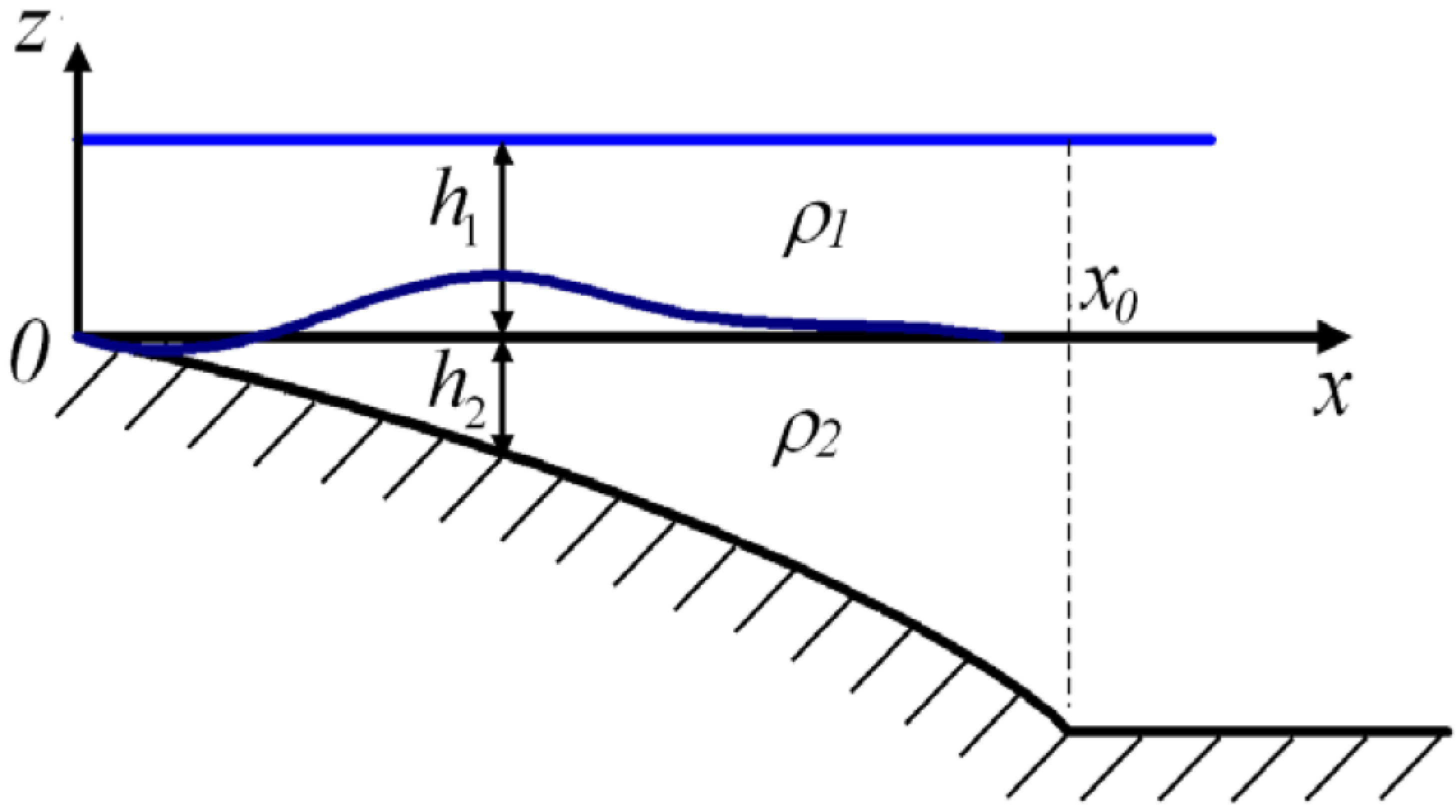
Breaking Interfacial "Runup" Height



$$R_{cr} = \frac{g'\beta^2}{\omega^2}$$

Conjoined bottom profile

(elimination of second singular point with Infinite depth)



Wave Field Matching

On constant depth

$$\eta(x,t) = A_i \exp\left[i\omega\left(t + \frac{x - x_0}{c_1} \right)\right] + A_r \exp\left[i\omega\left(t - \frac{x - x_0}{c_1} \right)\right]$$

On variable depth

$$\eta(x,t) = A_t \sqrt{\frac{c_1}{c(x)}}\left[e^{i\omega(t+\tau)} - e^{i\omega(t-\tau)} \right] = 2iA_t \sqrt{\frac{c_1}{c(x)}} \sin[\omega\tau(x)] \exp(i\omega t)$$

$$\tau(x) = \int_0^x \frac{dy}{c(y)} = \frac{3x_0}{c_1}\sqrt{\frac{c(x)}{c_1}} = \tau_0 \sqrt{\frac{c(x)}{c_1}}$$

Matching

$$A_t\left[\exp(i\omega\tau_0) - \exp(-i\omega\tau_0)\right] = A_i + A_r$$

$$-\frac{1}{3x_0}A_t\left[\exp(i\omega\tau_0) - \exp(-i\omega\tau_0)\right] + \frac{i\omega}{c_1}\left[\exp(i\omega\tau_0) + \exp(-i\omega\tau_0)\right] = \frac{i\omega}{c_1}\left[A_i - A_r\right],$$

Coefficients of reflection and transmission

$$A_r = -A_i \frac{\exp(-i\omega\tau_0) - \dfrac{\sin\omega\tau_0}{\omega\tau_0}}{\exp(i\omega\tau_0) - \dfrac{\sin\omega\tau_0}{\omega\tau_0}}$$

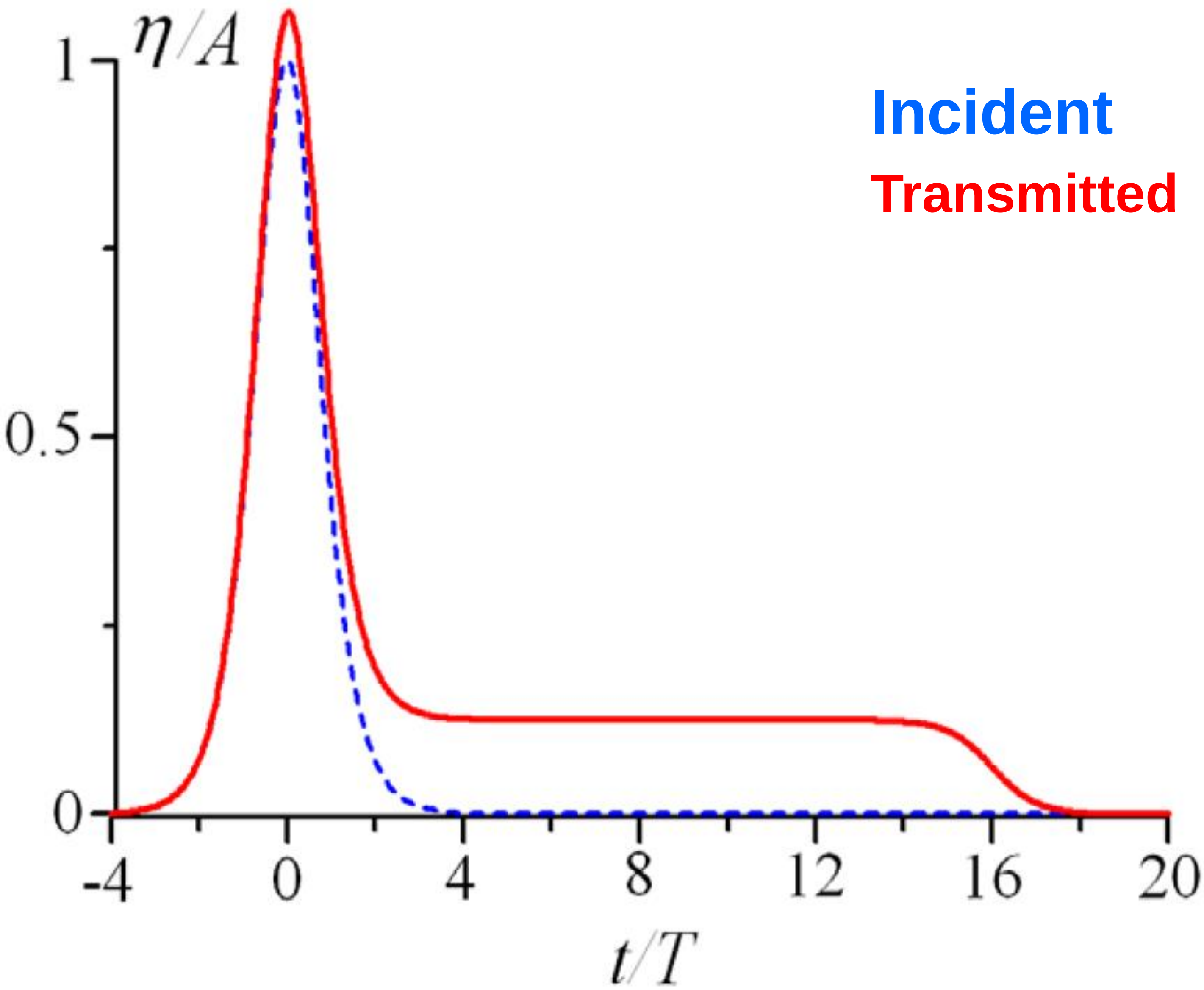
$$A_t = A_i\left[\frac{1}{\exp(i\omega\tau_0) - \dfrac{\sin(\omega\tau_0)}{\omega\tau_0}}\right]$$

Fourier Transform leads to
2d order Fredholm equations

$$\eta_t(t) - \frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0} \eta_t(\xi + t - \tau_0)\,d\xi = \eta_i(t)$$

$$\eta_r(t+\tau_0) - \frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0} \eta_r(t-\zeta)\,d\zeta = -\eta_i(t-\tau_0) + \frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0} \eta_i(t-\zeta)\,d\zeta$$

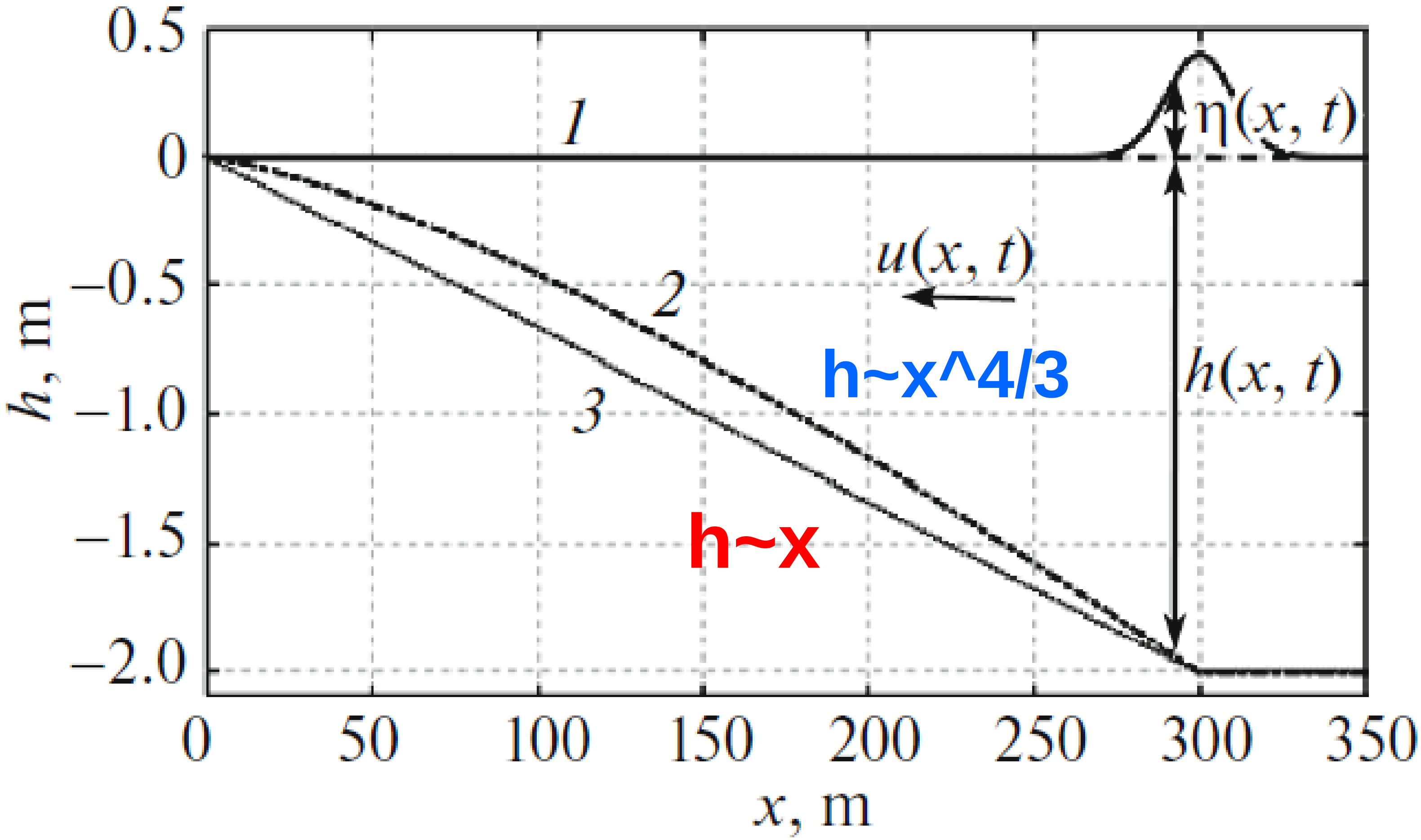
Example:



Long Wave Run-up on Plane and "Non-Reflecting" Slopes

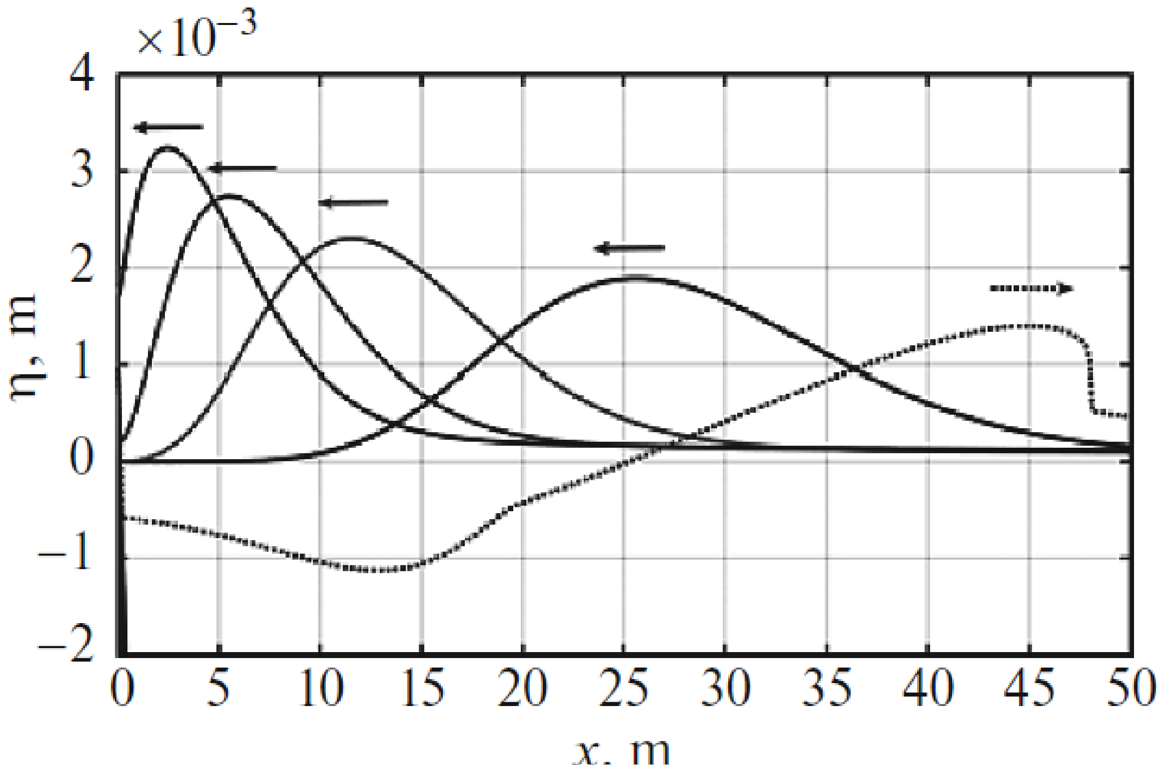
Fluid Dynamics, 2018, Vol. 53, 402–408.

I. I. Didenkulova[1,2,3*], E. N. Pelinovsky[1,2,4,5**], and A. A. Rodin[1]

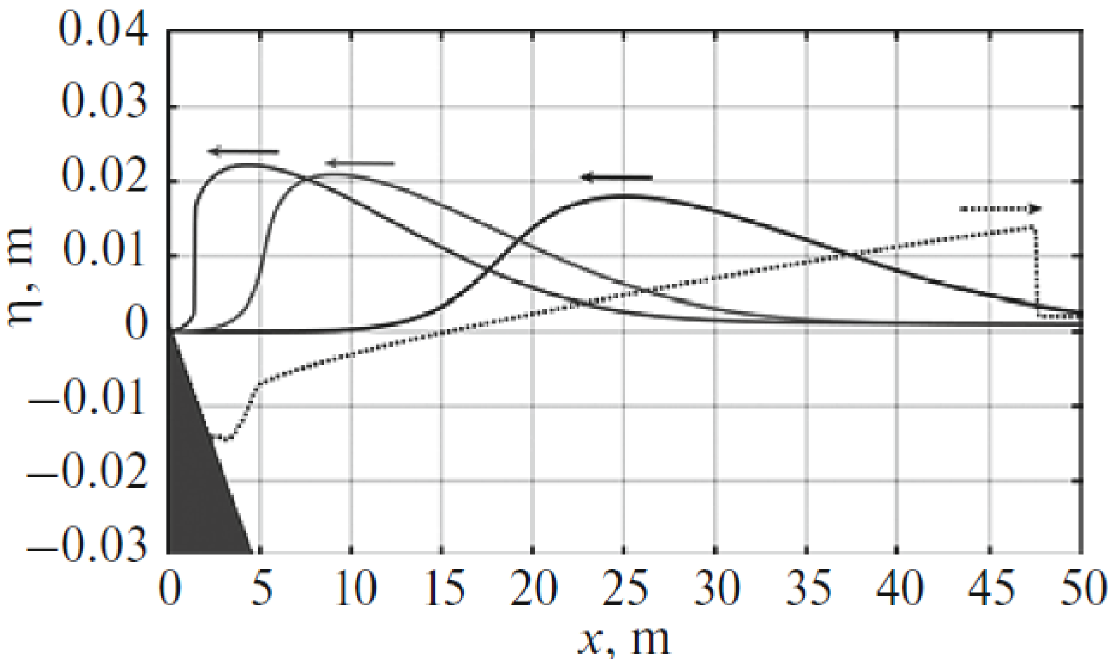


Numerical simulation

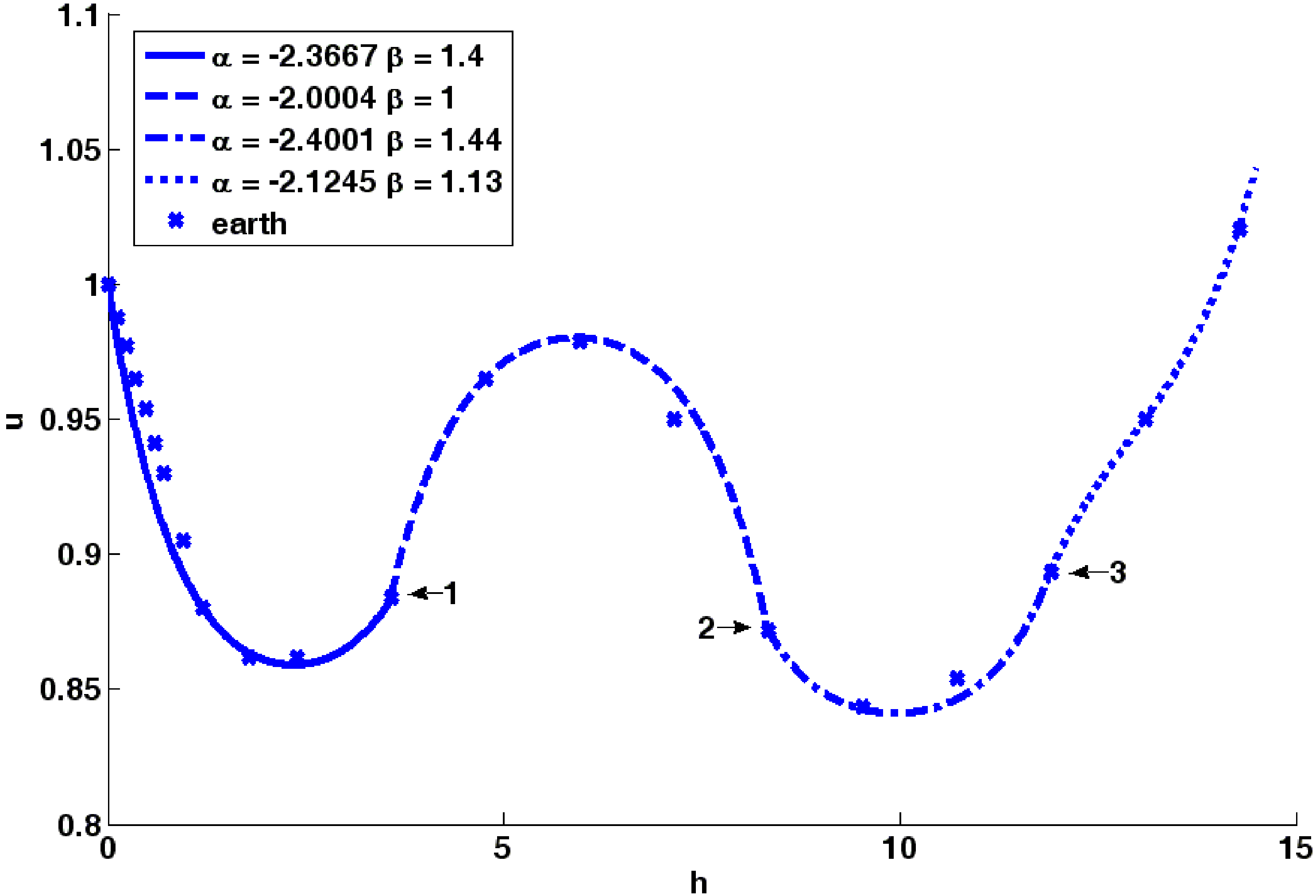
Amplitude 1mm
depth 1 m



Plane beach



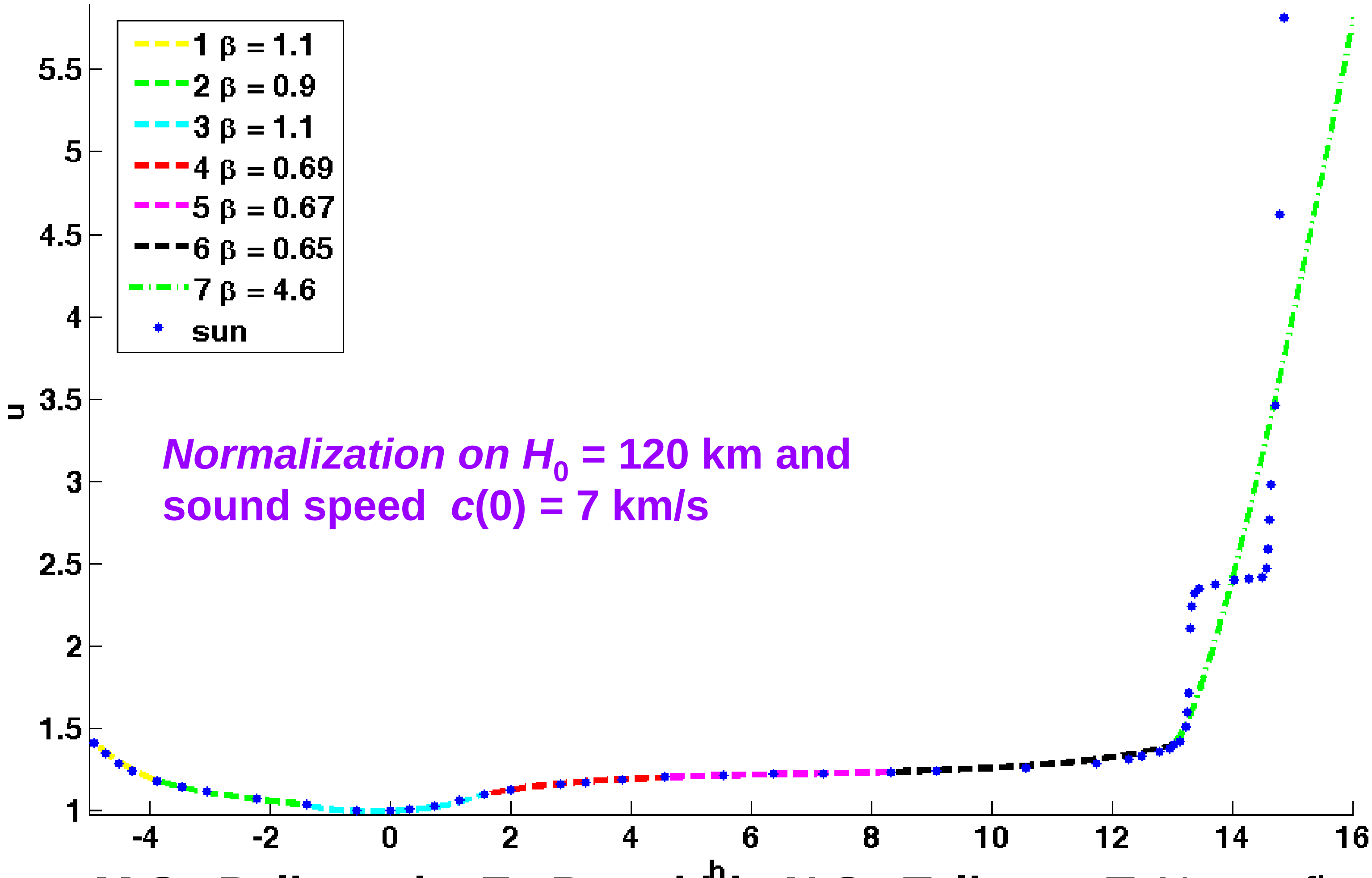
Non-reflected beach



Earth Atmosphere by 4 non-reflected layers

Petrukhin N.S., Pelinovsky E.N., Batsyna E.K. Reflectionless propagation of acoustic waves in the solar atmosphere. ***Astronomy Letters***, 2012, vol. 38, No. 6, 388 – 339.

Solar Atmosphere VAL3c



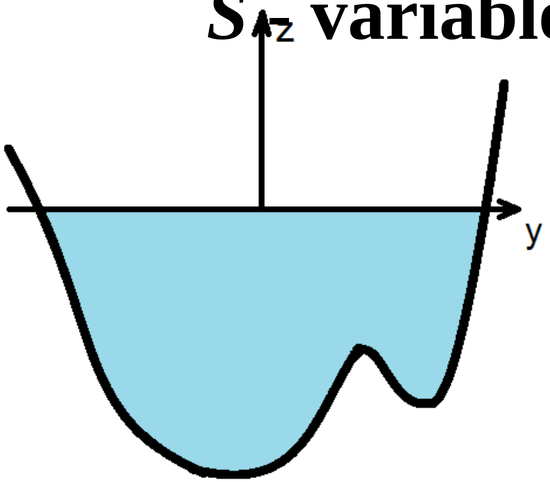
Ruderman M.S., Pelinovsky E., Petrukhin N.S., Talipova T. Non-reflective propagation of kink waves in solar magnetic tubes. ***Solar Physics***, 2013, vol. 286, 417-426.

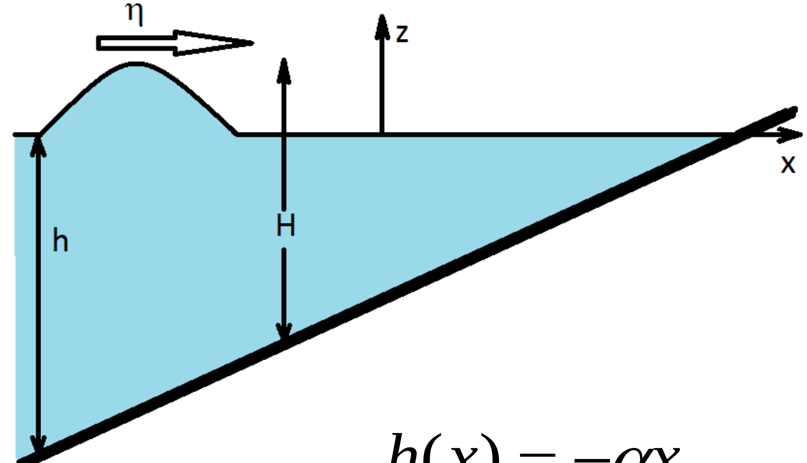
Nonlinear Travelling Waves

Narrow bays and channels

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial H}{\partial x} = g\frac{\partial h}{\partial x}} \qquad \boxed{\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}(Su) = 0}$$

***H* – total water depth, *u* - depth-averaged flow,**
***S* - variable water cross-section of the channel**





$$h(x) = -\alpha x$$

Rybkin, A. Pelinovsky, E., and Didenkulova, I. Nonlinear wave run-up in bays of arbitrary cross-section: generalization of the Carrier-Greenspan approach. ***J Fluid Mechanics***, vol. 748, 416-432

Linear Wave Equation

$$\left(\frac{\partial^2\Phi}{\partial\lambda^2}-\frac{\partial^2\Phi}{\partial\sigma^2}\right)-W(\sigma)\frac{\partial\Phi}{\partial\sigma}=0$$

$$W(\sigma)=\frac{2-dF/d\sigma}{F(\sigma)}$$

all physical variables are expressed through σ and λ:

$$\eta(x,t)=H(\sigma)-\frac{1}{2g}\int_0^\sigma F(\sigma')d\sigma'+\frac{1}{2g}\frac{\partial\Phi}{\partial\lambda}-\frac{u^2}{2g}$$

$$x=\frac{1}{2\alpha g}\left(\frac{\partial\Phi}{\partial\lambda}-u^2-\int_0^\sigma F(\sigma')d\sigma'\right)$$

$$t=\frac{1}{\alpha g}\left(\lambda-\frac{1}{F(\sigma)}\frac{\partial\Phi}{\partial\sigma}\right)$$

$$u=\frac{1}{F(\sigma)}\frac{\partial\Phi}{\partial\sigma}$$

3) bay of parabolic cross-section

$$\boxed{\frac{\partial^2\Phi}{\partial\lambda^2}-\frac{\partial^2\Phi}{\partial\sigma^2}-\frac{2}{\sigma}\frac{\partial\Phi}{\partial\sigma}=0}$$

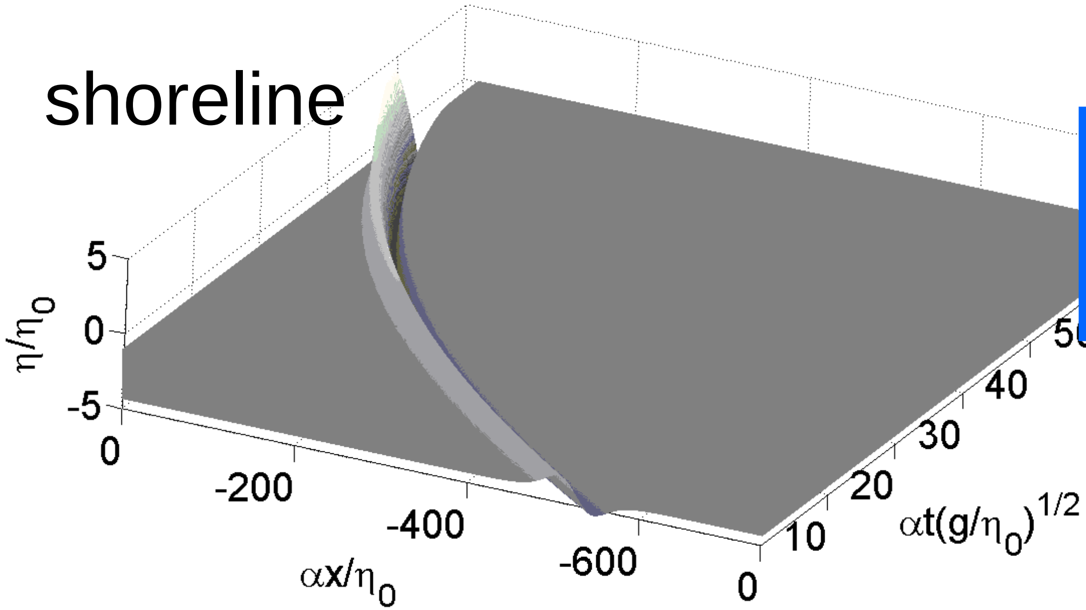
$[m=2]$

$$\boxed{\sigma\geq 0}$$

General solution = sum of for traveling waves

$$\Phi(\sigma,\lambda)=\frac{\Theta_1(\lambda+\sigma)+\Theta_2(\lambda-\sigma)}{\sigma}$$

3) bay of parabolic cross-section

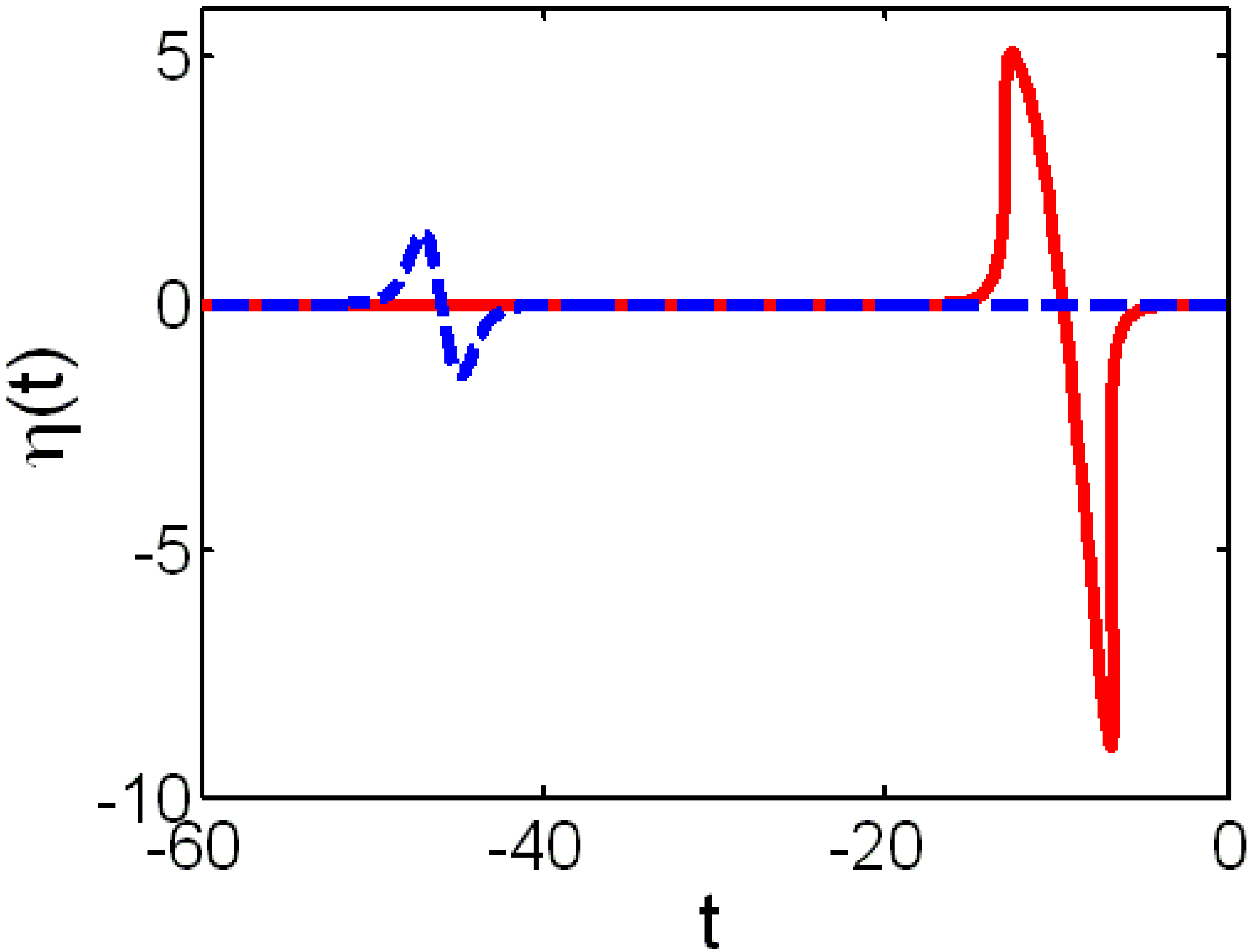


$$\Phi(\sigma,\lambda)=\frac{\Theta(\lambda+\sigma)}{\sigma}$$

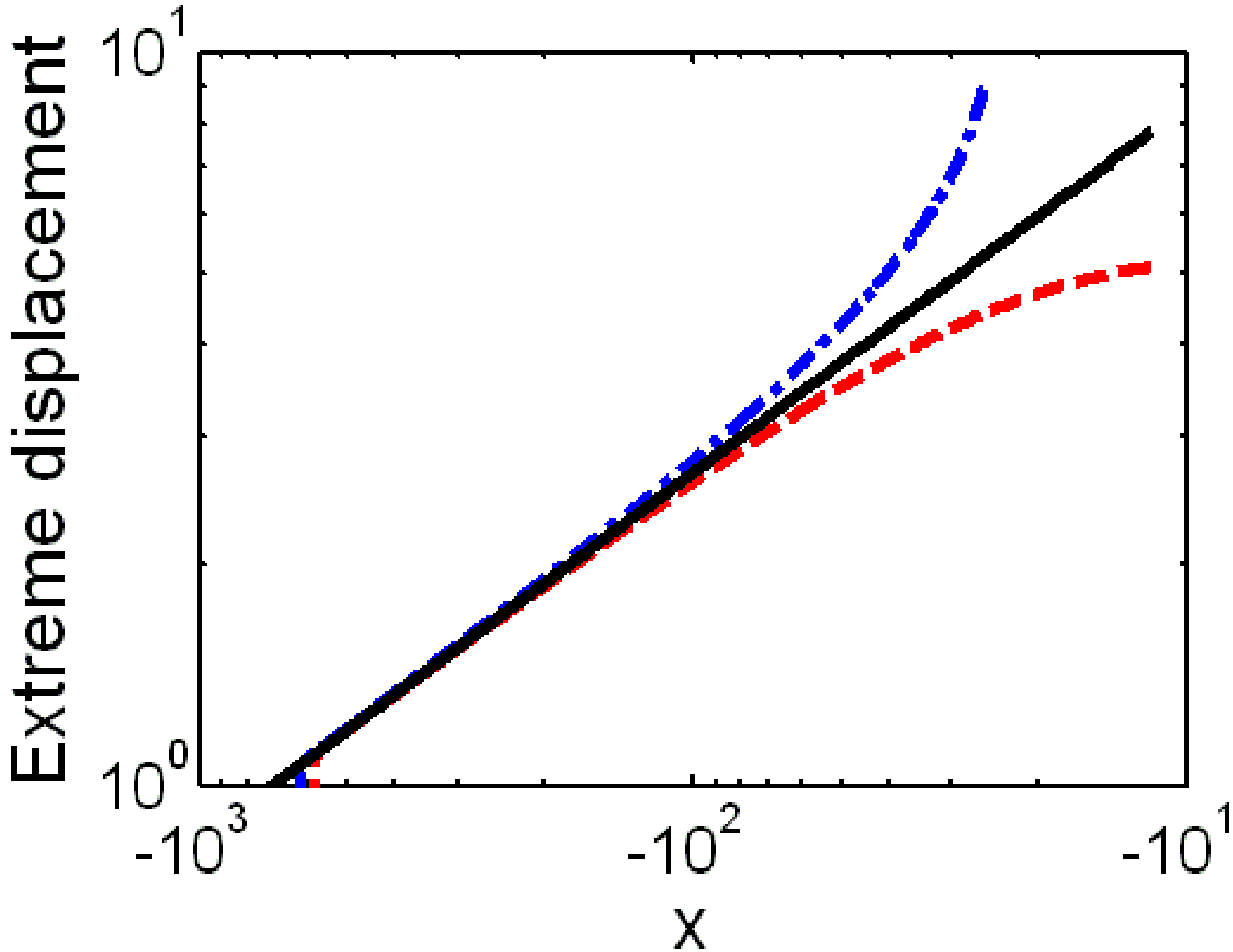
$$R(t)\approx 2\tau_0\,\frac{d\eta_{in}(t-\tau_0)}{dt}$$

- **simple expression for run-up height**
- **strong influence of wave steepness (asymmetry)**

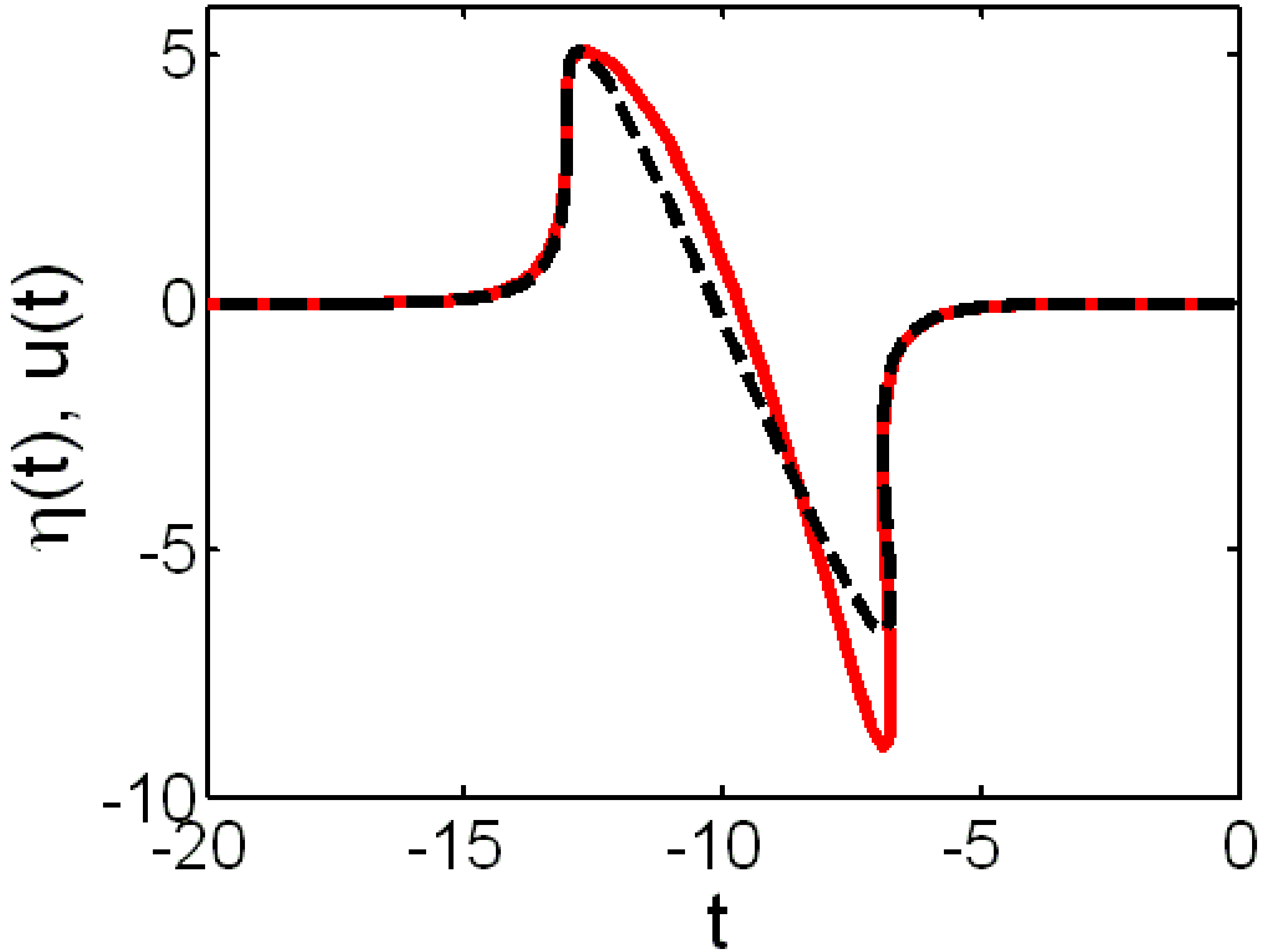
(Didenkulova & Pelinovsky 2009; 2011)



Deformation of the wave shape in approaching wave:
blue dashed and red solid lines correspond to an incident wave and the wave near the shoreline respectively

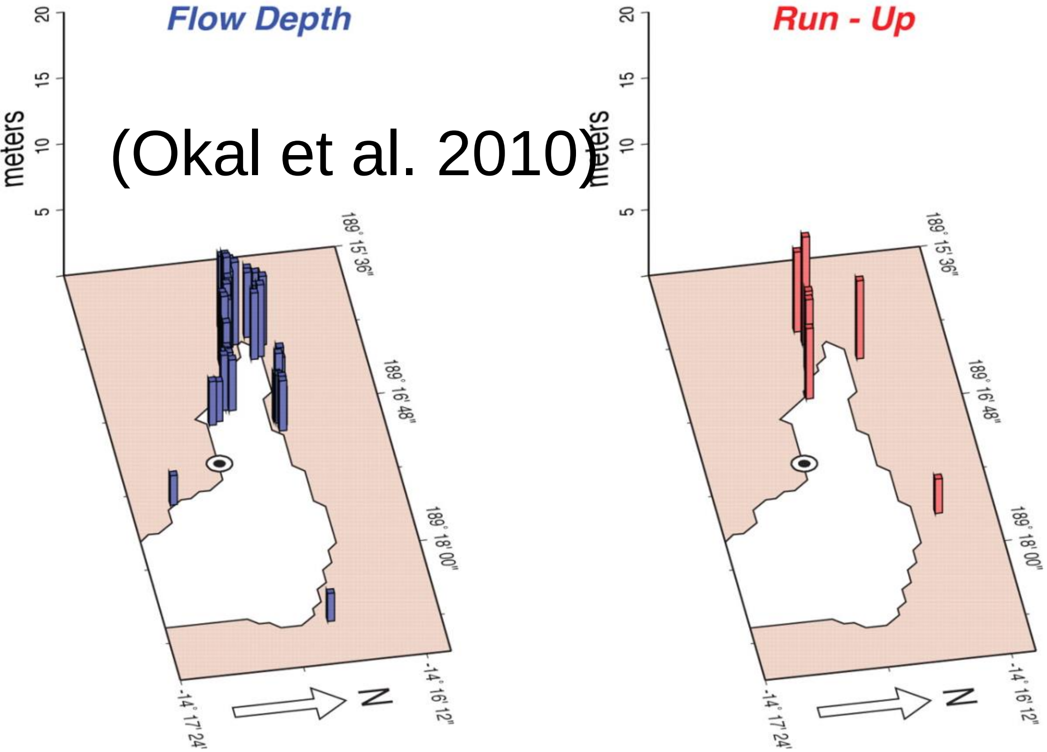


Variation of the positive (red) and negative (blue) amplitudes with distance; black solid line corresponds to the linear Green's law



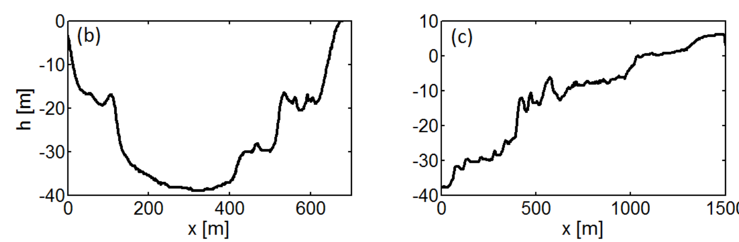
Shapes of water displacement (red) and velocity (black) near the shoreline

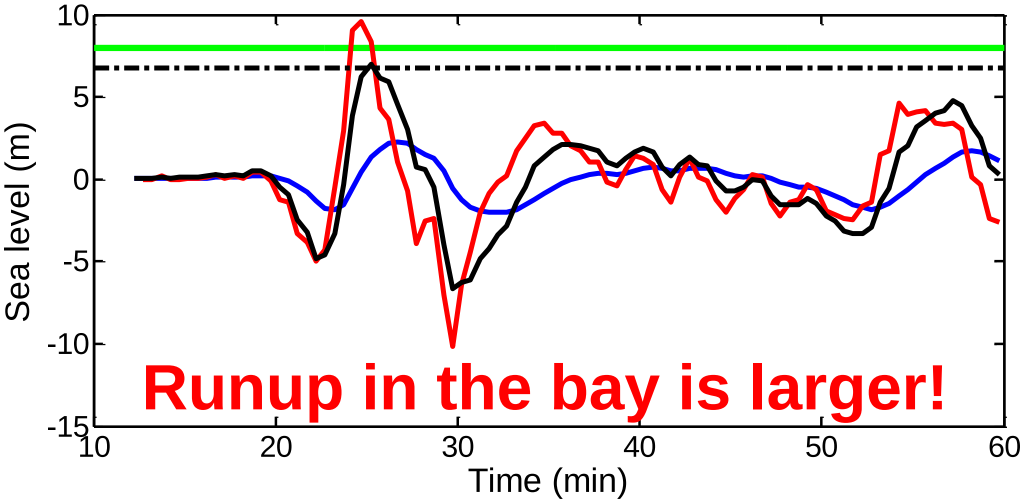
Samoa 2009 tsunami: Pago-Pago



(Okal et al. 2010)

Measured runup – 8m
Calculated for a plane beach – 6.5m

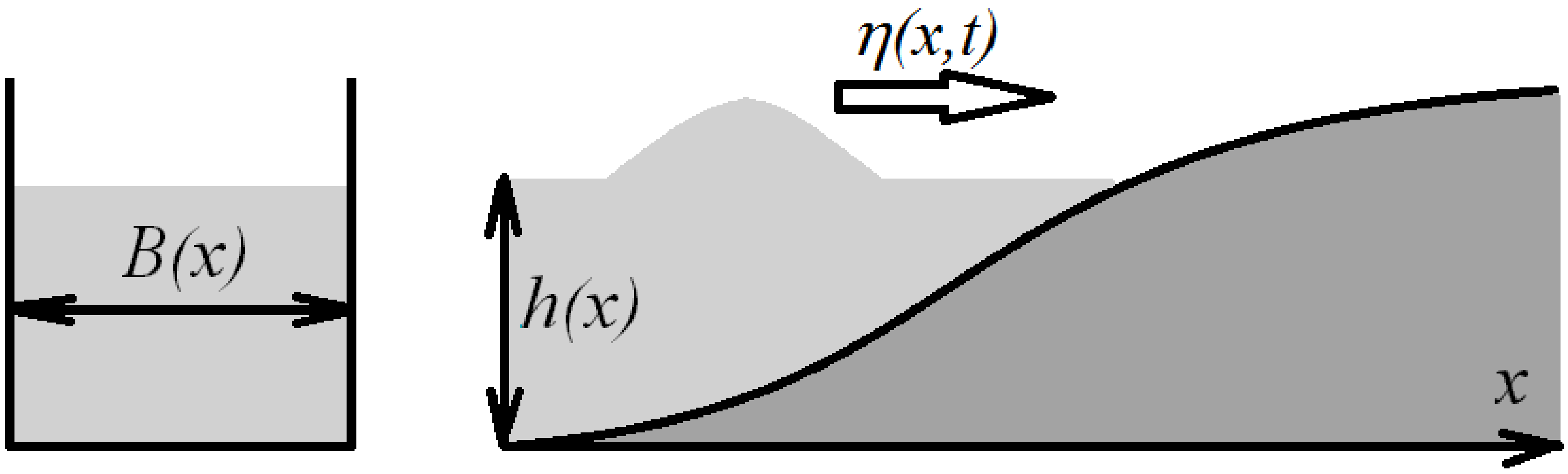




Runup in the bay is larger!

(Didenkulova 2013)

Nonlinear Travelling Waves in Rectangular Channel



$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} = 0$$

Firstly
Hyperbolic Shallow-Water System

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}(uS) = 0$$

$$S = H(x,t)B(x)$$

Beginning from linear case

$$B(x)\frac{\partial^2\eta}{\partial t^2}-\frac{\partial}{\partial x}\left(B(x)c^2(x)\frac{\partial\eta}{\partial x}\right)=0$$

$$c(x)=\sqrt{gh(x)}$$

Reduction to Klein-Gordon equation if

$$\frac{d}{dx}\left[\sqrt{\frac{c}{B}}\frac{d}{dx}(Bc)\right]+\sqrt{\frac{B}{c}}P=0$$

P is constant

Self-Consistent Non-reflected Channel

$$\frac{d}{dx}\left[\sqrt{\frac{c}{B}}\frac{d}{dx}(Bc)\right]+\sqrt{\frac{B}{c}}P=0$$

P = 0

$$B(x)c(x)=\mathrm{const}$$

In this case

$$B(x)\frac{\partial^2\eta}{\partial t^2}-\frac{\partial}{\partial x}\left(B(x)c^2(x)\frac{\partial\eta}{\partial x}\right)=0$$

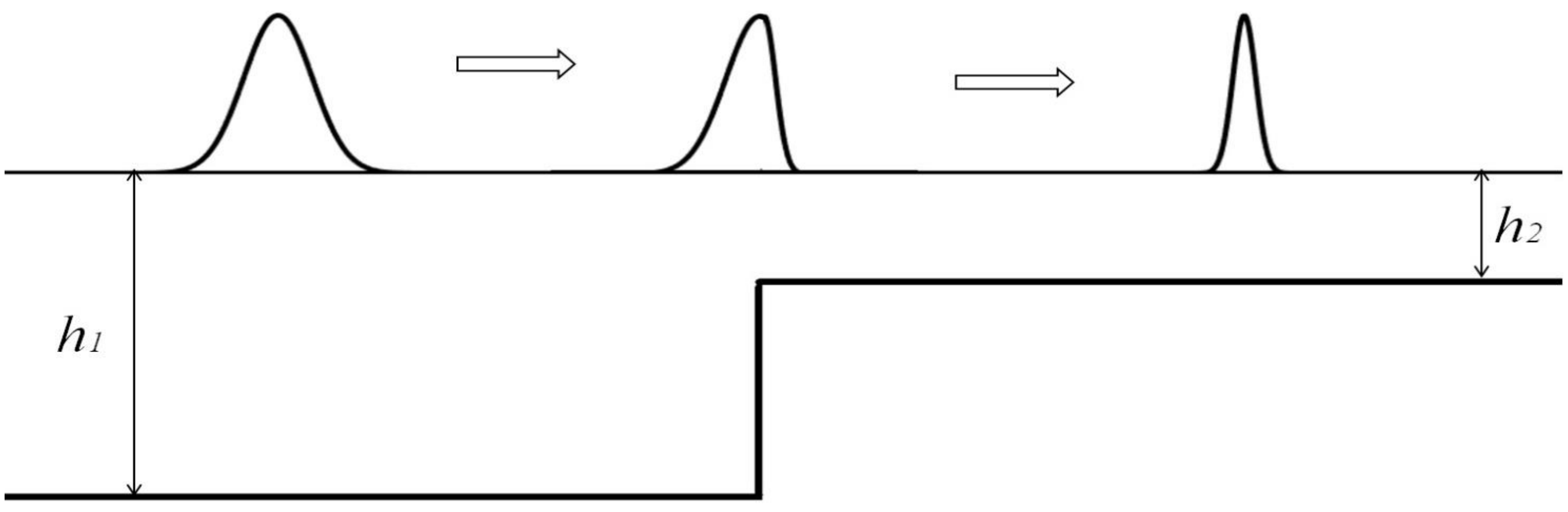
transforms to

$$\frac{\partial^2\eta}{\partial t^2}-c\frac{\partial}{\partial x}\left(c\frac{\partial\eta}{\partial x}\right)=0$$

Travelling Waves

$$\eta(x,t) = A_0\Phi[t-\tau(x)] = A_0\Phi\left[t - \int_{x_0}^{x}\frac{dy}{\sqrt{gh(y)}}\right]$$

$$u(x,t) = \frac{g}{c(x)}\eta[t-\tau(x)] = \sqrt{\frac{g}{h(x)}}\eta[t-\tau(x)]$$



Derived KdV-like equation for arbitrary h(x)

$$\frac{\partial \eta}{\partial t} + c\left[1 + \frac{ch}{4}\frac{d^2h}{dx^2} + \frac{3\eta}{2h}\right]\frac{\partial \eta}{\partial x} + \frac{c}{6h}\frac{\partial}{\partial x}\left(h^3 \frac{\partial^2 \eta}{\partial x^2}\right) = \frac{c\eta^2}{4h^2}\frac{dh}{dx}$$

Didenkulova, Pelinovsky, et al, Journal Physics A Math. Theory, 2017

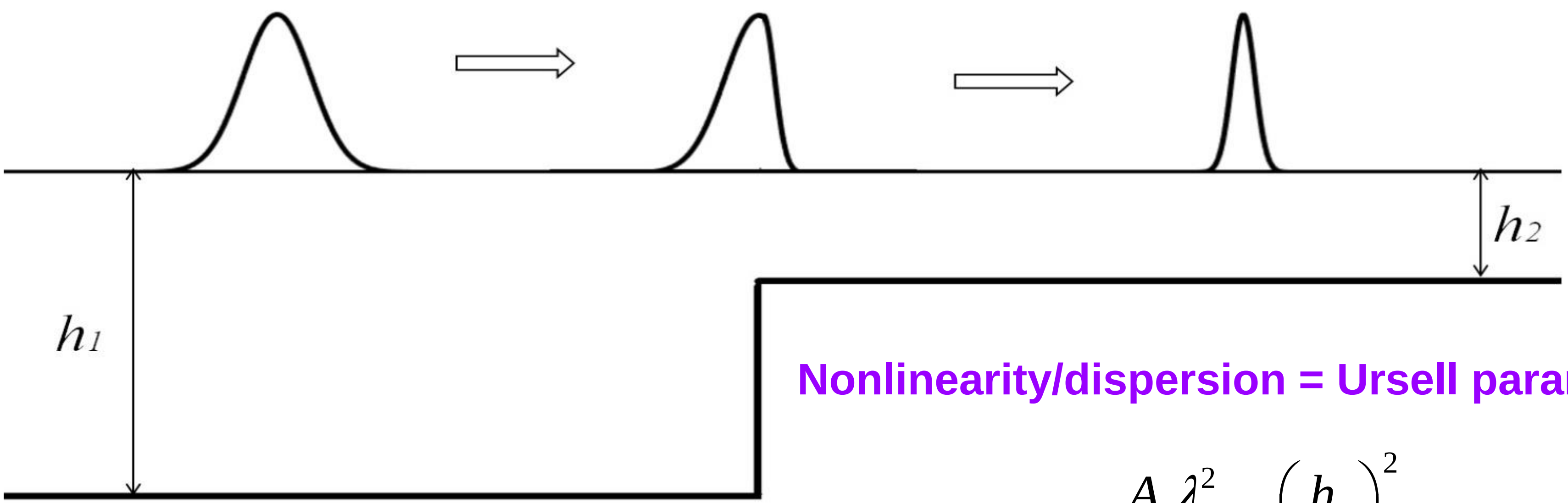
For smoothly varied bottom

$$\frac{\partial \eta}{\partial t} + c\left[1 + \frac{3}{2}\frac{\eta}{h(x)}\right]\frac{\partial \eta}{\partial x} + \frac{ch^2(x)}{6}\frac{\partial^3 \eta}{\partial x^3} = 0$$

as for flat bottom (no linear shoaling)

“Adiabatic” soliton $A(x) \sim h^{-2/3}$ $\lambda(x) \sim h^{11/6}$

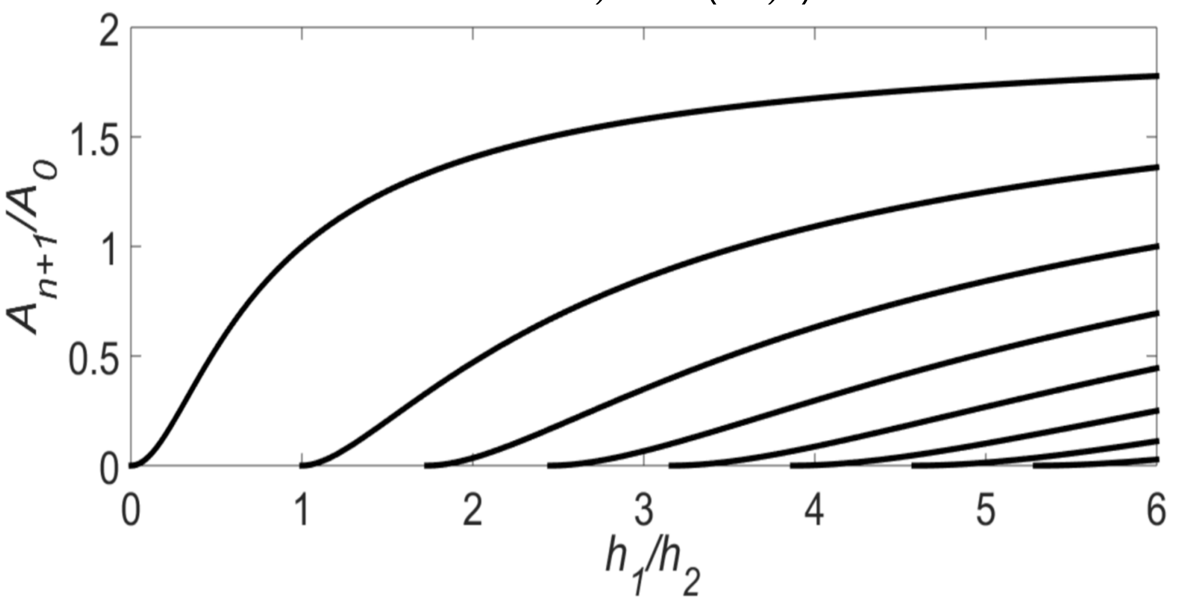
Soliton Fission on a Step



Nonlinearity/dispersion = Ursell parameter

$$U_2 = \frac{A_2 \lambda_2^2}{h_2^3} = \left(\frac{h_1}{h_2}\right)^2$$

$$\frac{A_{n+1}}{A_0} = \frac{1}{4U_2}\left[\sqrt{1+8U_2} - (1+2n)\right]^2$$



Nonlinear Dispersive Travelling Waves Exist!

Therefore, Rogue Waves appear in strongly inhomogeneous media due to similar mechanisms (of course, additionally to geometric focusing)

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **49** (2016) 194001 (11pp) doi:10.1088/1751-8113/49/19/194001

On shallow water rogue wave formation in strongly inhomogeneous channels

Ira Didenkulova[1,2] and Efim Pelinovsky[3,4]

Conclusions:

“Non-Reflected” Potential allows:

1. To get the BIG wave amplification

2. To analyze pointed and distributed reflection of solitary waves

3. To give simple algorithm to compute wave propagation above complicated relief

4. More mechanisms of rogue wave formation