Time-frequency analysis of nonlinear ship waves

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Ship waves
Overview

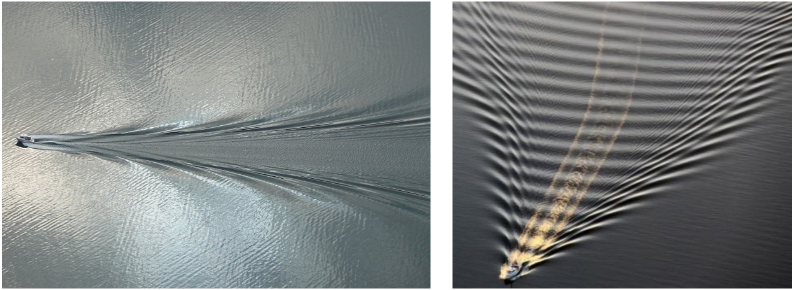


Figure: The wake of a boat

Spectrogram
Overview

- Relatively simple to set up
- Can analyse the different frequency components of a wave.
- Can be used to measure energy of a passing wave.

Figure: A LOG_aLevel echosounder mounted on a tripod[1].

[1] T. Torsvik, H. Herrmann, I. Didenkulova & A. Rodin, Analysis of ship wake transformation in the coastal zone using time-frequency methods. *Proc. Est. Acad. Sci.* **64** (2015) 379–388.

Computing the Spectrogram

- For a given signal $s(t)$ a short-time Fourier transform is given by

$$S(t,\omega) = \int_{-\infty}^{\infty} w(\tau - t)s(\tau)\mathrm{e}^{-i\omega\tau}\,\mathrm{d}\tau,$$

  where $w(t)$ is a window function.
- Plot $\log_{10}(|S(t,\omega)|^2)$.

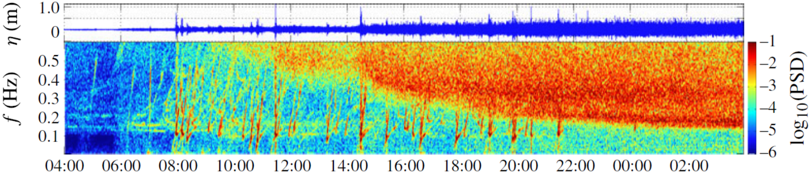
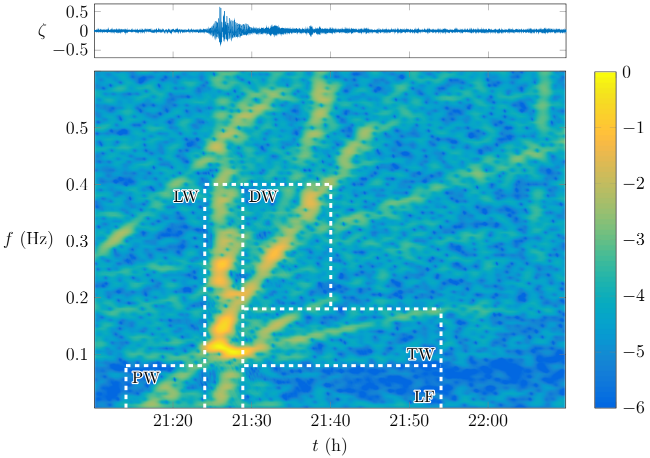


Figure: A spectrogram of the measured water level[2]

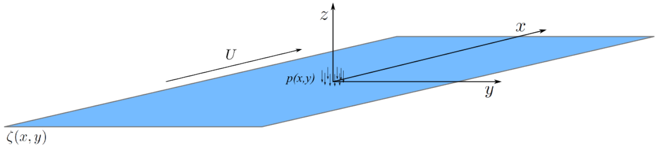
[2]T. Torsvik, T. Soomere, I. Didenkulova, & A. Sheremet, Identification of ship wake structures by a timefrequency method. *J. Fluid Mech.* **765** (2015) 229–251.

Torsvik *et al.* (2015) spectrogram



T. Torsvik, T. Soomere, I. Didenkulova, & A. Sheremet, Identification of ship wake structures by a timefrequency method. *J. Fluid Mech.* **765** (2015) 229–251.

The free surface problem



- Laplace's equation

$$\nabla^2\Phi = 0 \qquad \text{for} \quad z < \zeta(x, y)$$

- Far-field condition:

$$(\Phi_x, \Phi_y, \Phi_z) \rightarrow (1, 0, 0), \qquad \text{as} \quad z \rightarrow -\infty$$
$$(\Phi_x, \Phi_y, \Phi_z) \rightarrow (1, 0, 0), \quad \zeta \rightarrow 0 \qquad \text{as} \quad x \rightarrow -\infty$$

The free surface problem

- Boundary conditions

$$\Phi_x\zeta_x + \Phi_y\zeta_y = \Phi_z \qquad \text{on} \quad z = \zeta(x, y),$$

$$\frac{1}{2}(\Phi_x^2 + \Phi_y^2 + \Phi_z^2) + \zeta + \epsilon p(x, y) = \frac{1}{2} \qquad \text{on} \quad z = \zeta(x, y),$$

where

$$p(x, y) = \mathrm{e}^{-\pi^2 F^4 (x^2+y^2)},$$

and

$$F = \frac{U}{\sqrt{gL}}, \qquad \epsilon = \frac{P_0}{\rho U^2}.$$

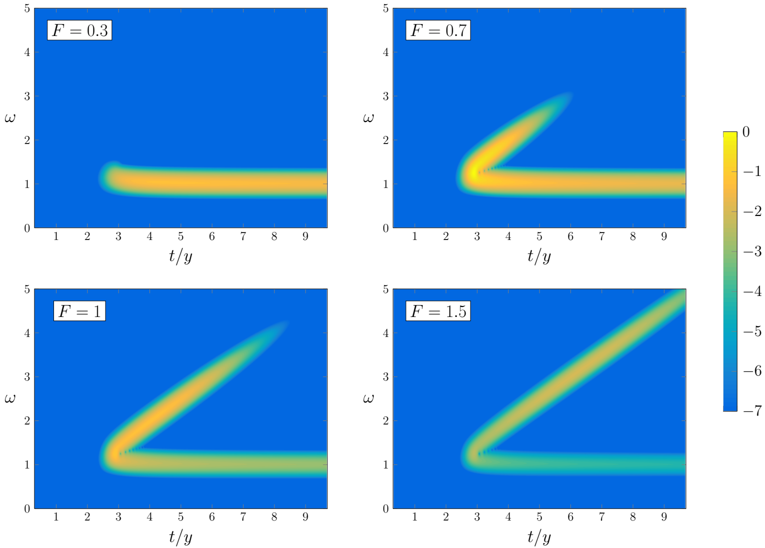
The exact linear solution

- Linear regime $\epsilon \ll 1$ (small amplitude waves)
- Exact solution:

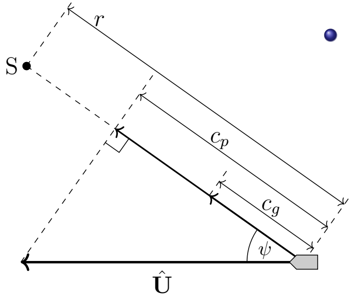
$$\zeta(x,y) = -\epsilon p(x,y) + \frac{\epsilon}{2\pi^2} \int_{-\pi/2}^{\pi/2} \int_{0}^{\infty} \frac{k^2 \tilde{p}(k,\psi) \cos(k[|x| \cos\psi + y \sin\psi])}{k - k_0} \, \mathrm{d}k \, \mathrm{d}\psi - \frac{\epsilon H(x)}{\pi} \int_{-\pi/2}^{\pi/2} k_0^2 \tilde{p}(k_0,\psi) \sin(k_0[x \cos\psi + y \sin\psi]) \, \mathrm{d}\psi,$$

where $k_0 = \sec^2 \psi$.

Small amplitude waves



Linear dispersion curve



- From the diagram, $c_p = \cos\psi$
- For fluid of infinite depth

$$\Omega(k) = \sqrt{k}$$
$$c_p = \frac{\Omega(k)}{k} = \frac{1}{\sqrt{k}}$$
$$c_g = \frac{\mathrm{d}\Omega}{\mathrm{d}k} = \frac{1}{2\sqrt{k}} = \frac{c_p}{2}$$

Linear dispersion curve parametrised in terms of the ship's time, t_s

$$(t, \omega) = \left(t_s + \frac{r}{c_g}, \Omega(k)\right)$$

Linear dispersion curve

- At a given t_s we have

$$r = \sqrt{t_s^2 + y^2}, \qquad \psi = -\tan^{-1}\frac{y}{t_s}$$

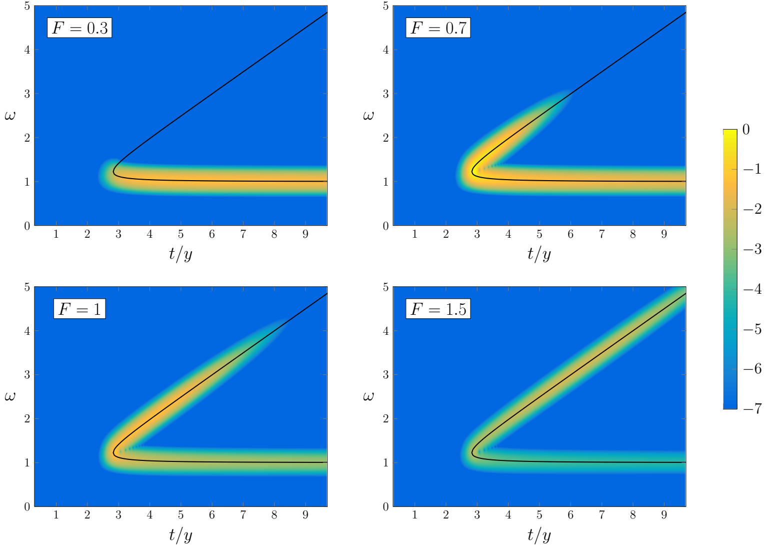
- Use $c_p = \cos\psi$ to give

$$k = \frac{t_s^2 + y^2}{t_s^2}$$

- Finally, we have

$$(t, \omega) = \left(t_s - 2\frac{t_s^2 + y^2}{t_s}, -\frac{\sqrt{t_s^2 + y^2}}{t_s}\right)$$

Small amplitude waves



Deriving an integral equation

using Green's second identity

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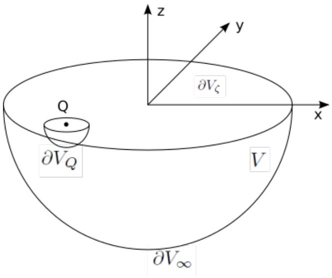


Green's Second Identity

$$\iiint_V (G\nabla^2 H - H\nabla^2 G)\,\mathrm{d}V = \oiint_{\partial V} \left(G\frac{\partial H}{\partial n} - H\frac{\partial G}{\partial n} \right) \mathrm{d}S$$

where

$$G(x^*, y^*, z^*; x, y, z) = \frac{1}{4\pi}\frac{1}{R},$$

$$R = \sqrt{(x^* - x)^2 + (y^* - y)^2 + (z^* - z)^2}.$$

$$H(x, y, z) = \Phi(x, y, z) - x,$$

Deriving an integral equation

in its full glory

Boundary integral equation:

$$2\pi(\phi(x,y)-x)=\int_0^\infty\int_{-\infty}^\infty(\phi(x^*,y^*)-\phi(x,y)-x^*+x)K_1\,\mathrm{d}x^*\,\mathrm{d}y^*$$

$$+\int_0^\infty\int_{-\infty}^\infty\zeta_x(x^*,y^*)K_2\,\mathrm{d}x^*\,\mathrm{d}y^*,$$

Bernoulli's equation

$$\frac{1}{2}\frac{(1+\zeta_x^2)\phi_y^2+(1+\zeta_y^2)\phi_x^2-2\zeta_x\zeta_y\phi_x\phi_y}{1+\zeta_x^2+\zeta_y^2}+\zeta+\epsilon p(x,y)=\frac{1}{2},\quad \text{on}\quad z=\zeta(x,y).$$

Towards a nonlinear system

applying the discretisation

- Create a surface mesh $x_1, \ldots, x_N$ and $y_1, \ldots, y_M$.
- We have $2(N+1)M$ non-linear equations

$$\mathbf{F}(\mathbf{u}) = \mathbf{0}$$

  that are solved using a preconditioned Jacobian-free Newton-Krylov method with preconditioner matrix **P**.
- The preconditioner is taken from the linearised system of equations.[3]

[3] R. Pethiyagoda, S. W. McCue, T. J. Moroney & J. M. Back, Jacobian-free Newton-Krylov methods with GPU acceleration for computing nonlinear ship wave patterns. *J. Comp. Phys.* **269**,(2014) 297–313.

Stitching method

We achieve large domains through an iterative stitching process

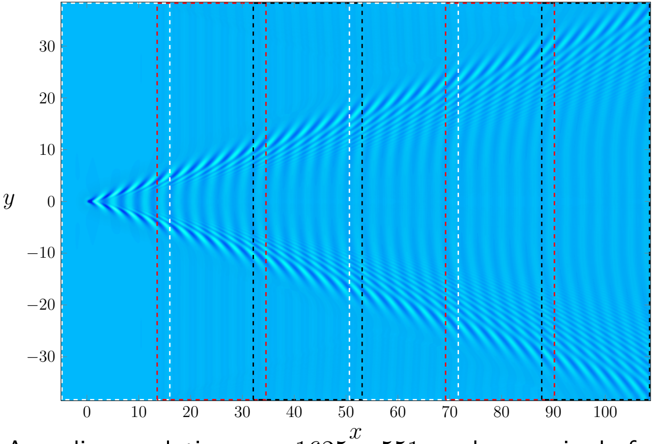
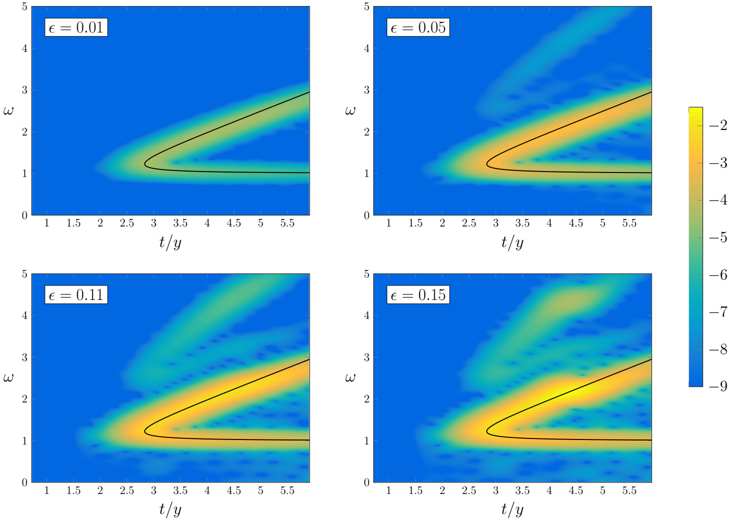


Figure: A nonlinear solution on a 1625×551 mesh comprised of six 301×551 panels.

$F = 1$



$\epsilon = 0.01$
$\epsilon = 0.05$
$\epsilon = 0.11$
$\epsilon = 0.15$
ω
t/y
0
1
2
3
4
5
1.5
2.5
3.5
4.5
5.5
−2
−3
−4
−5
−6
−7
−8
−9

Second-order dispersion curve

- A second order solution[4] to the ship wave problem can be given by

$$\zeta + \zeta_0 = \sum_{n=1}^{N} a_n \cos R_n + \sum_{r=1}^{N} \sum_{s=1}^{r} \left\{ b'_{rs+} \cos(R_r + R_s) + b'_{rs-} \cos(R_r - R_s) \right\},$$

where ζ_0 is the mean fluid height and $R_n = k_n(x \cos\psi_n + y \sin\psi_n)$.

- Assume only two waves exist, given by upper (ω_1) and lower (ω_2) branches.
- The second order dispersion curves, are given by

$$\omega_{3,4} = 2\omega_{1,2},$$
$$\omega_{5,6} = \omega_1 \pm \omega_2.$$

[4] N. Hogben, Nonlinear distortion of the Kelvin ship-wave pattern. *J. Fluid Mech.* **55** (1972) 513–528.

Second-order dispersion curve

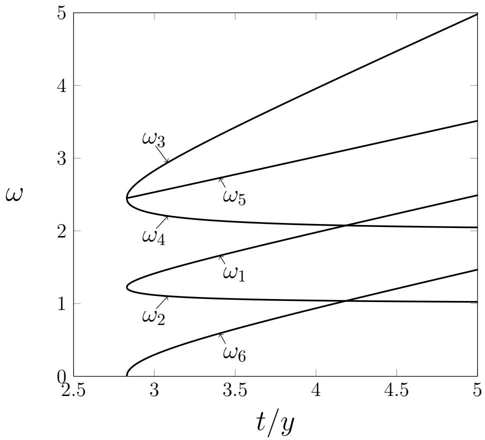
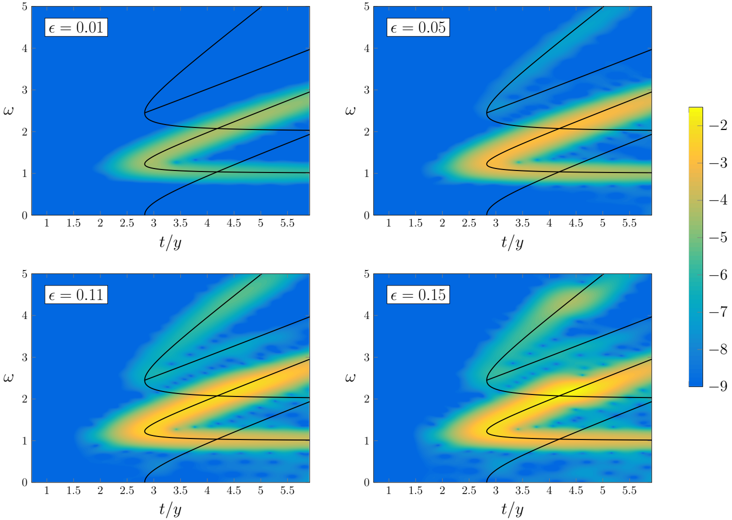
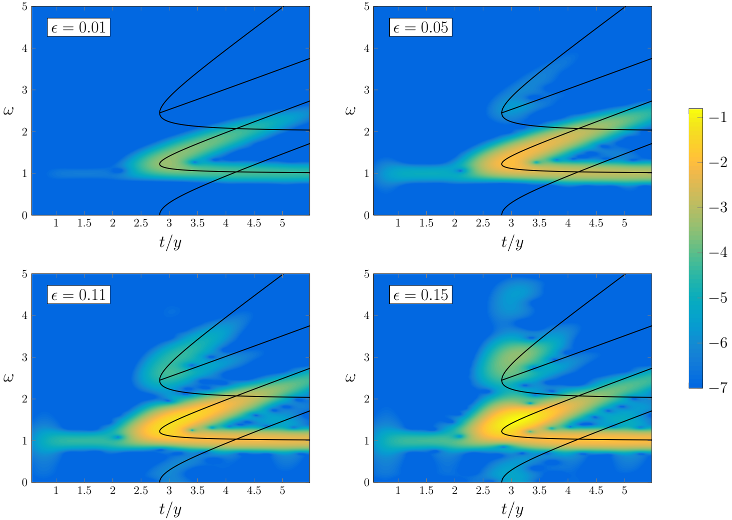


Figure: The linear ($\omega_{1,2}$) and second-order ($\omega_{3,4,5,6}$) dispersion curves.

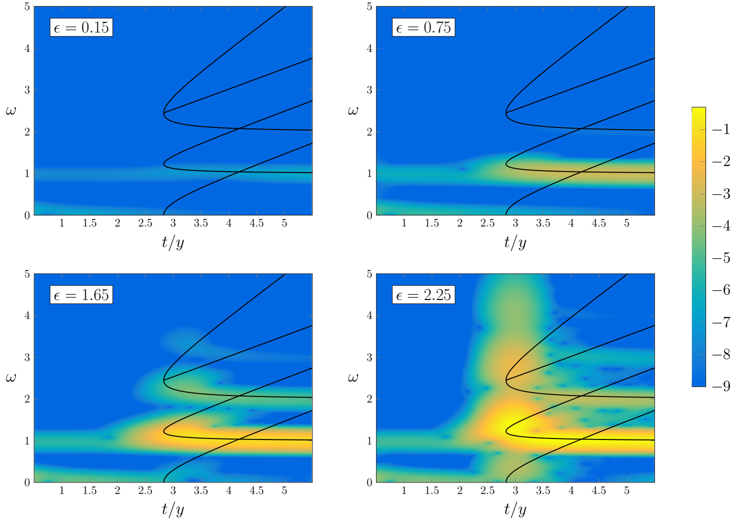
$F = 1$


$\epsilon = 0.01$
$\epsilon = 0.05$
$\epsilon = 0.11$
$\epsilon = 0.15$
ω
t/y
−2
−3
−4
−5
−6
−7
−8
−9

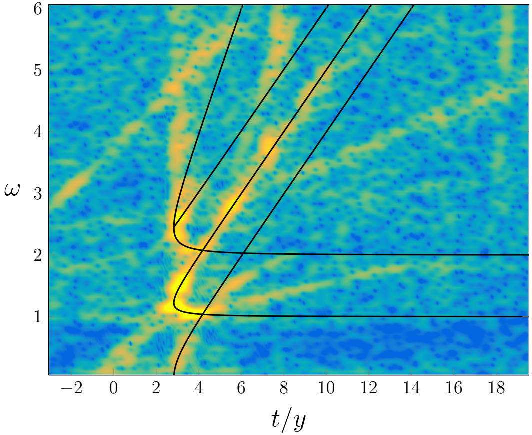
$F = 0.7$


$\epsilon = 0.01$
$\epsilon = 0.05$
$\epsilon = 0.11$
$\epsilon = 0.15$
ω
t/y
0
1
2
3
4
5
1.5
2.5
3.5
4.5
-1
-2
-3
-4
-5
-6
-7

$F = 0.2$



Torsvik *et al.* (2015) spectrogram again



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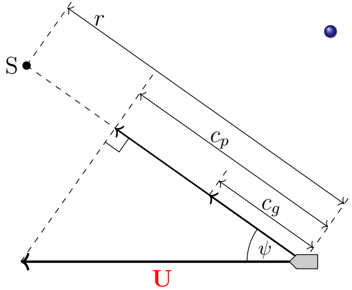
An accelerating ship



- From the diagram, $c_p = U \cos\psi$
- For fluid of infinite depth

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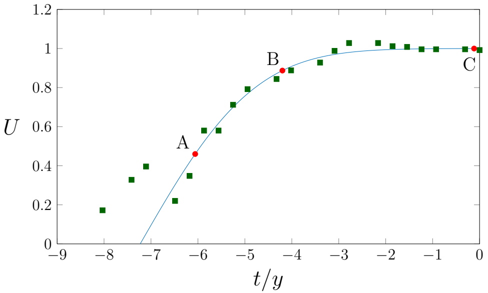
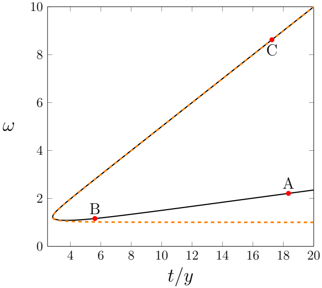
$$c_p = \frac{\Omega(k)}{k} = \frac{1}{\sqrt{k}}$$
$$c_g = \frac{\mathrm{d}\Omega}{\mathrm{d}k} = \frac{1}{2\sqrt{k}} = \frac{c_p}{2}$$

Linear dispersion curve parametrised in terms of the ship's time, t_s

$$(t, \omega) = \left(t_s + \frac{r}{c_g}, \Omega(k) \right)$$

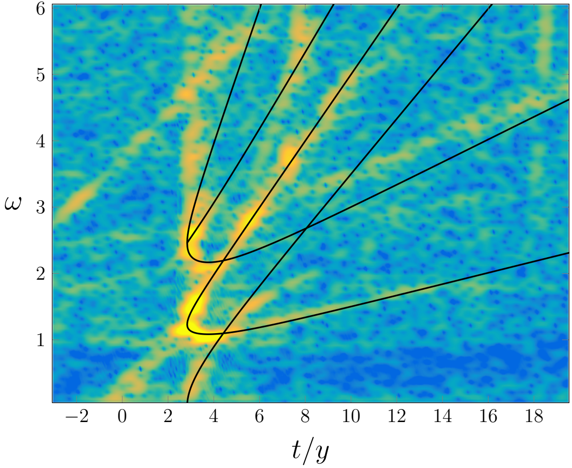
An accelerating ship





MarineTraffic AIS Vessel Tracking AIS Positions Maps. Retrieved 26 February 2016, from http://www.marinetraffic.com/en/ais/home/shipid:352956/zoom:10.

An accelerating ship



Linear dispersion curve

- From the diagram, $c_p = \cos\psi$
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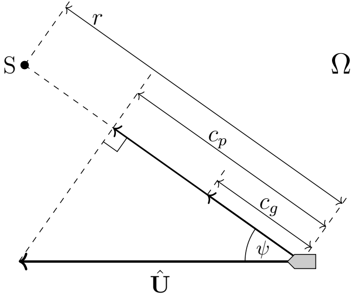


$$\Omega(k) = \sqrt{k \tanh \frac{k}{F_H^2}}$$

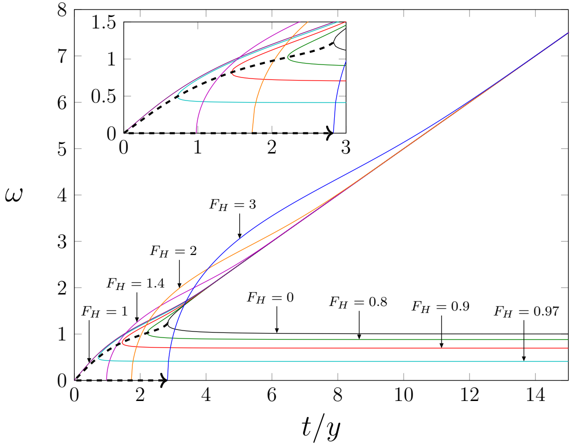
$$c_p = \frac{\Omega(k)}{k} = \frac{\sqrt{\tanh \frac{k}{F_H^2}}}{\sqrt{k}}$$

$$c_g = \frac{\mathrm{d}\Omega}{\mathrm{d}k} = \frac{\sqrt{\tanh \frac{k}{F_H^2}}}{2\sqrt{k}} \left(1 + \frac{2k}{F_H^2 \sinh \frac{2k}{F_H^2}}\right)$$

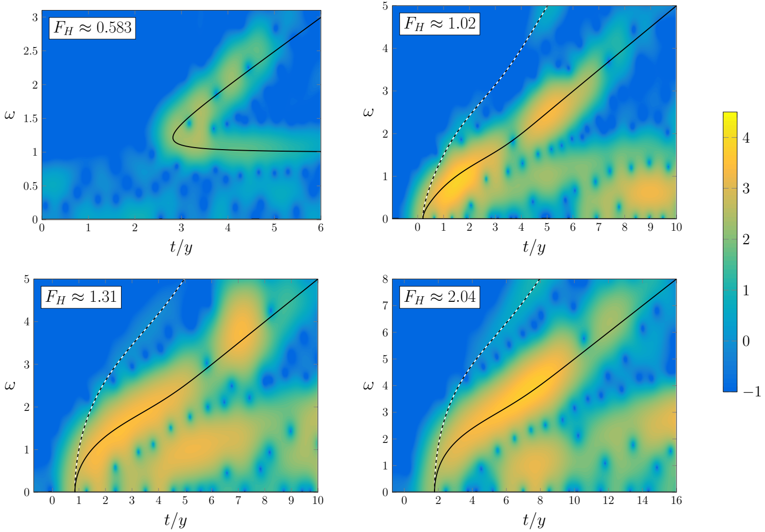
Linear dispersion curve parametrised in terms of the ship's time, t_s

$$(t, \omega) = \left(t_s + \frac{r}{c_g}, \Omega(k)\right)$$

Finite depth linear dispersion curve



Finite depth



Conclusion

- Spectrograms can be used to analyse the wake components of a ship.
- The linear and second order dispersion curves predict the location of intensity in the spectrogram.
- The visible second-order intensities depend on the intensity along the linear dispersion curve.
- The linear dispersion relation cannot predict the distortion of the intensity away from the dispersion curve.
- Acceleration is a possible explanation for the difference between the experimental spectrogram and the steady linear dispersion curve
- The dispersion curve procedure holds for finite depth examples

Acknowledgements

- Tomas Torsvik
- Gregor Macfarlane

References

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R.Pethiyagoda, S. W. McCue & T. J. Moroney, Spectrograms of ship wakes: identifying linear and nonlinear wave signals. *J. Fluid Mech.* **811** (2017) 189–209.

R.Pethiyagoda, T. J. Moroney, G. J. Macfarlane, J. R. Binns & S. W. McCue. Time-frequency analysis of ship wave patterns in shallow water: modelling and experiments. *Ocean Engng* **158** (2018) 123–131.

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