Interactions of vector solitons in the model of a particle chain

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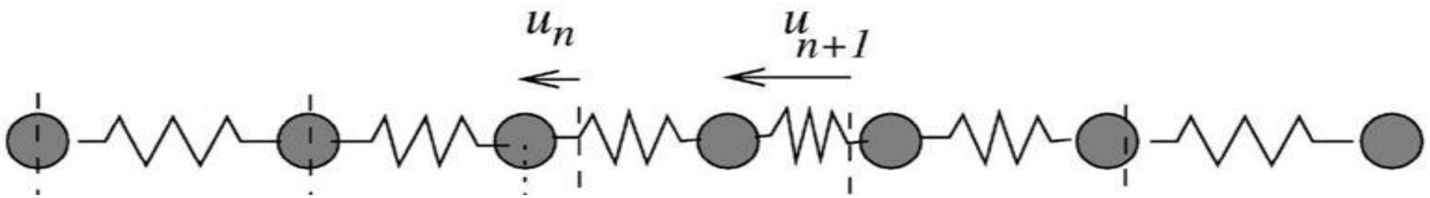
One-Dimensional Case (FPU Model)

In the <u>one-dimensional case</u> in application to a chain of atoms the equation of motion for <u>longitudinal</u> modes is <u>scalar</u> describing atom vibrations in the direction of wave propagation (*Fermi et al., 1955; Toda, 1989*),

(Raj, N., Obregon, M. and Stepanyants, Y. (2012) Numerical study of nonlinear wave processes by means of discrete chain models.**)**



Transverse Modes

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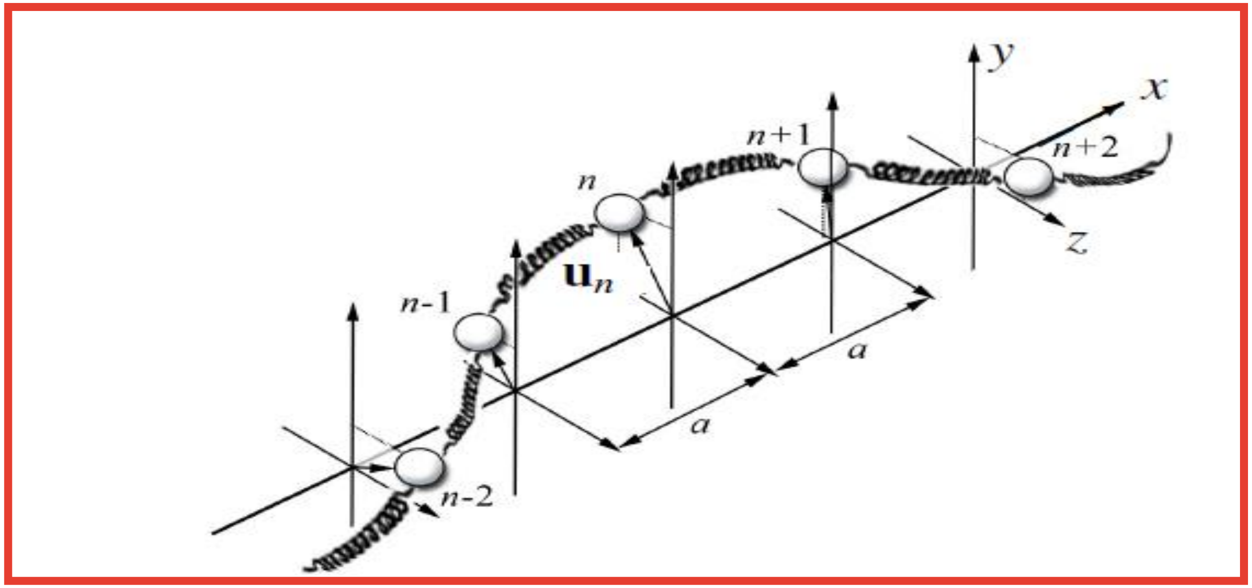
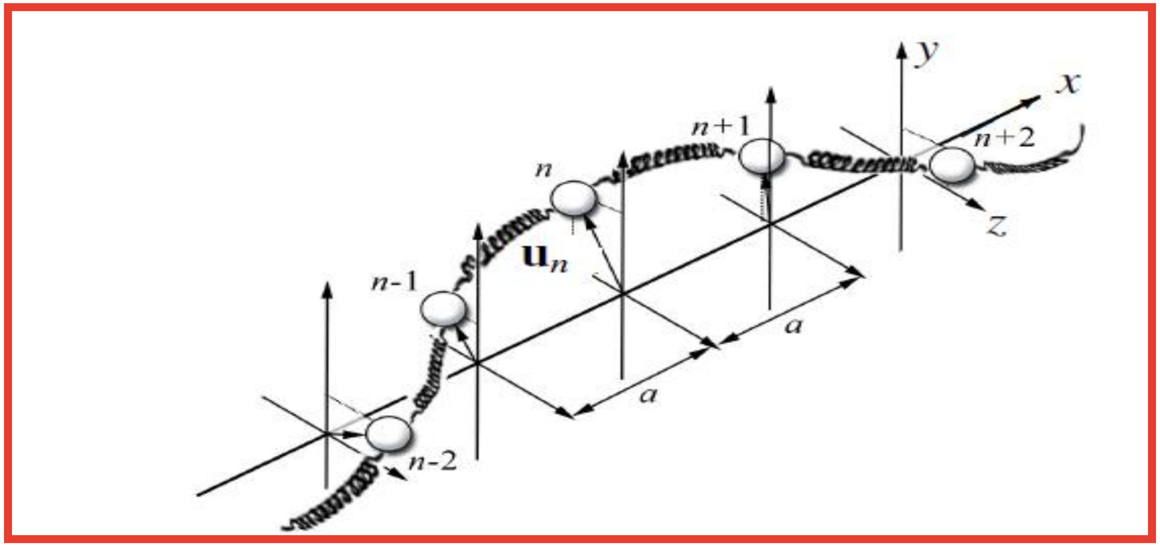
However, when the transverse modes are considered the equation of motion becomes vector describing particle displacements in two perpendicular directions transverse to the direction of wave propagation. (Gorbacheva & Ostrovsky, 1983).



A schematic example of a transverse flexural perturbation on the chain of particles linked by elastic springs.

Plasma Waves



One of the first example of helical solitons was reported for circularly polarised waves in solid-state plasma described by a rather specific nonlinear wave equation (Gorshkov, Kozlov & Ostrovskii, 1974).

Other plasma wave studies: (Kuehl, 1977), (Shukla, 1977), (Dysthe, Mjølhus, Pecseli & Stenflo, 1978)

Vector Equation

Equation of Motion for the Atom

Considering the chain of equal mass atoms the equation of motion for the atom with number $\boxed{n}$ can be written as:

$$\boxed{m\frac{d^2\xi_n}{dt^2} = \mathbf{F}_{n-2} + \mathbf{F}_{n-1} + \mathbf{F}_{n+1} + \mathbf{F}_{n+2}}$$

$\boxed{m}$ - is the mass of each atom;

$\boxed{\xi_n = (y_n, z_n)}$ - is the two-component transverse displacement vector with the y and z-components orthogonal to the axis x, the axis along which perturbations propagate.

$\boxed{\mathbf{F}_n}$ - are transverse forces exerting on the n–th atom

Vector Equation

The transverse force exerting on the atom is given as:

$$\boxed{F_{n\pm j} = \pm\beta_j \left[T + jaK\left(\frac{1}{\cos\alpha_{n\pm j}} - 1 \right) \right] \sin\alpha_{n\pm j}}$$

$\boxed{\beta_j}$ - coefficients which characterize the corresponding force

$\boxed{T}$ - uniform tension of the chain

$\boxed{K}$ - analogue of Hooke's constant

$\boxed{\alpha_n}$ - local angle between the chain and the x-axis

$\boxed{j=1}$ - nearest two neighbor atoms and $\boxed{j=2}$ for the next two atoms

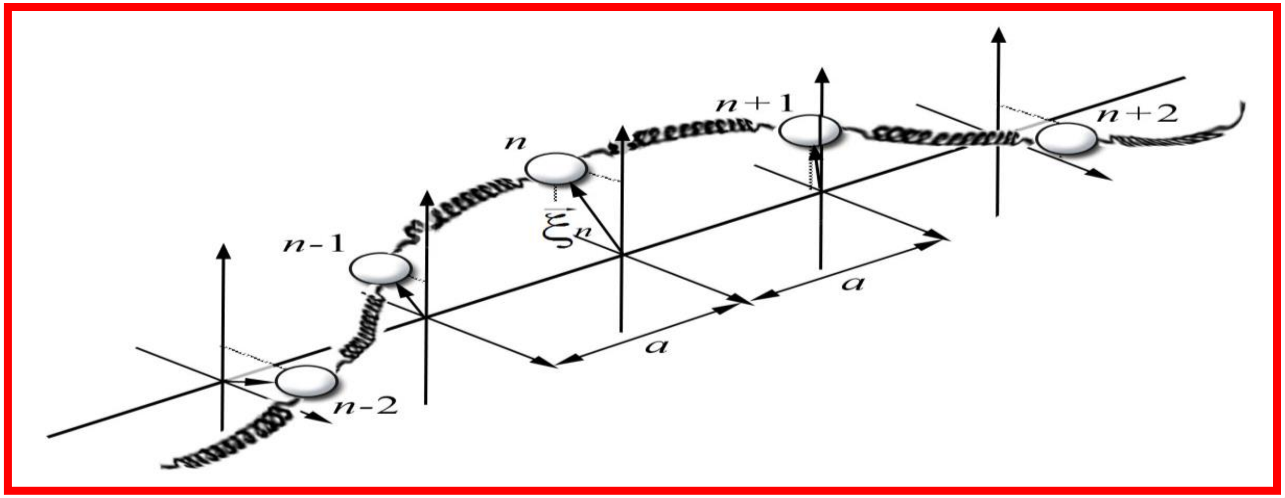
[Gorbacheva & Ostrovsky (1983)]

Vector Equation

Total Force Equation

When the angle $\alpha << 1$, the total force $\mathbf{F} = (\mathbf{Y}, \mathbf{Z})$ can be presented as:

$$\mathbf{F}_{n\pm j} \approx \pm \frac{\xi_{n\pm j} - \xi_n}{ja}\left[T + \frac{jaK - T}{2(ja)^2}\left|\xi_{n\pm j} - \xi_n\right|^2\right]$$



Vector Equation

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Newtonian Equation of Motion for the *n*-th Particle

After manipulations, the following vector equation of motion

is derived:

$$\frac{d^2\bar{\xi}_n}{d\tau^2} = \bar{\xi}_{n+1} - 2\bar{\xi}_n + \bar{\xi}_{n-1} + \frac{1}{2}(\mu-1)\left[\left|\bar{\xi}_{n+1} - \bar{\xi}_n\right|^2\left(\bar{\xi}_{n+1} - \bar{\xi}_n\right) - \left|\bar{\xi}_n - \bar{\xi}_{n-1}\right|^2\left(\bar{\xi}_n - \bar{\xi}_{n-1}\right)\right] +$$

$$\frac{\beta_2}{2}\left\{\bar{\xi}_{n+2} - 2\bar{\xi}_n + \bar{\xi}_{n-2} + \frac{1}{8}(2\mu-1)\left[\left|\bar{\xi}_{n+2} - \bar{\xi}_n\right|^2\left(\bar{\xi}_{n+2} - \bar{\xi}_n\right) - \left|\bar{\xi}_n - \bar{\xi}_{n-2}\right|^2\left(\bar{\xi}_n - \bar{\xi}_{n-2}\right)\right]\right\}$$

where the following normalised variables were introduced:

$$\tau = (T/am)^{\frac{1}{2}}\, t,\ \ \overline{\xi_n} = \xi_n / a\ ,\ \ \mu = aK/T$$

(S.P. Nikitenkova, N. Raj, Y.A. Stepanyants, 2015)

Non-Linear PDEs

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In the long-wave approximation, the governing equation reduces to the given PDE:

$$\boxed{\frac{\partial^2\overline{\boldsymbol{\xi}}}{\partial\tau^2}=(1+2\beta_2)\frac{\partial^2\overline{\boldsymbol{\xi}}}{\partial x^2}+\frac{1+8\beta_2}{12}\frac{\partial^4\overline{\boldsymbol{\xi}}}{\partial x^4}+\frac{1+32\beta_2}{360}\frac{\partial^6\overline{\boldsymbol{\xi}}}{\partial x^6}+\frac{\mu(1+4\beta_2)-(1+2\beta_2)}{2}\frac{\partial}{\partial x}\left(\left|\frac{\partial\overline{\boldsymbol{\xi}}}{\partial x}\right|^2\frac{\partial\overline{\boldsymbol{\xi}}}{\partial x}\right)}$$

The basic equation then can be presented in terms of $\boxed{\mathbf{u}=\frac{\partial\overline{\boldsymbol{\xi}}_n}{\partial x}}$

$$\boxed{\frac{\partial^2\mathbf{u}}{\partial\tau^2}-(1+2\beta_2)\frac{\partial^2\mathbf{u}}{\partial x^2}-\frac{1+8\beta_2}{12}\frac{\partial^4\mathbf{u}}{\partial x^4}-\frac{1+32\beta_2}{360}\frac{\partial^6\mathbf{u}}{\partial x^6}-\frac{\mu(1+4\beta_2)-(1+2\beta_2)}{2}\frac{\partial^2}{\partial x^2}\left(|\mathbf{u}|^2\mathbf{u}\right)=0}$$

(S.P. Nikitenkova, N. Raj, Y.A. Stepanyants, 2015)

Non-Linear PDEs

When $\boxed{\beta_2 \neq -\frac{1}{2} \text{ so and } \beta_2 \neq -\frac{1}{8}}$,

then for unidirectional wave propagation, the equation can be further reduced to vector mKdV equation:

$$\boxed{\frac{\partial \mathbf{u}}{\partial \tau} + \sqrt{1+2\beta_2}\,\frac{\partial \mathbf{u}}{\partial x} + \frac{1+8\beta_2}{24\sqrt{1+2\beta_2}}\frac{\partial^3 \mathbf{u}}{\partial x^3} + \frac{\mu\left(1+4\beta_2\right)-\left(1+2\beta_2\right)}{4\sqrt{1+2\beta_2}}\frac{\partial}{\partial x}\left(\left|\mathbf{u}\right|^2 \mathbf{u}\right) = 0}$$

(S.P. Nikitenkova, N. Raj, Y.A. Stepanyants, 2015)

Non-Linear PDEs

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The derived vector mKdV equation is non-integrable but is very close to the completely integrable equation:

$$\frac{\partial \mathbf{u}}{\partial \tau} + \alpha |\mathbf{u}|^2 \frac{\partial \mathbf{u}}{\partial x} + \beta \frac{\partial |\mathbf{u}|^3}{\partial x^3} = -\alpha \mathbf{u} \frac{\partial |\mathbf{u}|^2}{\partial x}$$

$$\alpha = \frac{\mu(1+4\beta_2) - c_0{}^2}{4c_0}, \qquad \beta = \frac{1+8\beta_2}{24c_0}, \qquad c_0 = \sqrt{1+2\beta_2}$$

(S.P. Nikitenkova, N. Raj, Y.A. Stepanyants, 2015)

Vector modified Korteweg-de Vries Equation

In the typical case, wave propagation can be described by the vector modified Korteweg-de Vries (mKdV) equation:

$$\boxed{\frac{\partial \mathbf{u}}{\partial t} + \alpha |\mathbf{u}|^2 \frac{\partial \mathbf{u}}{\partial x} + \beta \frac{\partial^3 \mathbf{u}}{\partial x^3} = -\alpha_1 \mathbf{u} \frac{\partial |\mathbf{u}|^2}{\partial x}}$$

Here $\boxed{\alpha_1 = \alpha}$ but for the sake of flexibility we use different notations for the non linear coefficients which allows to consider the case when $\boxed{\alpha_1 = 0}$

Vector modified Korteweg-de Vries Equation

Integrable and Non-Integrable Cases have been investigated within the framework of the derived vector mKdV.

When $\alpha_1 = \alpha$:

$$\frac{\partial \mathbf{u}}{\partial t} + \alpha |\mathbf{u}|^2 \frac{\partial \mathbf{u}}{\partial x} + \beta \frac{\partial^3 \mathbf{u}}{\partial x^3} = -\alpha_1 \mathbf{u} \frac{\partial |\mathbf{u}|^2}{\partial x}$$

When $\alpha_1 = 0$:

$$\frac{\partial \mathbf{u}}{\partial t} + \alpha |\mathbf{u}|^2 \frac{\partial \mathbf{u}}{\partial x} + \beta \frac{\partial^3 \mathbf{u}}{\partial x^3} = 0$$

Results

In our study it was confirmed that solitary and periodic stationary solutions exist within the framework of the derived equation.

Some of them represent helical periodic waves, others represent plane solitary waves which can propagate

along the chain at different angles with respect to each other.

Each solitary wave performs a chain displacement in a certain plane only.

Interaction of solitary waves is elastic in the integrable case but is non-elastic in general in non integrable case.

Results - Plane

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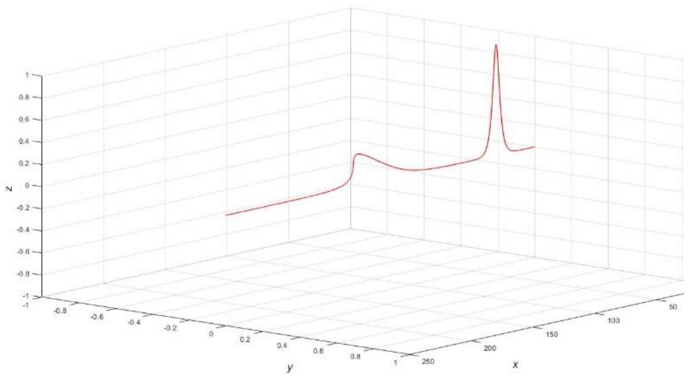
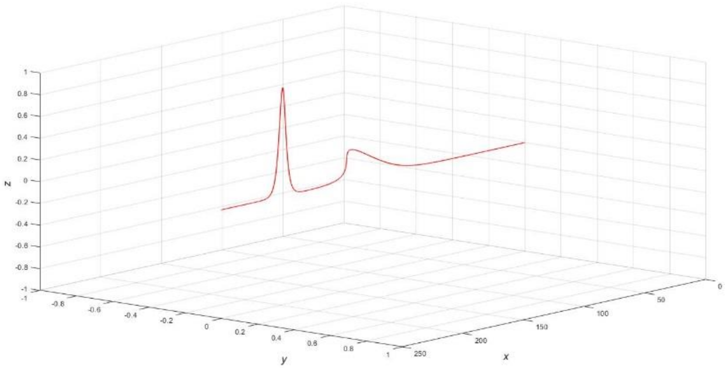
We present the results of numerical study of interactions between plane solitary waves of different polarization, i.e. having different orientation.

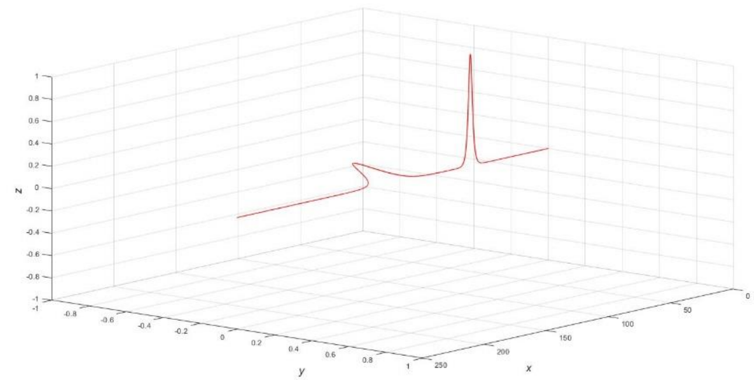
Plane Soliton Interaction at 30 Degrees

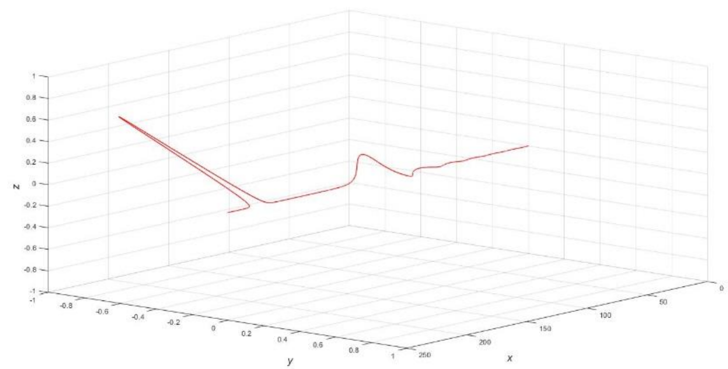
Integrable





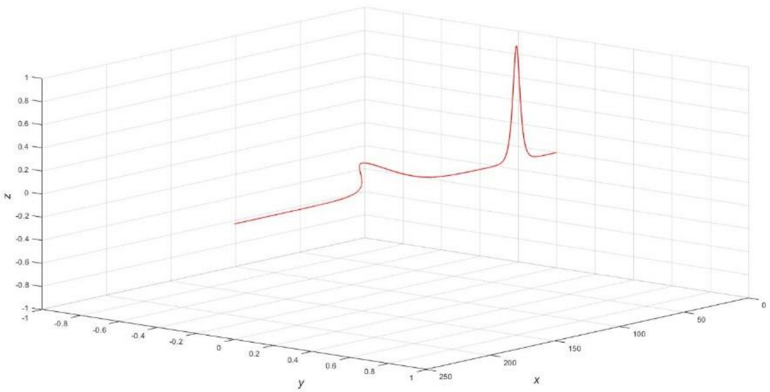
Non-Integrable

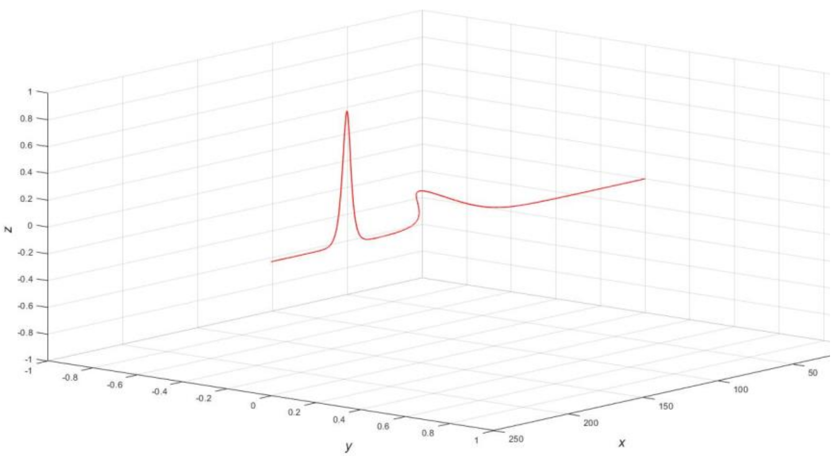




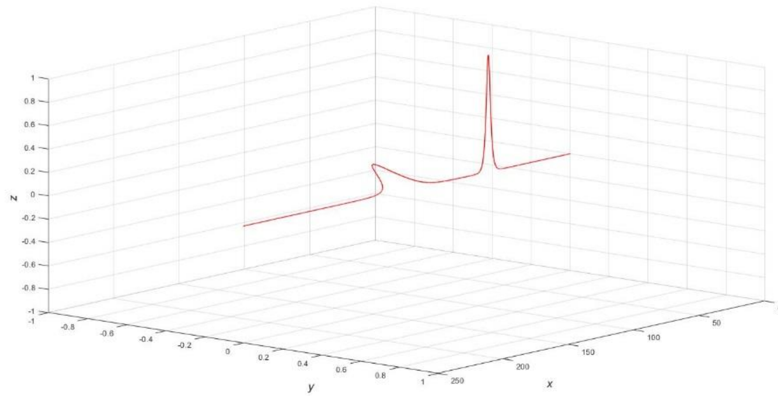
Plane Soliton Interaction at 45 Degrees

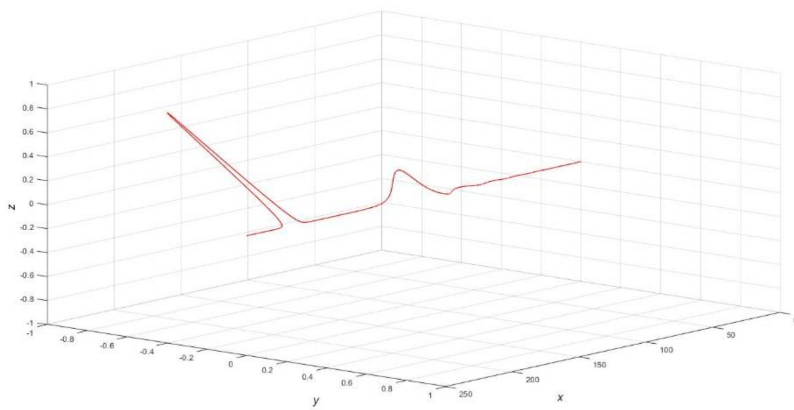
Integrable





Non-Integrable





Soliton Interaction at 60 Degrees

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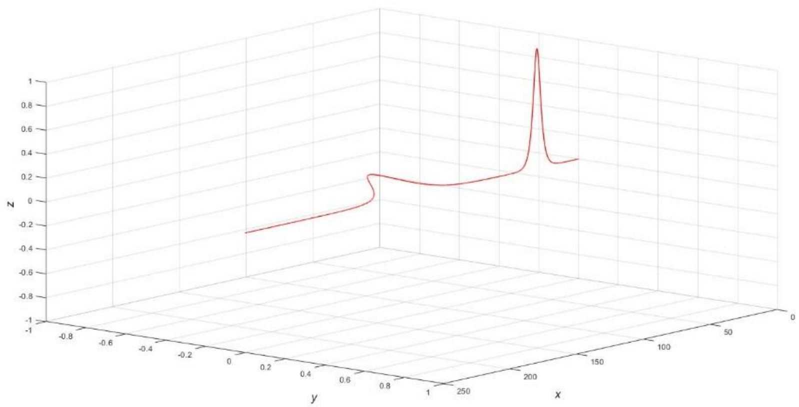
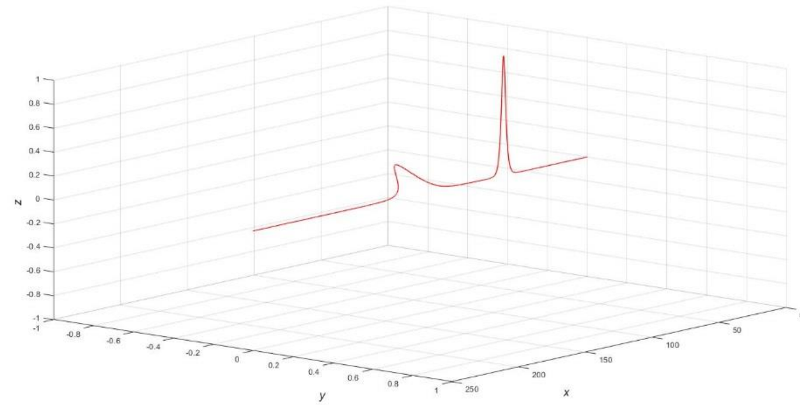
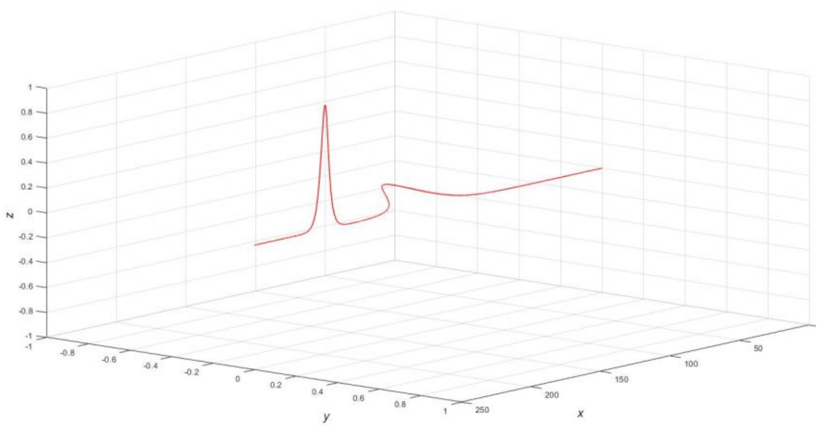
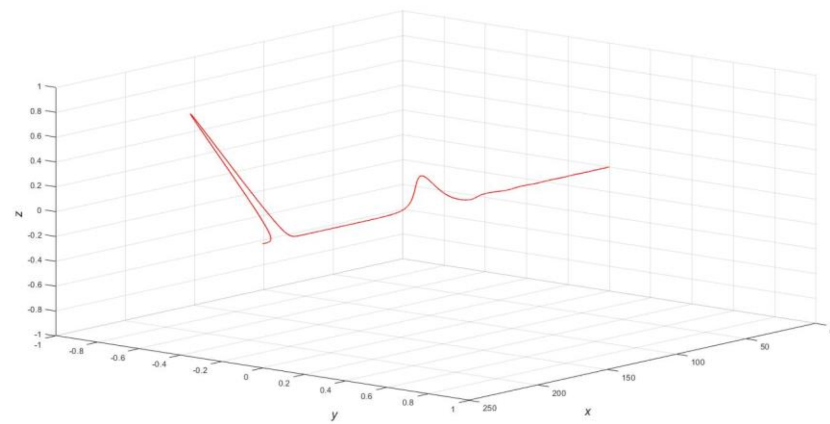
Integrable

Non-Integrable





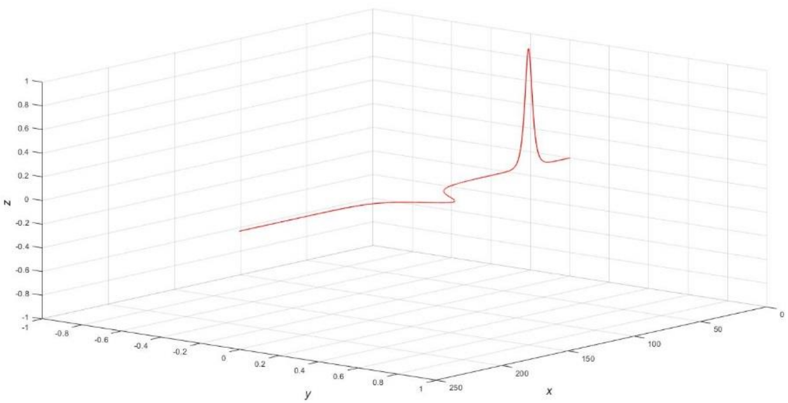


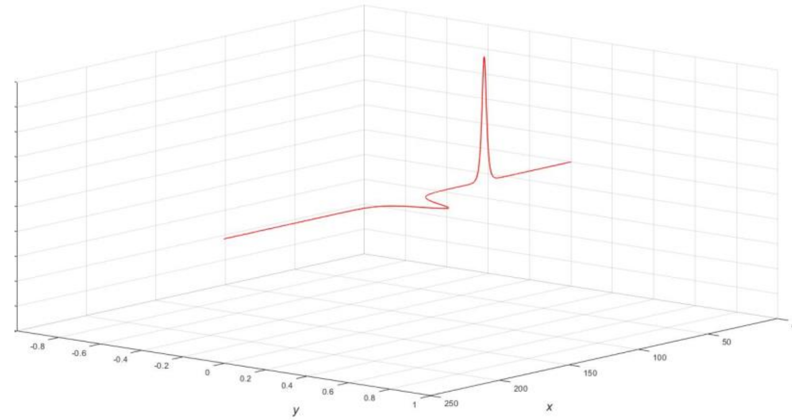


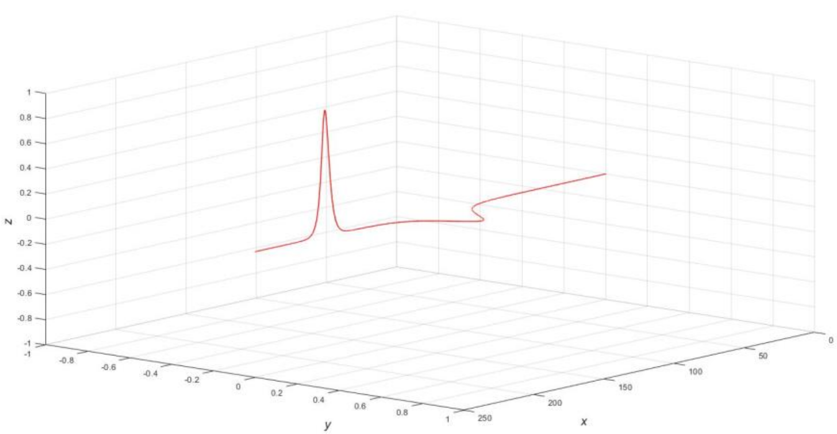
Plane Soliton Interaction at 90 Degrees

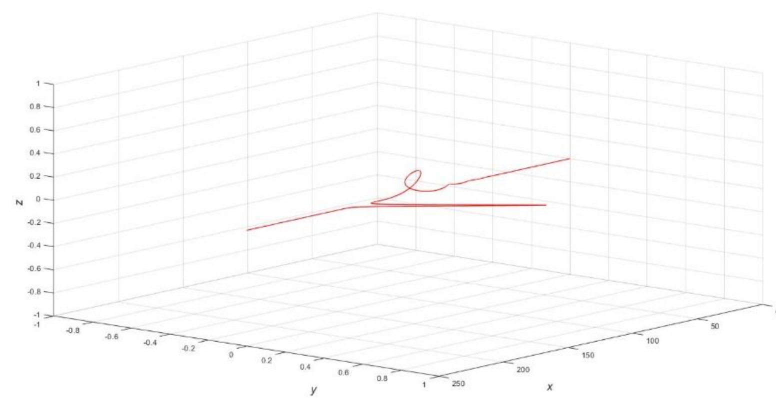
Integrable

Non-Integrable







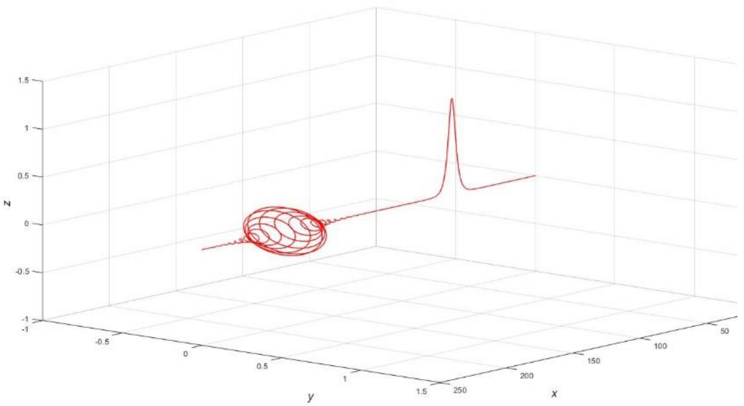


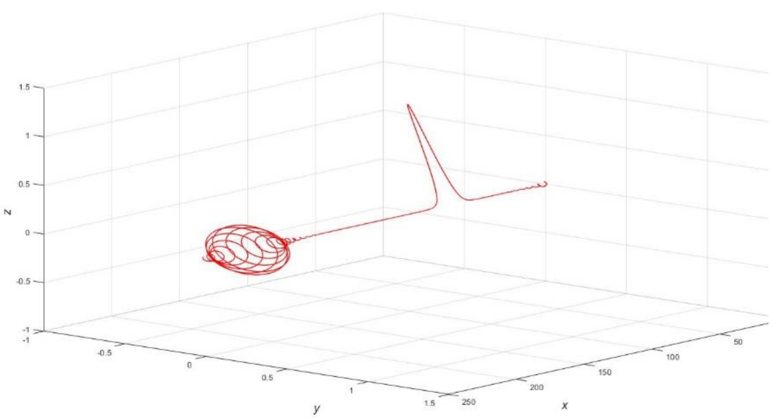
Results – Plane and Helical

We also consider a family of helical solitary waves and study interactions between plane and helical.

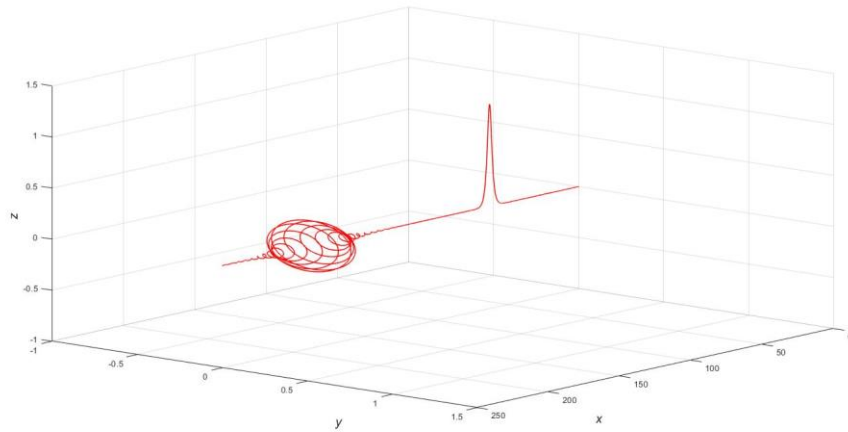
Helical and Plane Soliton Interaction

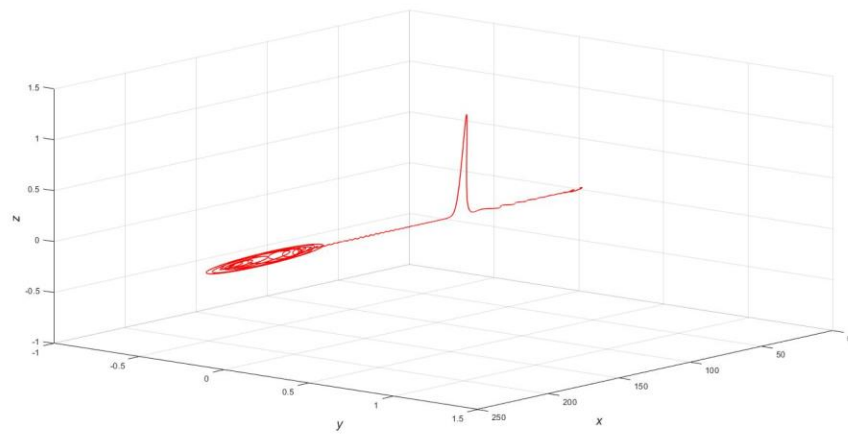
Integrable





Non-Integrable





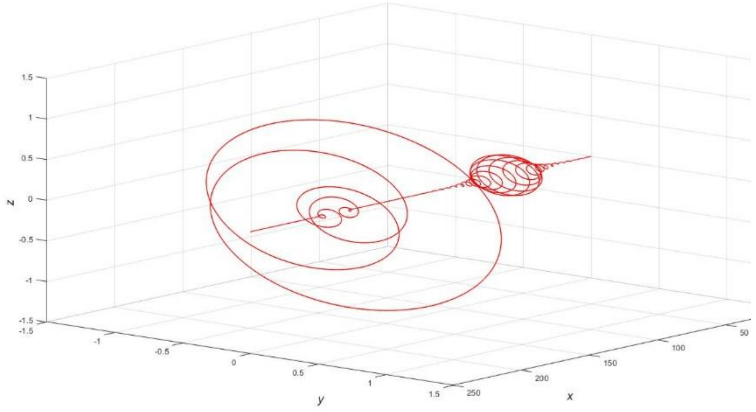
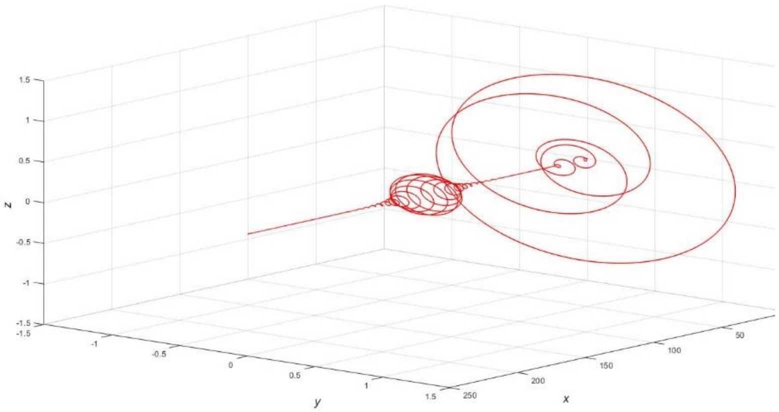
Results – Helical and Helical

Next we consider a family of helical solitary waves and study interactions between them in the cases of same and opposite helicity.

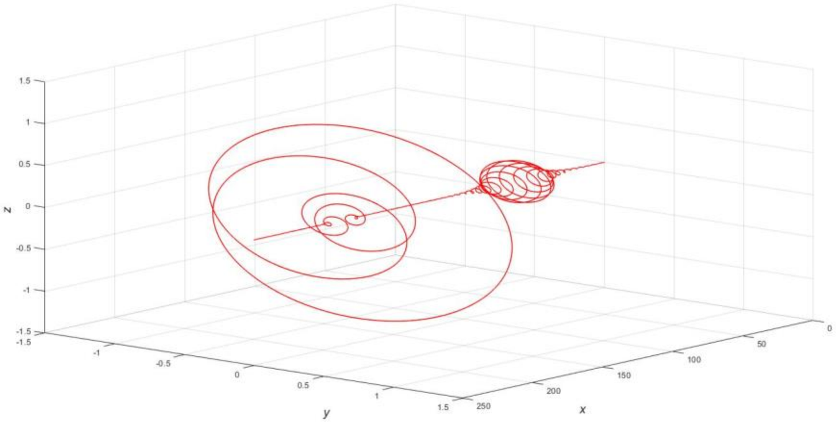
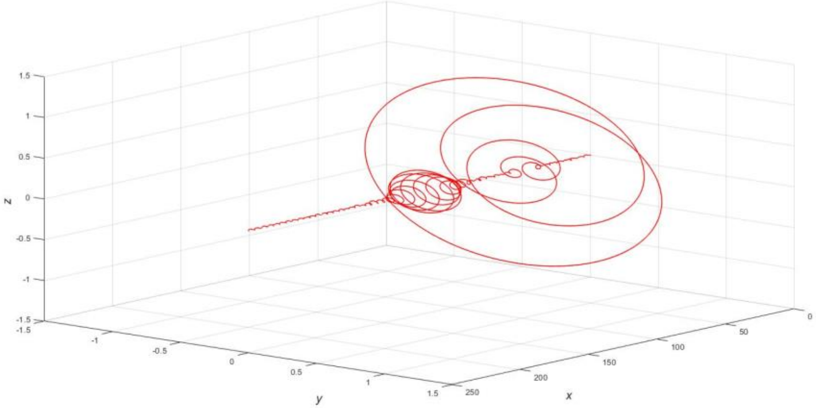
The result as shown depends upon the direction and strength of helicity.

Helical and Helical Soliton Interaction With Same Helicity

Integrable

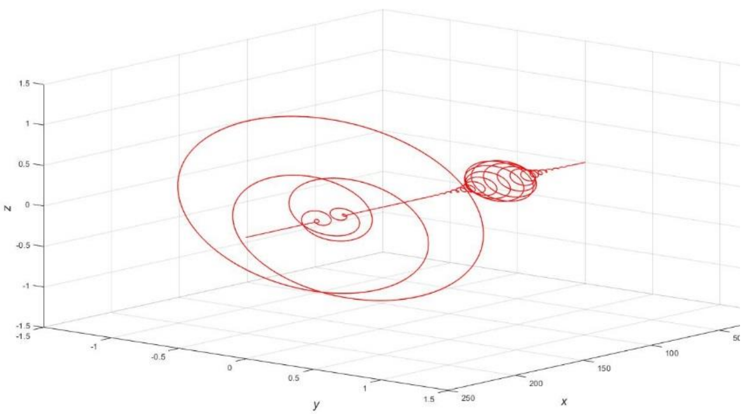
Non-Integrable

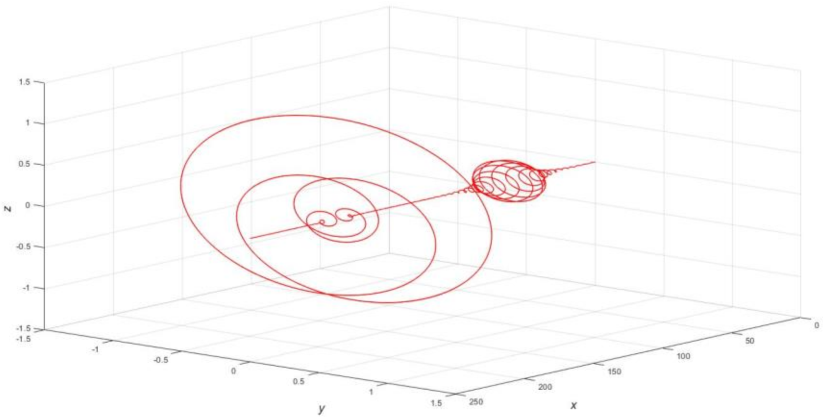



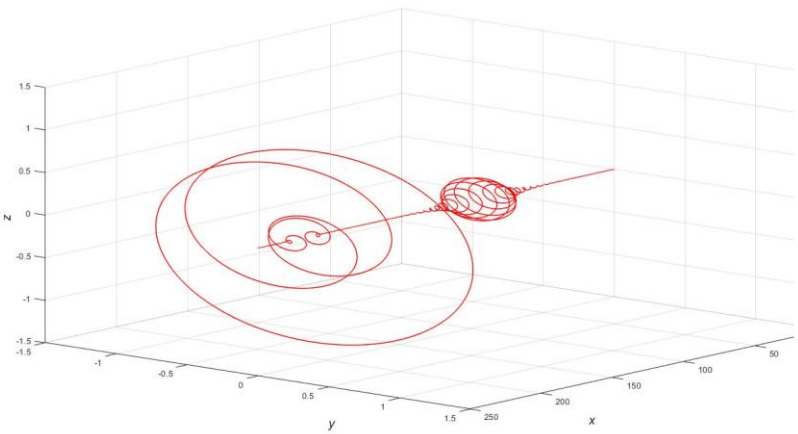
Helical and Helical Soliton Interaction With Opposite Helicity

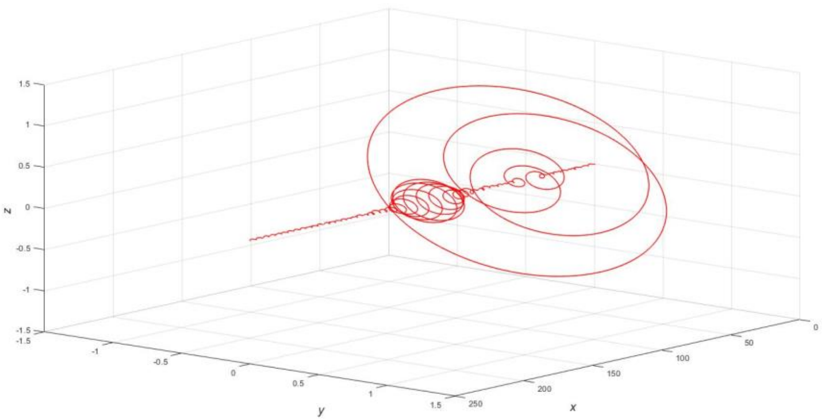
Integrable

Non-Integrable



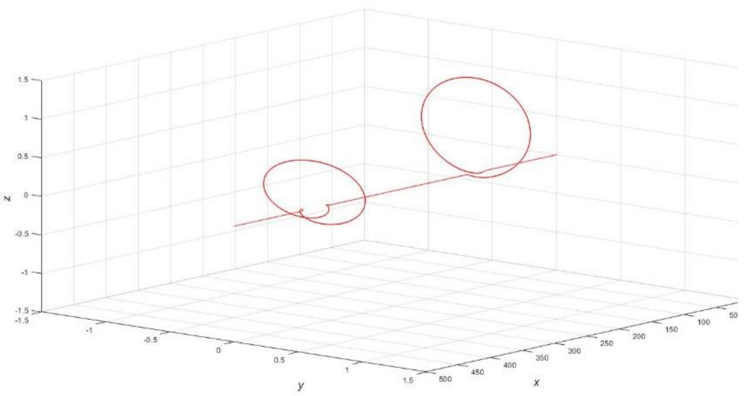


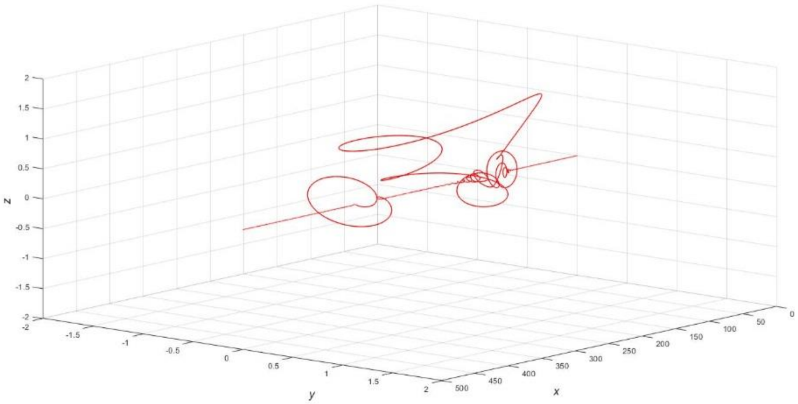




Helical and Helical Soliton Interaction With Reduced Helicity

Integrable





This study is currently underway and we are looking forward to more results.

Animations (WEBPAGE)

Animations of plane with plane, helical with plane and helical with helical solitons in both cases of integrable and non integrable are given on a webpage designed by the authors is shown by the following link:

https://eportfolio.usq.edu.au/view/view.php?t=dj2HQ3ioUZEOW0SfArbL

Particle Oscillation in a chain (Beads Demo – Relevant Video)

Amazing bead chain experiment in slow motion
(Steve Mould, YouTube, http://youtu.be/6ukMId5fIi0)

Main References

1. *Destrade M. and Saccomandi G*., 2008. Nonlinear transverse waves in deformed dispersive solids, *Wave Motion*, v. 45, 325–336.

2. Gorbacheva O.B. and *Ostrovsky L.A., 1983.* Nonlinear vector waves in a mechanical model of a molecular chain. *Physica D*, v. 8, 223–228.

3. *Raj, N., Obregon, M. and Stepanyants, Y., 2012* Numerical study of nonlinear wave processes by means of discrete chain models. In: 4th International Conference on Computational Methods (ICCM 2012), 25-28 Nov 2012, Gold Coast, Australia.

4. Raj N., Nikitenkova S.P., Stepanyants Y.A., 2015. Nonlinear vector waves of a flexural mode in a chain model of atomic particles. Nonlinear Science and Numerical Simulation, vol. 21, 238-249.