Exact soliton, periodic and superposition solutions to the fifth-order Korteweg-de Vries equation and derivation of the new equation for an uneven bottom

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Shallow water problem

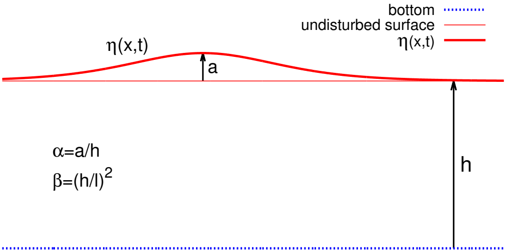


Figure: Schematic view of the geometry of the problem.

Assumptions:

Fluid is incompressible and inviscid, Motion is irrotational,
$\alpha \ll 1$, $\beta \ll 1$ - small parameters of the same order.

Euler equations

Euler equations in scaled dimensionless coordinates

$$
\begin{aligned}
\beta\phi_{2x}+\phi_{2z} &= 0, \quad \text{for} \quad 0<z<1+\alpha\eta(x,t) && (1)\\
\frac{1}{\beta}\phi_z-(\alpha\eta_x\phi_x+\eta_t) &= 0, \quad \text{for} \quad z=1+\alpha\eta(x,t) && (2)\\
\phi_t+\frac{1}{2}(\alpha\phi_x^2+\frac{\alpha}{\beta}\phi_z^2)+\eta &= 0, \quad \text{for} \quad z=1+\alpha\eta(x,t) && (3)\\
\phi_z &= 0, \quad \text{for} \quad z=0. && (4)
\end{aligned}
$$

Velocity potential

$$\Phi(x,z,t)=\sum_{m=0}^{\infty} z^m\phi^{(m)}(x,t). \tag{5}$$

Eqs. (1), (4) and (5) imply $(f(x,t)\equiv\phi^{(0}(x,t))$

$$\phi=f-\frac{1}{2}\beta z^2 f_{2x}+\frac{1}{24}\beta^2 z^4 f_{4x}-\frac{1}{720}\beta^3 z^6 f_{6x}+\frac{1}{40320}\beta^4 z^8 f_{8x}\cdots . \tag{6}$$

Boussinesq's equations

Insertion of $\phi(x,z,t)$ (6) into (2) gives

$$\eta_t + w_x + \alpha(\eta w)_x - \frac{1}{6}\beta w_{3x} - \frac{1}{2}\alpha\beta(\eta w_{2x})_x + \frac{1}{120}\beta^2 w_{5x} = 0. \tag{7}$$

Insertion of $\phi(x,z,t)$ (6) into (2) yields

$$w_t + \eta_x + \beta\left(ww_x - \frac{1}{2}w_{2xt}\right) + \alpha\beta\left(-(\eta w_{xt})_x + \frac{1}{2}w_x w_{2x} - \frac{1}{2}ww_{3x}\right) + \frac{1}{24}\beta^2 w_{4xt} = 0. \tag{8}$$

The equations (7) and (8) constitute the coupled set of Boussinesq's equations (obtained after neglecting terms of order higher than the second) for two unknown functions:

$$\eta(x,t) \qquad \text{and} \qquad w(x,t) \equiv f_x(x,t) \equiv \phi_x^{(0)}(x,t).$$

KdV equation

In the lowest (zeroth) order

$$\eta_t + w_x = 0, \quad w_t + \eta_x = 0 \quad \Longrightarrow \quad w = \eta, \quad \eta_t + \eta_x = 0. \tag{9}$$

In the first order one tries

$$w = \eta + \alpha Q^{(\alpha)} + \beta Q^{(\beta)}. \tag{10}$$

Substituting (10) into the Boussinesq system, subtracting the results and using $Q_t = -Q_x$ one obtains

$$\alpha(2Q_x^{(\alpha)} + \eta\eta_x) + \beta(2Q_x^{(\beta)} - \frac{2}{3}\eta_{3x}) = 0 \tag{11}$$

which gives

$$Q_x^{(\alpha)} = -\frac{1}{2}\eta\eta_x \quad \Longrightarrow \quad Q^{(\alpha)} = -\frac{1}{4}\eta^2, \tag{12}$$

$$Q_x^{(\beta)} = \frac{1}{3}\eta_{2x} \quad \Longrightarrow \quad Q^{(\beta)} = \frac{1}{3}\eta_{3x}. \tag{13}$$

KdV and extended KdV equations

First order: Korteweg - de Vries equation in fixed reference frame (1895)

$$w = \eta - \alpha \frac{1}{4}\eta^2 + \frac{1}{6}\beta\, \eta_{2x} \tag{14}$$

$$\eta_t + \eta_x + \frac{3}{2}\alpha\, \eta\eta_x + \frac{1}{6}\beta\, \eta_{3x} = 0. \tag{15}$$

Second order: KdV2 equation (Marchant & Smyth 1990)

$$w = \eta - \frac{1}{4}\alpha\, \eta^2 + \frac{1}{3}\beta\, \eta_{2x} + \frac{1}{8}\alpha^2\eta^3 + \frac{1}{10}\beta^2\eta_{4x} + \alpha\beta\left(\frac{3}{16}\eta_x^2 + \frac{1}{2}\eta\eta_{2x}\right). \tag{16}$$

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{3x} - \frac{3}{8}\alpha^2\eta^2\eta_x + \alpha\beta\left(\frac{23}{24}\eta_x\eta_{2x} + \frac{5}{12}\eta\eta_{3x}\right) + \frac{19}{360}\beta^2\eta_{5x} = 0, \tag{17}$$

Solutions to KdV

- Single soliton solution

$$\eta(x,t) = A\,\mathrm{sech}[B(x-vt)]^2 \tag{18}$$

- Periodic (cnoidal) solution

$$\eta(x,t) = A\,\mathrm{cn}[B(x-vt)]^2 + D \tag{19}$$

- Periodic ("superposition") solution, Khare & Saxena, Phys. Lett. A, **377** 2761 (2013)

$$\eta_{\pm}(y) = \frac{A}{2}\left[\mathrm{dn}^2(By,m) \pm \sqrt{m}\,\mathrm{cn}(By,m)\,\mathrm{dn}(By,m)\right] + D, \tag{20}$$

- Multi-soliton solutions, IST method, Hirota method ...

Solutions to KdV2

Hypothesis

There exist the following analytic solutions to KdV2:

- **Single soliton solution** $A\,\mathrm{sech}^2[B(x-vt)]$
- **Periodic (cnoidal) solution** $A\,\mathrm{cn}^2[B(x-vt)]+D$
- **Periodic ("superposition") solution** $\frac{A}{2}[\,\mathrm{dn}^2 \pm \sqrt{m}\,\mathrm{cn}\,\mathrm{dn}]+D$

of the same form as the corresponding KdV solutions, but with different coefficients A, B, v, m**.**

Soliton solutions to KdV2 $\eta = A\,\mathrm{sech}^2[B(x-vt)]$

With $\eta(x,t)=\eta(x-vt)$, we have $\eta_t = -v\eta_x$. Then from (17) one has

$$(1-v)\eta + \alpha\,\frac{3}{4}\eta^2 + \beta\,\frac{1}{6}\eta_{2x} - \frac{1}{8}\alpha^2\eta^3 + \alpha\beta\left(\frac{13}{48}\eta_x^2 + \frac{5}{12}\eta\eta_{2x}\right) + \beta^2\,\frac{19}{360}\eta_{4x} = 0.$$

Substitution of $\eta = A\,\mathrm{sech}^2[B(x-vt)]$ gives

$$C_2\,\mathrm{sech}^2(By) + C_4\,\mathrm{sech}^4(By) + C_6\,\mathrm{sech}^6(By) = 0, \tag{21}$$

Solution exists when simultaneously

$$C_2 = 0 \quad = \quad (1-v) + \frac{2}{3}B^2\beta + \frac{38}{45}B^4\beta^2 \tag{22}$$

$$C_4 = 0 \quad = \quad \frac{3A\alpha}{4} - B^2\beta + \frac{11}{4}A\alpha\,B^2\beta - \frac{19}{3}B^4\beta^2 \tag{23}$$

$$C_6 = 0 \quad = \quad -\left(\frac{1}{8}\right)(A\alpha)^2 - \frac{43}{12}A\alpha\,B^2\beta + \frac{19}{3}B^4\beta^2 \tag{24}$$

Single soliton solutions to KdV2

From (24), denoting $z = \dfrac{\beta B^2}{\alpha A}$ we obtain

$$\frac{19}{3} z^2 - \frac{43}{12} z - \frac{1}{8} = 0, \qquad (25)$$

with solutions (only z_2 is physically relevant)

$$z_1 = \frac{43 - \sqrt{2305}}{152} \approx -0.033 < 0, \qquad z_2 = \frac{43 + \sqrt{2305}}{152} \approx 0.599 > 0. \qquad (26)$$

Inserting $\beta B^2 = \alpha A\, z$ into (23) we have:

$$A = \frac{z_2 - \frac{3}{4}}{\alpha\, z_2 (\frac{11}{4} - \frac{19}{3} z_2)} \approx \frac{0.2424}{\alpha}, \qquad (27)$$

$$B = \sqrt{\frac{\alpha\, A\, z_2}{\beta}} \approx \sqrt{\frac{0.14514}{\beta}}, \qquad v = 1 + \beta B^2 \left(\frac{2}{3} + \frac{38}{45} \beta B^2 \right) \approx 1.11455. \qquad (28)$$

Comparison of KdV and KdV2 solitons

The same α, β values

KdV

A – arbitrary

For $A = 1 \quad B = \sqrt{\frac{3}{4}\frac{\alpha}{\beta}}$

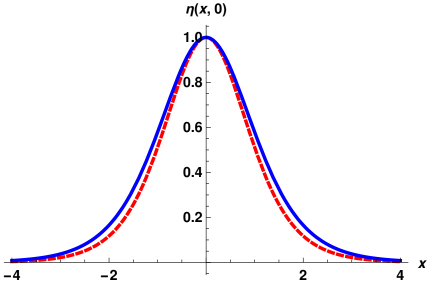
$$v = 1 + \frac{\alpha}{2} A$$

KdV2

$A \approx \frac{0.2424}{\alpha}$ – fixed

$$B \approx \sqrt{0.599 \frac{\alpha}{\beta}}$$

$$v \approx 1.11455.$$



Soliton's profiles for $A = 1$.
KdV soliton - red dashed line
KdV2 soliton - blue line.

Cnoidal solutions $\eta = A\,\mathrm{cn}^2(B(x-vt),m)+D$

Substitution of $\eta = A\,\mathrm{cn}^2(B(x-vt),m)+D$ gives

$$F_0 + F_2\,\mathrm{cn}^2(By) + F_4\,\mathrm{cn}^4(By) = 0, \tag{29}$$

Denoting $z = \dfrac{\beta B^2}{\alpha A} m$ one obtains the equation $F_4 = 0$ equivalent to (25) with the same roots z_1, z_2.

Equations

$$F_0 = 0 \qquad \text{and} \qquad F_2 = 0$$

have to be suplemented by volume conservation condition

$$\int_0^L (A\,\mathrm{cn}^2(By,m)+D)\,dy = 0. \tag{30}$$

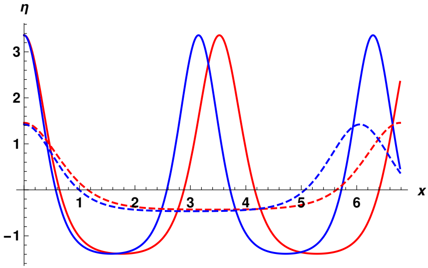
Altogether there are 4 conditions on 4 unknowns A, B, D, v in KdV2 case.

In KdV case there are only 3 conditions.

Comparison of KdV and KdV2 cnoidal solutions

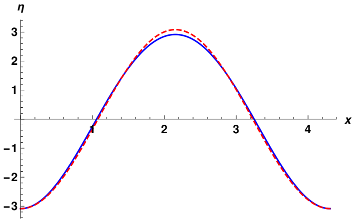
KdV: $m \in [0, 1]$ – all values admissible

KdV2: $z = z_2 \Longrightarrow m \in (0.96, 1]$ 'normal' cnoidal solution
$z = z_1 \Longrightarrow m \in [0,\ 0.2)$ 'inverted' cnoidal solution



Profiles of KdV2 solutions (red) and KdV solutions (blue) for case $m = 0.98$ (solid) and $m = 0.995$ (dashed).

Here $\alpha = 0.5$, $\beta = 0.4$.



Profiles of KdV2 solution (blue line) with the cosine wave of the same amplitude and wavelength (red, dashed line).

Here $\alpha = 0.3$, $\beta = 0.5$ and $m = 0.2$.

'Superposition' solutions to KdV

Assume solution in the form

$$\eta_\pm = \frac{A}{2}\left[\mathrm{dn}^2(By,m) \pm \sqrt{m}\,\mathrm{cn}(By,m)\,\mathrm{dn}(By,m)\right] + D, \quad y = x - vt \qquad (31)$$

Insertion into KdV gives

$$F_0 + F_2\,\mathrm{cn}^2 + F_{11}\,\mathrm{cn}(By,m)\,\mathrm{dn} = 0, \qquad (32)$$

Equations $F_2 = 0$ and $F_4 = 0$ are equivalent and give $B = \sqrt{\frac{3}{4}\frac{\alpha}{\beta}A}$

Volume conservation supplies $D = -\frac{A}{2}\frac{E(m)}{K(m)}$. Then

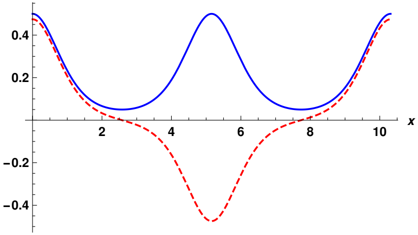
$$v = 1 + \frac{\alpha A}{8}\left[5 - m - 6\,\frac{E(m)}{K(m)}\right] \qquad \text{– superposition solution}$$

$$v = 1 + \frac{\alpha A}{8}\left[4(2-m) - 6\,\frac{E(m)}{K(m)}\right] \qquad - \ \mathrm{cn}^2 \text{ or } \mathrm{dn}^2 \text{ solution}$$

Properties of $\eta_+(By, m)$ and $\eta_-(By, m)$ functions

From periodicity of the Jacobi elliptic functions it follows that

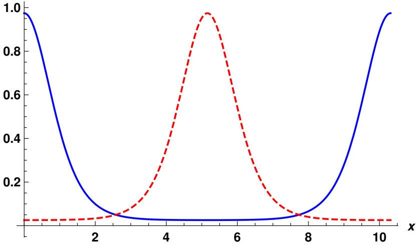
$$\eta_+(x,t) = \eta_-(x \pm L/2, t). \tag{33}$$



Profiles of functions

$\frac{A}{2}\,\mathrm{dn}^2(x,m)$ – blue solid line and

$\frac{A}{2}\sqrt{m}\,\mathrm{cn}(x,m)\,\mathrm{dn}(x,m)$ – red dashed line.



Profiles of functions

η_+ – blue solid line and

η_- – red dashed line.

'Superposition' solutions to KdV2

Insertion of $\eta_\pm$ into KdV2 gives

$$F_0 + F_2\,\mathrm{cn}^2 + F_4\,\mathrm{cn}^4 + F_{11}\,\mathrm{cn}\,\mathrm{dn} + F_{31}\,\mathrm{cn}^3\,\mathrm{dn} = 0, \tag{34}$$

Equations $F_4 = 0$ and $F_{31} = 0$ are equivalent.

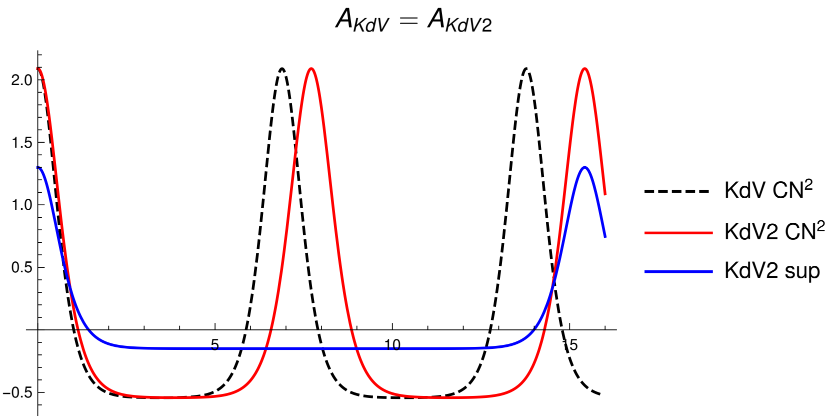
Denoting as previously $z = \dfrac{\beta B^2}{\alpha A}$ we obtain the same roots z_1, z_2.

Equations $F_4 = 0$ and $F_{31} = 0$ are equivalent, as well.

Altogether there are 4 conditions on 4 unknowns A, B, D, v in KdV2 case.

In KdV case there are only 3 conditions.

Comparison of periodic solutions



Case: $\alpha = \beta = 0.2424, \quad m = 0.999.$

$v_{KdV} = 1.12132, \quad v_{KdV2} = 1.08965, \quad v_{sup} = 1.10992.$

Higher order equation – KdV3

In third order the wave equation is

$$\eta_t + \eta_x + \alpha \frac{3}{2}\eta\eta_x + \beta \frac{1}{6}\eta_{3x} - \alpha^2 \frac{3}{8}\eta^2\eta_x + \alpha\beta \left(\frac{23}{24}\eta_x\eta_{2x} + \frac{5}{12}\eta\eta_{3x} \right) + \beta^2 \frac{19}{360}\eta_{5x}$$
$$+ \alpha^3 \left(\frac{3}{16}\eta^3\eta_x \right) + \alpha^2\beta \left(\frac{19}{32}\eta_x^3 + \frac{23}{16}\eta\eta_x\eta_{2x} + \frac{5}{16}\eta^2\eta_{3x} \right) \tag{35}$$
$$+ \alpha\beta^2 \left(\frac{317}{288}\eta_{2x}\eta_{3x} + \frac{1079}{1440}\eta_x\eta_{4x} + \frac{19}{80}\eta\eta_{5x} \right) + \beta^3 \frac{55}{3024}\eta_{7x} = 0.$$

Substitution $\eta(x,t) = A\,\mathrm{sech}[B(x-vt)]^2$ yields

$$C\,\Big(C_0 + C_2\cosh^2[B(x-vt)] + C_4\cosh^4[B(x-vt)] + C_6\cosh^6[B(x-vt)]\Big) = 0. \tag{36}$$

Then $C_0 = C_2 = C_4 = C_6 = 0.$ Denote $z = \dfrac{\beta\, B^2}{\alpha\, A}$ as previously.

Equation $C_2 = 0$ has three roots z_1, z_2, z_3, all complex. Then there are no physically relevant solutions in the form of single solitons.

Conclusions

- **There exist three classes of exact solutions to KdV2 which have the same form as the corresponding solutions to KdV but with slightly different coefficients. These are:**
  solitary waves of the form $A\,\mathrm{sech}^2$**,**
  cnoidal waves $A\,\mathrm{cn}^2 + D$ **and**
  periodic waves of the form $\frac{A}{2}[\,\mathrm{dn}^2 \pm \sqrt{m}\,\mathrm{cn}\,\mathrm{dn}] + D$ **.**
- **KdV2 imposes one more condition on coefficients of the exact solutions than KdV.**
- **For next (third) order approximation to the Euler equations (the wave equation KdV3) exact solutions of the same forms do not exist.**

New equation for uneven bottom

Consider the case of $\alpha = O(\beta)$ and $\delta = O(\beta^2)$.

Then $\alpha = A\,\beta, \quad \delta = D\,\beta^2$, A, D of the order of 1.
Now, we insert the general form of the velocity potential ($F = \phi_x^{(1)}$)

$$\begin{aligned}\phi = \phi^{(0)} - \frac{1}{2}\beta z^2 \phi^{(0)}_{2x} + \frac{1}{24}\beta^2 z^4 \phi^{(0)}_{4x} - \frac{1}{720}\beta^3 z^6 \phi^{(0)}_{6x} + \cdots \\ + \beta z F - \frac{1}{6}\beta^2 z^3 F_{2x} + \frac{1}{120}\beta^3 z^5 F_{4x} + \cdots\,,\end{aligned} \tag{37}$$

into the bottom boundary condition which in this case is

$$\phi_z - D\beta^3\,(h_x\,\phi_x) = 0, \quad \text{for} \quad z = D\beta^2 h(x) \tag{38}$$

obtaining the relation

$$F = D\beta^2 (hf_x)_x + \frac{1}{2}D^2\beta^5 (h^2 F_x)_x - \frac{1}{6}D^3\beta^7 (h^3 f_{3x})_x \cdots \tag{39}$$

New equation for uneven bottom

Now inserting (37) into (2) and (3) we obtain the Boussinesq equations

$$\eta_t + w_x + \beta\left(A(\eta w)_x - \frac{1}{6}w_{3x}\right) + \beta^2\left(-A\frac{1}{2}(\eta w_{2x})_x + \frac{1}{120}w_{5x} - D\,(hw)_x\right) = 0,$$

$$w_t + \eta_x + \beta\left(Aww_x - \frac{1}{2}w_{2xt}\right) + \beta^2\left(-A(\eta w_{xt})_x + A\frac{1}{2}w_x w_{2x} - A\frac{1}{2}ww_{3x} + \frac{1}{24}w_{4xt}\right) = 0.$$

They can be made compatible with

$$w = \eta + \beta\left(-A\frac{1}{4}\eta^2 + \frac{1}{3}\eta_{2x}\right) + \beta^2\left(A^2\frac{1}{8}\eta^3 + A\frac{3}{16}\eta_x^2 + A\frac{1}{2}\eta\eta_{2x} + \frac{1}{10}\eta_{4x} + \frac{1}{2}D(h\,\eta)\right)$$

with the final KdV2 – type equation for the uneven bottom

$$\eta_t + \eta_x + \beta\left(A\frac{3}{2}\eta\eta_x + \frac{1}{6}\eta_{3x}\right) + \beta^2\left(-A^2\frac{3}{8}\eta^2\eta_x + A\frac{23}{24}\eta_x\eta_{2x} + A\frac{5}{12}\eta_x\eta_{3x} + \frac{19}{360}\eta_{5x}\right) + \beta^2\left(-\frac{1}{2}D(h\,\eta)_x\right) = 0. \tag{40}$$

New wave equation, original notation

Substituting $A\,\beta = \alpha$ and $D\,\beta = \delta$ we can come back to original parameters

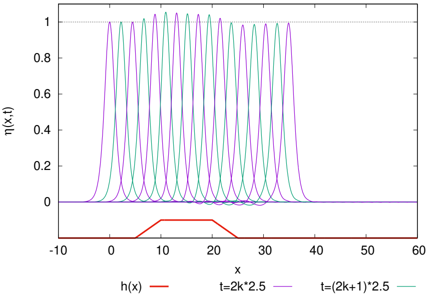
$$w = \eta - \frac{1}{4}\alpha\eta^2 + \frac{1}{3}\beta\eta_{2x} + \frac{1}{8}\alpha^2\eta^3 + \alpha\beta\left(\frac{3}{16}\eta_x^2 + \frac{1}{2}\eta\eta_{2x}\right) + \frac{1}{10}\beta^2\eta_{4x} + \frac{1}{2}\delta\, h\,\eta \tag{41}$$

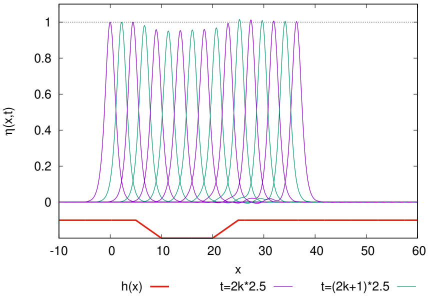
$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{3x} - \frac{3}{8}\alpha^2\eta^2\eta_x + \alpha\beta\left(\frac{23}{24}\eta_x^2 + \frac{5}{12}\eta\eta_{2x}\right) + \beta^2\left(\frac{19}{360}\eta_{5x}\right) - \frac{1}{2}\delta\,(h\,\eta)_x = 0. \tag{42}$$

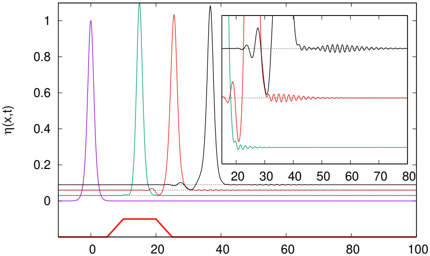
The black terms are as in the extended KdV equation (KdV2).

The blue ones come directly from interaction with the uneven bottom.

Examples in numerical simulations







Parameters of calculations:

Top: $\alpha = \beta = 0.2424 \quad \delta = 2\beta^2 = 0.1175$

Bottom: $\alpha = \beta = 0.2424 \quad \delta = 3\beta^2 = 0.176$

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