Exact soliton, periodic and superposition solutions to the fifth-order Korteweg-de Vries equation and derivation of the new equation for an uneven bottom

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Introduction, origin of Korteweg - de Vries equation

Shallow water problem

\[ \eta(x,t) \]

\[ \alpha = \frac{a}{h} \]
\[ \beta = (h/l)^2 \]

**Assumptions:**

Fluid is incompressible and inviscid, Motion is irrotational,
\[ \alpha \ll 1, \beta \ll 1 \] - small parameters of the same order.

**Figure:** Schematic view of the geometry of the problem.
Euler equations

Euler equations in scaled dimensionless coordinates

\[ \beta \phi_{2x} + \phi_{2z} = 0, \quad \text{for} \quad 0 < z < 1 + \alpha \eta(x, t) \quad (1) \]

\[ \frac{1}{\beta} \phi_z - (\alpha \eta_x \phi_x + \eta_t) = 0, \quad \text{for} \quad z = 1 + \alpha \eta(x, t) \quad (2) \]

\[ \phi_t + \frac{1}{2} (\alpha \phi_x^2 + \frac{\alpha}{\beta} \phi_z^2) + \eta = 0, \quad \text{for} \quad z = 1 + \alpha \eta(x, t) \quad (3) \]

\[ \phi_z = 0, \quad \text{for} \quad z = 0. \quad (4) \]

Velocity potential

\[ \Phi(x, z, t) = \sum_{m=0}^{\infty} z^m \phi^{(m)}(x, t). \quad (5) \]

Eqs. (1), (4) and (5) imply \((f(x, t) \equiv \phi^{(0)}(x, t))\)

\[ \phi = f - \frac{1}{2} \beta z^2 f_{2x} + \frac{1}{24} \beta^2 z^4 f_{4x} - \frac{1}{720} \beta^3 z^6 f_{6x} + \frac{1}{40320} \beta^4 z^8 f_{8x} \cdots. \quad (6) \]
Boussinesq’s equations

Insertion of $\phi(x, z, t)$ (6) into (2) gives

$$
\eta_t + w_x + \alpha(\eta w)_x - \frac{1}{6} \beta w_{3x} - \frac{1}{2} \alpha \beta (\eta w_{2x})_x + \frac{1}{120} \beta^2 w_{5x} = 0. \quad (7)
$$

Insertion of $\phi(x, z, t)$ (6) into (2) yields

$$
w_t + \eta_x + \beta \left( w w_x - \frac{1}{2} w_{2xt} \right) + \alpha \beta \left( - (\eta w_{xt})_x + \frac{1}{2} w_x w_{2x} - \frac{1}{2} w w_{3x} \right) + \frac{1}{24} \beta^2 w_{4xt} = 0. \quad (8)
$$

The equations (7) and (8) constitute the coupled set of Boussinesq’s equations (obtained after neglecting terms of order higher than the second) for two unknown functions:

$$
\eta(x, t) \quad \text{and} \quad w(x, t) \equiv f_x(x, t) \equiv \phi^{(0)}_x(x, t).
$$
KdV equation

In the lowest (zeroth) order

\[ \eta_t + w_x = 0, \quad w_t + \eta_x = 0 \quad \implies \quad w = \eta, \quad \eta_t + \eta_x = 0. \quad (9) \]

In the first order one tries

\[ w = \eta + \alpha Q^{(\alpha)} + \beta Q^{(\beta)}. \quad (10) \]

Substituting (10) into the Boussinesq system, subtracting the results and using \( Q_t = -Q_x \) one obtains

\[ \alpha (2Q_x^{(\alpha)} + \eta \eta_x) + \beta (2Q_x^{(\beta)} - \frac{2}{3} \eta \eta_x) = 0 \quad (11) \]

which gives

\[ Q_x^{(\alpha)} = -\frac{1}{2} \eta \eta_x \quad \implies \quad Q^{(\alpha)} = -\frac{1}{4} \eta^2, \quad (12) \]

\[ Q_x^{(\beta)} = \frac{1}{3} \eta 2x \quad \implies \quad Q^{(\beta)} = \frac{1}{3} \eta 3x. \quad (13) \]
KdV and extended KdV equations

First order: Korteweg - de Vries equation in fixed reference frame (1895)

\[
w = \eta - \alpha \frac{1}{4} \eta^2 + \frac{1}{6} \beta \eta_{2x}
\]  

(14)

\[
\eta_t + \eta_x + \frac{3}{2} \alpha \eta \eta_x + \frac{1}{6} \beta \eta_{3x} = 0.
\]  

(15)

Second order: KdV2 equation (Marchant & Smyth 1990)

\[
w = \eta - \frac{1}{4} \alpha \eta^2 + \frac{1}{3} \beta \eta_{2x} + \frac{1}{8} \alpha^2 \eta^3 + \frac{1}{10} \beta^2 \eta_{4x} + \alpha \beta \left( \frac{3}{16} \eta_x^2 + \frac{1}{2} \eta \eta_{2x} \right).
\]  

(16)

\[
\eta_t + \eta_x + \frac{3}{2} \alpha \eta \eta_x + \frac{1}{6} \beta \eta_{3x} - \frac{3}{8} \alpha^2 \eta^2 \eta_x + \alpha \beta \left( \frac{23}{24} \eta_x \eta_{2x} + \frac{5}{12} \eta \eta_{3x} \right) + \frac{19}{360} \beta^2 \eta_{5x} = 0,
\]  

(17)
Solutions to KdV

- Single soliton solution
  \[ \eta(x, t) = A \operatorname{sech}[B(x - vt)]^2 \] (18)

- Periodic (cnoidal) solution
  \[ \eta(x, t) = A \operatorname{cn}[B(x - vt)]^2 + D \] (19)

  \[ \eta_{\pm}(y) = \frac{A}{2} \left[ \operatorname{dn}^2(By, m) \pm \sqrt{m} \operatorname{cn}(By, m) \operatorname{dn}(By, m) \right] + D, \] (20)

- Multi-soliton solutions, IST method, Hirota method ...
Hypothesis

There exist the following analytic solutions to KdV2:

- **Single soliton solution**  \( A \, \text{sech}^2[B(x - vt)] \)
- **Periodic (cnoidal) solution**  \( A \, \text{cn}^2[B(x - vt)] + D \)
- **Periodic ("superposition") solution**  \( \frac{A}{2} \left[ \text{dn}^2 \pm \sqrt{m} \, \text{cn} \, \text{dn} \right] + D \)

of the same form as the corresponding KdV solutions, but with different coefficients \( A, B, \nu, m \).
Soliton solutions to KdV2 \[ \eta = A \text{sech}^2[B(x - vt)] \]

With \[ \eta(x, t) = \eta(x - vt), \] we have \[ \eta_t = -v \eta_x. \] Then from (17) one has

\[
(1 - v)\eta + \alpha \frac{3}{4} \eta^2 + \beta \frac{1}{6} \eta_{2x} - \frac{1}{8} \alpha^2 \eta^3 + \alpha \beta \left( \frac{13}{48} \eta_x^2 + \frac{5}{12} \eta \eta_{2x} \right) + \beta^2 \frac{19}{360} \eta_{4x} = 0.
\]

Substitution of \[ \eta = A \text{sech}^2[B(x - vt)] \] gives

\[
C_2 \text{sech}^2(By) + C_4 \text{sech}^4(By) + C_6 \text{sech}^6(By) = 0,
\]

Solution exists when simultaneously

\[
C_2 = 0 = (1 - v) + \frac{2}{3} B^2 \beta + \frac{38}{45} B^4 \beta^2 \quad (22)
\]

\[
C_4 = 0 = \frac{3A\alpha}{4} - B^2 \beta + \frac{11}{4} A\alpha B^2 \beta - \frac{19}{3} B^4 \beta^2 \quad (23)
\]

\[
C_6 = 0 = - \left( \frac{1}{8} \right) (A\alpha)^2 - \frac{43}{12} A\alpha B^2 \beta + \frac{19}{3} B^4 \beta^2 \quad (24)
\]
From (24), denoting \( z = \frac{\beta B^2}{\alpha A} \) we obtain \( \frac{19}{3} z^2 - \frac{43}{12} z - \frac{1}{8} = 0 \), (25) with solutions (only \( z_2 \) is physically relevant)

\[
z_1 = \frac{43 - \sqrt{2305}}{152} \approx -0.033 < 0, \quad z_2 = \frac{43 + \sqrt{2305}}{152} \approx 0.599 > 0.
\] (26)

Inserting \( \beta B^2 = \alpha A z \) into (23) we have:

\[
A = \frac{z_2 - \frac{3}{4}}{\alpha z_2 \left( \frac{11}{4} - \frac{19}{3} z_2 \right)} \approx \frac{0.2424}{\alpha},
\] (27)

\[
B = \sqrt{\frac{\alpha A z_2}{\beta}} \approx \sqrt{\frac{0.14514}{\beta}}, \quad v = 1 + \beta B^2 \left( \frac{2}{3} + \frac{38}{45} \beta B^2 \right) \approx 1.11455.
\] (28)
Comparison of KdV and KdV2 solitons

The same $\alpha, \beta$ values

**KdV**

$A$ – arbitrary

For $A = 1$ \[ B = \sqrt{\frac{3\alpha}{4\beta}} \]

$\nu = 1 + \frac{\alpha}{2}A$

**KdV2**

$A \approx \frac{0.2424}{\alpha}$ – fixed

$B \approx \sqrt{0.599\frac{\alpha}{\beta}}$

$\nu \approx 1.11455.$

Soliton’s profiles for $A = 1$.

KdV soliton - red dashed line

KdV2 soliton - blue line.
Cnoidal solutions  \( \eta = A \, \text{cn}^2(B(x - vt), m) + D \)

Substitution of \( \eta = A \, \text{cn}^2(B(x - vt), m) + D \) gives

\[
F_0 + F_2 \, \text{cn}^2(By) + F_4 \, \text{cn}^4(By) = 0, \tag{29}
\]

Denoting \( z = \frac{\beta B^2}{\alpha A} m \) one obtains the equation \( F_4 = 0 \) equivalent to (25) with the same roots \( z_1, z_2 \).

Equations

\[
F_0 = 0 \quad \text{and} \quad F_2 = 0
\]

have to be suplemented by volume conservation condition

\[
\int_0^L (A \, \text{cn}^2(By, m) + D) \, dy = 0. \tag{30}
\]

Altogether there are 4 conditions on 4 unknowns \( A, B, D, \nu \) in KdV2 case.

In KdV case there are only 3 conditions.
Comparison of KdV and KdV2 cnoidal solutions

**KdV:** \( m \in [0, 1] \) – all values admissible

**KdV2:**
- \( z = z_2 \implies m \in (0.96, 1) \) ‘normal’ cnoidal solution
- \( z = z_1 \implies m \in [0, 0.2) \) ‘inverted’ cnoidal solution

Profiles of KdV2 solutions (red) and KdV solutions (blue) for case \( m = 0.98 \) (solid) and \( m = 0.995 \) (dashed).

Here \( \alpha = 0.5, \beta = 0.4 \).

Profiles of KdV2 solution (blue line) with the cosine wave of the same amplitude and wavelength (red, dashed line).

Here \( \alpha = 0.3, \beta = 0.5 \) and \( m = 0.2 \).
'Superposition' solutions to KdV and KdV2

Assume solution in the form
\[ \eta_{\pm} = \frac{A}{2} \left[ \text{dn}^2(By, m) \pm \sqrt{m} \text{cn}(By, m) \text{dn}(By, m) \right] + D, \quad y = x - vt \quad (31) \]

Insertion into KdV gives
\[ F_0 + F_2 \text{cn}^2 + F_{11} \text{cn}(By, m) \text{dn} = 0, \quad (32) \]

Equations $F_2 = 0$ and $F_4 = 0$ are equivalent and give $B = \sqrt{\frac{3}{4} \frac{\alpha}{\beta}} A$

Volume conservation supplies $D = -\frac{A}{2} \frac{E(m)}{K(m)}$. Then
\[ \nu = 1 + \frac{\alpha A}{8} \left[ 5 - m - 6 \frac{E(m)}{K(m)} \right] \quad \text{– superposition solution} \]

\[ \nu = 1 + \frac{\alpha A}{8} \left[ 4(2 - m) - 6 \frac{E(m)}{K(m)} \right] \quad \text{– cn}^2 \text{ or dn}^2 \text{ solution} \]
Properties of $\eta_+(By, m)$ and $\eta_-(By, m)$ functions

From periodicity of the Jacobi elliptic functions it follows that

$$
\eta_+(x, t) = \eta_-(x \pm L/2, t).
$$

(33)

Profiles of functions

$A^2 \text{dn}^2(x, m)$ – blue solid line and $A^2 \sqrt{m} \text{cn}(x, m) \text{dn}(x, m)$ – red dashed line.

Profiles of functions

$\eta_+$ – blue solid line and $\eta_-$ – red dashed line.
'Superposition' solutions to KdV2

Insertion of $\eta_\pm$ into KdV2 gives

$$F_0 + F_2 \cn^2 + F_4 \cn^4 + F_{11} \cn \dn + F_{31} \cn^3 \dn = 0,$$

(34)

Equations $F_4 = 0$ and $F_{31} = 0$ are equivalent.

Denoting as previously $z = \frac{\beta B^2}{\alpha A}$ we obtain the same roots $z_1, z_2$.

Equations $F_4 = 0$ and $F_{31} = 0$ are equivalent, as well.

Altogether there are 4 conditions on 4 unknowns $A, B, D, \nu$ in KdV2 case. In KdV case there are only 3 conditions.
Comparison of periodic solutions

\[ A_{KdV} = A_{KdV^2} \]

Case: \( \alpha = \beta = 0.2424, \quad m = 0.999. \)

\( v_{KdV} = 1.12132, \quad v_{KdV^2} = 1.08965, \quad v_{sup} = 1.10992. \)
In third order the wave equation is

\[ \eta_t + \eta_x + \frac{3}{2} \eta \eta_x + \beta \frac{1}{6} \eta_{3x} - \alpha^2 \frac{3}{8} \eta^2 \eta_x + \alpha \beta \left( \frac{23}{24} \eta_x \eta_{2x} + \frac{5}{12} \eta \eta_{3x} \right) + \beta^2 \frac{19}{360} \eta_{5x} \]

\[ + \alpha^3 \left( \frac{3}{16} \eta^3 \eta_x \right) + \alpha^2 \beta \left( \frac{19}{32} \eta^3 + \frac{23}{16} \eta \eta_x \eta_{2x} + \frac{5}{16} \eta^2 \eta_{3x} \right) \]

\[ + \alpha \beta^2 \left( \frac{317}{288} \eta_{2x} \eta_{3x} + \frac{1079}{1440} \eta_x \eta_{4x} + \frac{19}{80} \eta \eta_{5x} \right) + \beta^3 \frac{55}{3024} \eta_{7x} = 0. \quad (35) \]

Substitution \( \eta(x, t) = A \text{ sech}[B(x - vt)]^2 \) yields

\[ C \left( C_0 + C_2 \cosh^2[B(x - vt)] + C_4 \cosh^4[B(x - vt)] + C_6 \cosh^6[B(x - vt)] \right) = 0. \quad (36) \]

Then \( C_0 = C_2 = C_4 = C_6 = 0 \). Denote \( z = \frac{\beta B^2}{\alpha A} \) as previously.

**Equation** \( C_2 = 0 \) **has three roots** \( z_1, z_2, z_3 \), all complex. Then there are no physically relevant solutions in the form of single solitons.
There exist three classes of exact solutions to KdV2 which have the same form as the corresponding solutions to KdV but with slightly different coefficients. These are:
- solitary waves of the form \( A \, \text{sech}^2 \),
- cnoidal waves \( A \, \text{cn}^2 + D \) and
- periodic waves of the form \( A \left[ \text{dn}^2 \pm \sqrt{m} \, \text{cn} \, \text{dn} \right] + D \).

KdV2 imposes one more condition on coefficients of the exact solutions than KdV.

For next (third) order approximation to the Euler equations (the wave equation KdV3) exact solutions of the same forms do not exist.
Consider the case of \( \alpha = O(\beta) \) and \( \delta = O(\beta^2) \).

Then \( \alpha = A \beta, \quad \delta = D \beta^2, \quad A, D \) of the order of 1.

Now, we insert the general form of the velocity potential \((F = \phi_x(1))\)

\[
\phi = \phi^{(0)} - \frac{1}{2} \beta z^2 \phi^{(0)}_{2x} + \frac{1}{24} \beta^2 z^4 \phi^{(0)}_{4x} - \frac{1}{720} \beta^3 z^6 \phi^{(0)}_{6x} + \ldots
\]

\[
+ \beta zF - \frac{1}{6} \beta^2 z^3 F_{2x} + \frac{1}{120} \beta^3 z^5 F_{4x} + \ldots ,
\]

(37)

into the bottom boundary condition which in this case is

\[
\phi_z - D \beta^3 (h_x \phi_x) = 0, \quad \text{for} \quad z = D \beta^2 h(x)
\]

(38)

obtaining the relation

\[
F = D \beta^2 (hf_x)_x + \frac{1}{2} D^2 \beta^5 (h^2 F_x)_x - \frac{1}{6} D^3 \beta^7 (h^3 f_{3x})_x \ldots
\]

(39)
New KdV2 equation for an uneven bottom

Now inserting (37) into (2) and (3) we obtain the Boussinesq equations

\[
\eta_t + w_x + \beta \left( A(\eta w)_x - \frac{1}{6} w_{3x} \right) + \beta^2 \left( -A \frac{1}{2} (\eta w_{2x})_x + \frac{1}{120} w_{5x} - D(hw)_x \right) = 0,
\]

\[
w_t + \eta_x + \beta \left( Aww_x - \frac{1}{2} w_{2xt} \right) + \beta^2 \left( -A(\eta w_{xt})_x + A \frac{1}{2} w_x w_{2x} - A \frac{1}{2} w w_{3x} + \frac{1}{24} w_{4xt} \right) = 0.
\]

They can be made compatible with

\[
w = \eta + \beta \left( -A \frac{1}{4} \eta^2 + \frac{1}{3} \eta_{2x} \right) + \beta^2 \left( A^2 \frac{1}{8} \eta^3 + A \frac{3}{16} \eta_x^2 + A \frac{1}{2} \eta \eta_{2x} + \frac{1}{10} \eta_{4x} + \frac{1}{2} D(h \eta) \right)
\]

with the final KdV2 – type equation for the uneven bottom

\[
\eta_t + \eta_x + \beta \left( A \frac{3}{2} \eta \eta_x + \frac{1}{6} \eta_{3x} \right) + \beta^2 \left( -A^2 \frac{3}{8} \eta^2 \eta_x + A \frac{23}{24} \eta_x \eta_{2x} + A \frac{5}{12} \eta_x \eta_{3x} + \frac{19}{360} \eta_{5x} \right) + \beta^2 \left( -\frac{1}{2} D(h \eta)_x \right) = 0.
\]

(40)
Substituting $A \beta = \alpha$ and $D \beta = \delta$ we can come back to original parameters

$$w = \eta - \frac{1}{4} \alpha \eta^2 + \frac{1}{3} \beta \eta_{2x} + \frac{1}{8} \alpha^2 \eta^3 + \alpha \beta \left( \frac{3}{16} \eta_x^2 + \frac{1}{2} \eta \eta_{2x} \right) + \frac{1}{10} \beta^2 \eta_{4x} + \frac{1}{2} \delta h \eta$$

(41)

$$\eta_t + \eta_x + \frac{3}{2} \alpha \eta \eta_x + \frac{1}{6} \beta \eta_{3x} - \frac{3}{8} \alpha^2 \eta^2 \eta_x + \alpha \beta \left( \frac{23}{24} \eta_x^2 + \frac{5}{12} \eta \eta_{2x} \right) + \beta^2 \left( \frac{19}{360} \eta^5 \right)$$

$$- \frac{1}{2} \delta \left( h \eta \right)_x = 0.$$

(42)

The black terms are as in the extended KdV equation (KdV2).

The blue ones come directly from interaction with the uneven bottom.
Examples in numerical simulations

Parameters of calculations:

Top: \( \alpha = \beta = 0.2424 \quad \delta = 2\beta^2 = 0.1175 \)

Bottom: \( \alpha = \beta = 0.2424 \quad \delta = 3\beta^2 = 0.176 \)
References


