Breathers into the stratified fluid: generation, dynamics and stability

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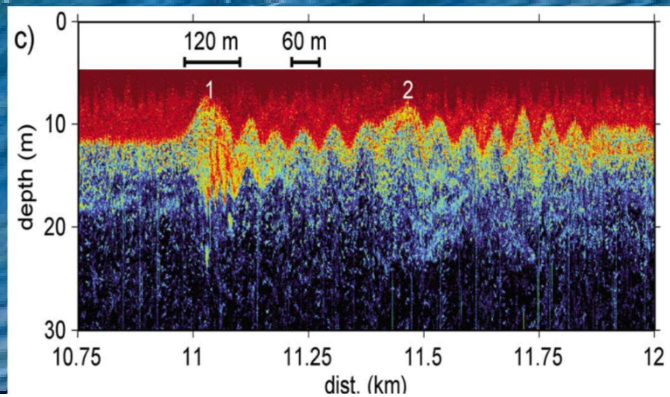
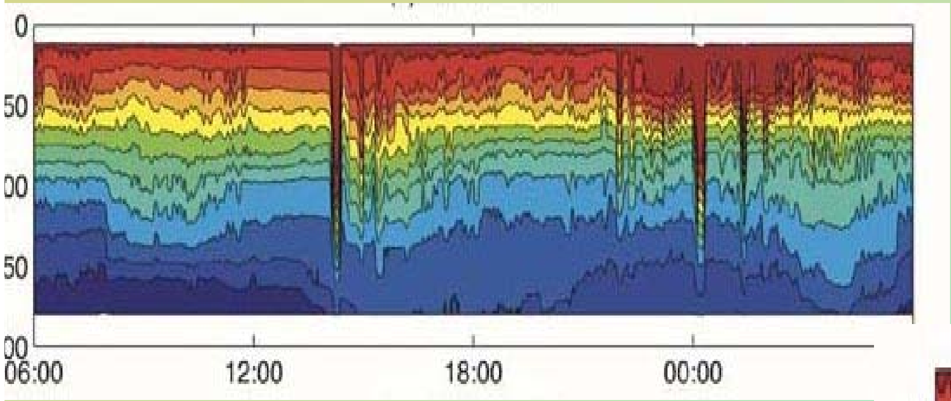
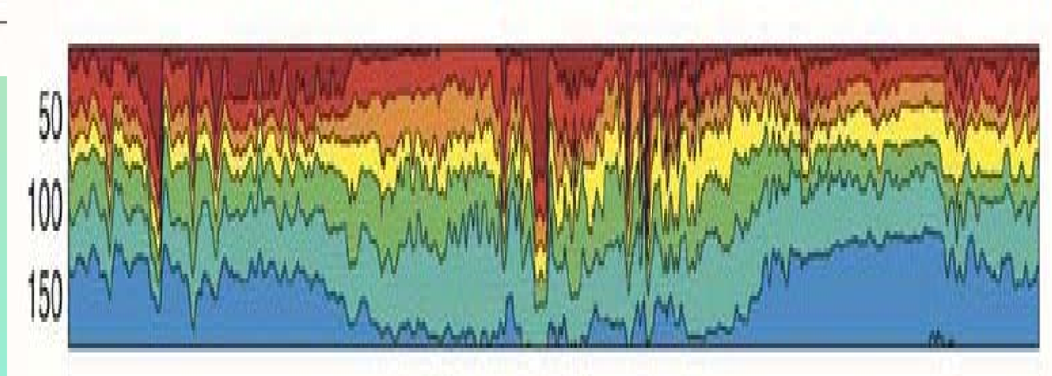
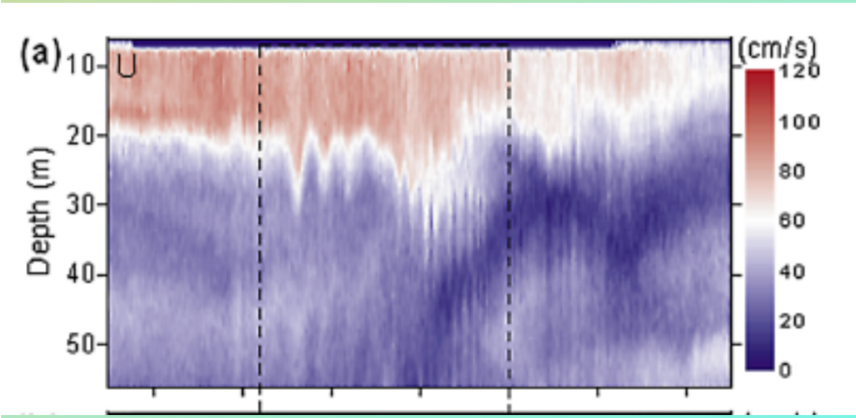


Intensive Internal Wave Observations



South China Sea(Duda et al., 2004)





South Korea shelf (Lee et al, 2006)

Generation of intensive internal waves at the ocean shelves is due to barotropic tide, with main period of 12.4 часа

The Gardner equation in the theory of extreme internal waves

The Gardner equation is well known as an appropriate physical model for the description of the weak nonlinear and weak dispersive wave (internal waves, plasma) field. It is derived from governing equations with use of a perturbation expansion series procedure and is valid to the certain conditions when $\alpha u \sim \beta u^2$

in the reference system

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \alpha_1 u^2 \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0$$

Cauchy Problem - Method of Inverse Scattering

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + 6qu^2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

$q = \pm 1$

$$\frac{\partial \hat{L}}{\partial t} = [\hat{L}\hat{A}] = \hat{L}\hat{A} - \hat{A}\hat{L}$$

$q > 0$

$$\hat{L}\left(\partial/\partial x, u(x,t)\right)\Psi = \lambda\Psi$$

$$\frac{\partial \Psi}{\partial t} = \hat{A}\left(\partial/\partial x, u(x,t)\right)\Psi$$

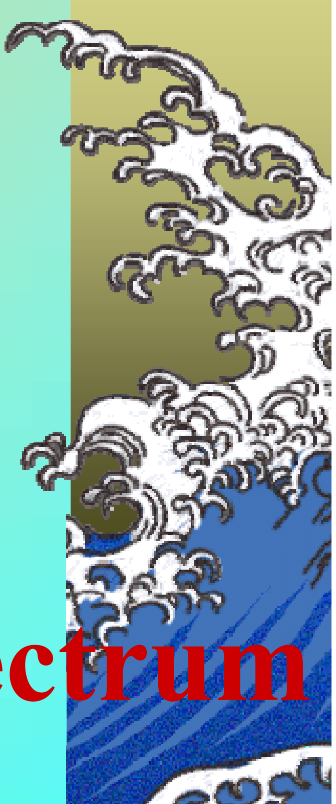
$$\hat{L} = \begin{pmatrix} -\frac{\partial}{\partial x} & -\frac{1}{2} + u \\ \frac{1}{2} - u & \frac{\partial}{\partial x} \end{pmatrix}$$

Cauchy Problem

First Step: t = 0

Direct Spectral Problem

$$u(x,0) \longrightarrow \hat{L}\Psi = \lambda\Psi \longrightarrow \text{spectrum}$$



Discrete spectrum: real roots correspond to solitons, complex roots – to breathers

- To find the discrete spectrum we find the steady – state solutions of the Gardner equation, the nonlinear waves of quasi-stationary shapes. They are the one – parametric family (solitons) and two-parametric family (breathers)

Gardner's Solitons

sign of α_1

$$u(x,t) = \frac{A}{1 + B\cosh(\gamma(x - Vt))},$$

$$A = \frac{6\beta\gamma^2}{\alpha},$$

$$B^2 = 1 + \frac{6\alpha_1\beta\gamma^2}{\alpha^2},$$

$$V = \beta\gamma^2$$

$$\alpha_1 < 0$$

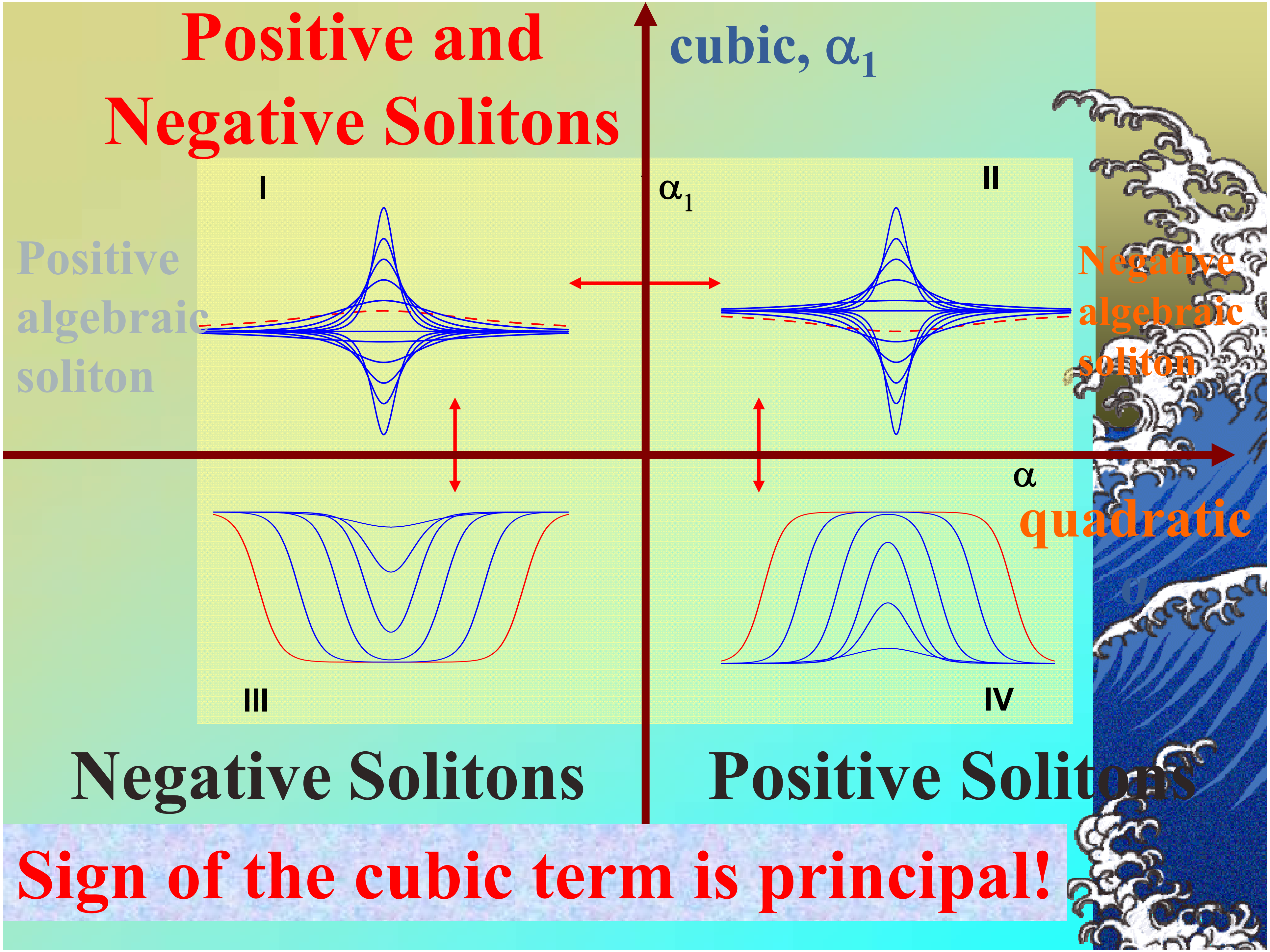
Limited amplitude

$a_{\lim} = -\alpha/\alpha_1$

$$a = \frac{A}{1+B}$$

$$\alpha_1 > 0$$

Two branches of solitons of both polarities, algebraic soliton $a_{\lim} = -2\,\alpha/\alpha_1$


Positive and Negative Solitons
cubic, α_1
I
α_1
II
Positive algebraic soliton
Negative algebraic soliton
α
quadratic
III
IV
Negative Solitons
Positive Solitons
Sign of the cubic term is principal!

Gardner's Breathers cubic, $\alpha_1 > 0$

$\beta = 1$, $\alpha = 12q$, $\alpha_1 = 6$, where q is arbitrary)

$$u = 2\frac{\partial}{\partial x}\operatorname{atan}\frac{l\,\mathrm{ch}(\Psi)\cos(\theta) - k\cos(\Phi)\mathrm{sh}(\kappa)}{l\,\mathrm{sh}(\Psi)\sin(\theta) + k\sin(\Phi)\mathrm{ch}(\kappa)}$$

θ and κ are the phases of carrier wave and envelope

$$\theta = k(x - wt) + \theta_0, \quad \kappa = l(x - vt) + \kappa_0$$

propagating with speeds $w = -k^2 + 3l^2, \quad v = -3k^2 + l^2$

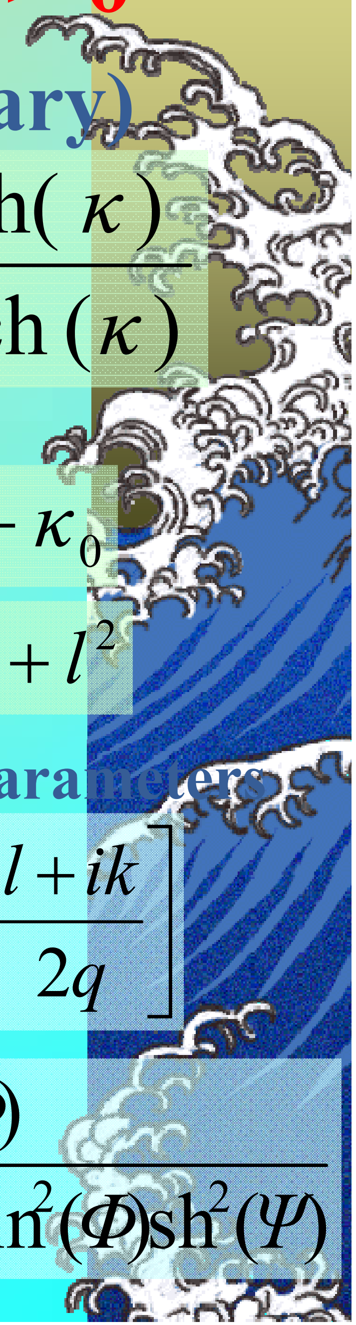
There are 4 free parameters: θ_0, κ_0 and two energetic parameters

Pelinovsky D &Grimshaw, 1997

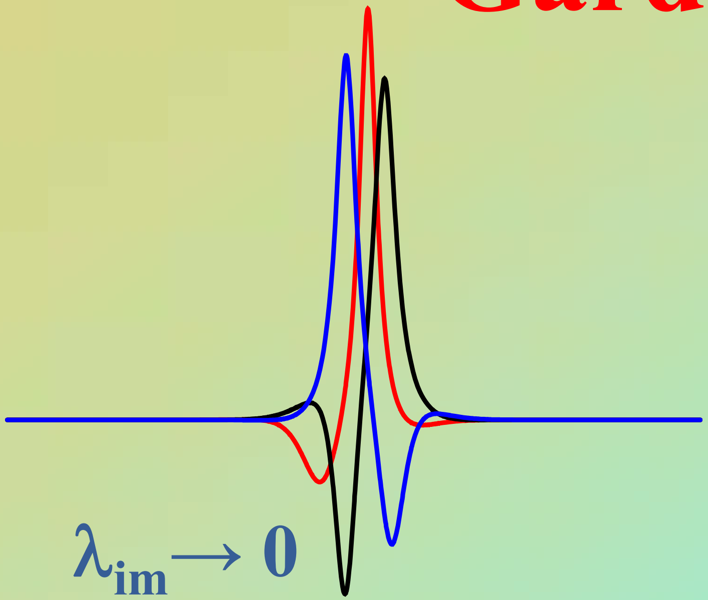
$$\Phi + i\Psi = \tan^{-1}\left[\frac{l + ik}{2q}\right]$$

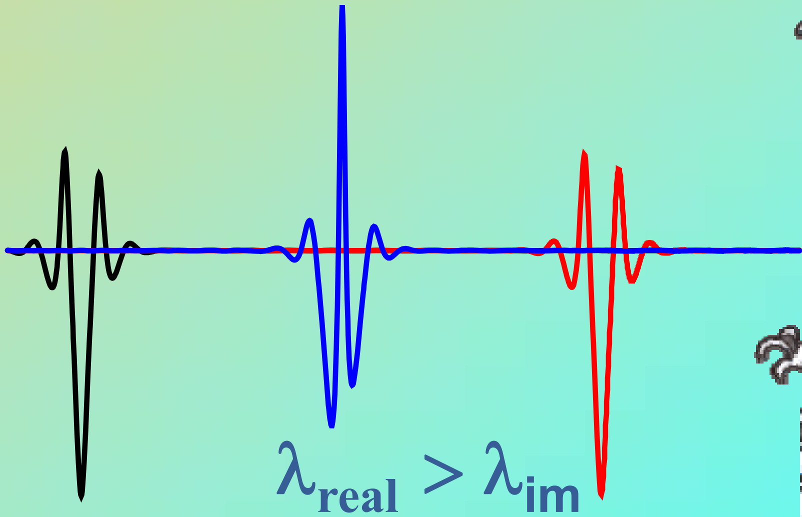
$$k = q\frac{\mathrm{sh}(2\Psi)}{\cos^2(\Phi)\mathrm{ch}^2(\Psi) + \sin^2(\Phi)\mathrm{sh}^2(\Psi)}$$

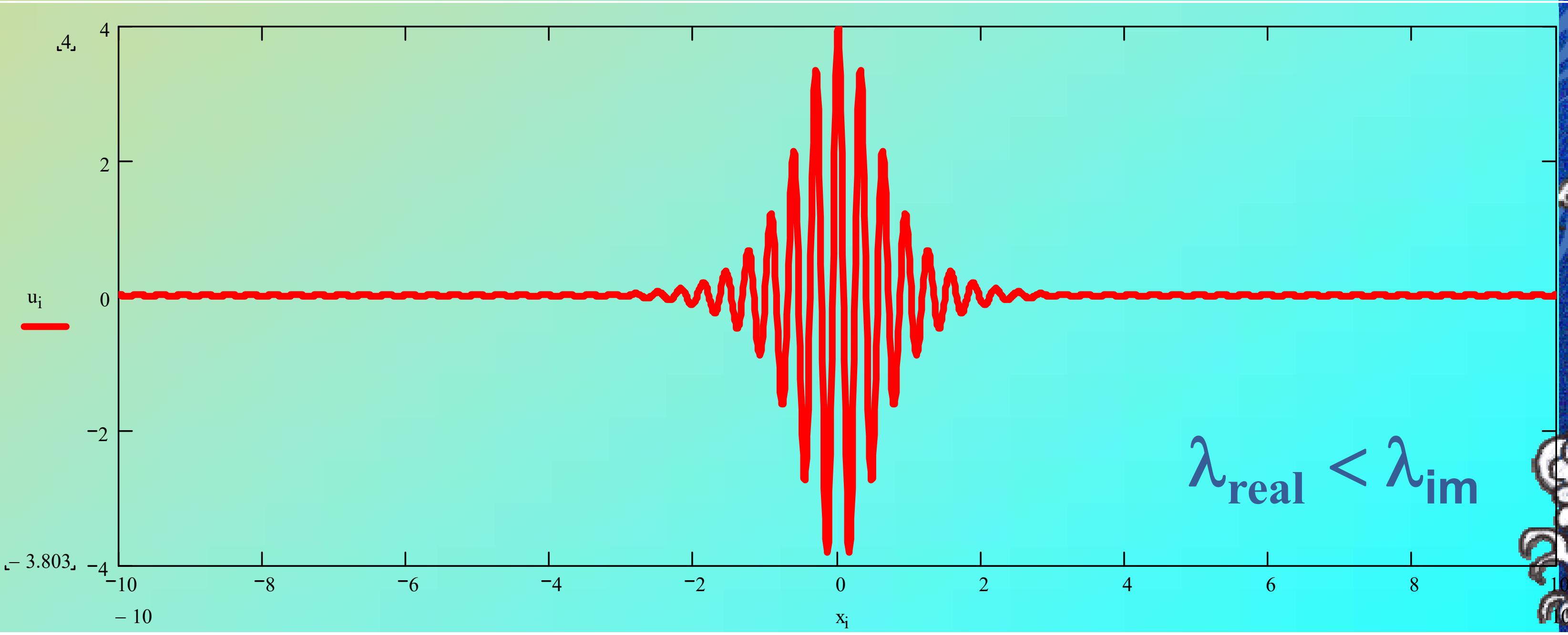
$$l = q\frac{\sin(2\Phi)}{\cos^2(\Phi)\mathrm{ch}^2(\Psi) + \sin^2(\Phi)\mathrm{sh}^2(\Psi)}$$

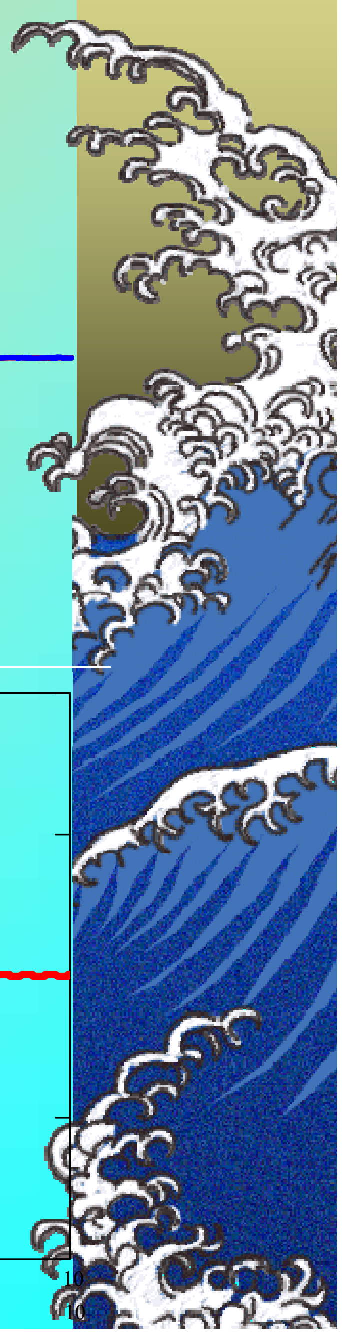


Gardner Breathers









The long internal waves give the real life to the full integrable Korteweg – de Vries equation and its extensions as the Gardner equation which is also the full integrable model.

Solitons and breathers in the frames of the Gardner equation are studied well.

Coefficients of the Gardner Equation

$$\frac{d^2\Phi}{dz^2}+\frac{N^2(x,z)}{c^2(x)}=0 \qquad \Phi(0)=\Phi(H)=0 \quad \Phi_{\max}=1$$

$$N^2(z)=-\frac{g}{\rho(z)}\frac{d\rho(z)}{dz}$$

$$\alpha=\left(\frac{3c}{2}\right)\frac{\int_0^H (d\Phi/dz)^3\,dz}{\int_0^H (d\Phi/dz)^2\,dz} \qquad \beta=\left(\frac{c}{2}\right)\frac{\int_0^H \Phi^2\,dz}{\int_0^H (d\Phi/dz)^2\,dz}$$

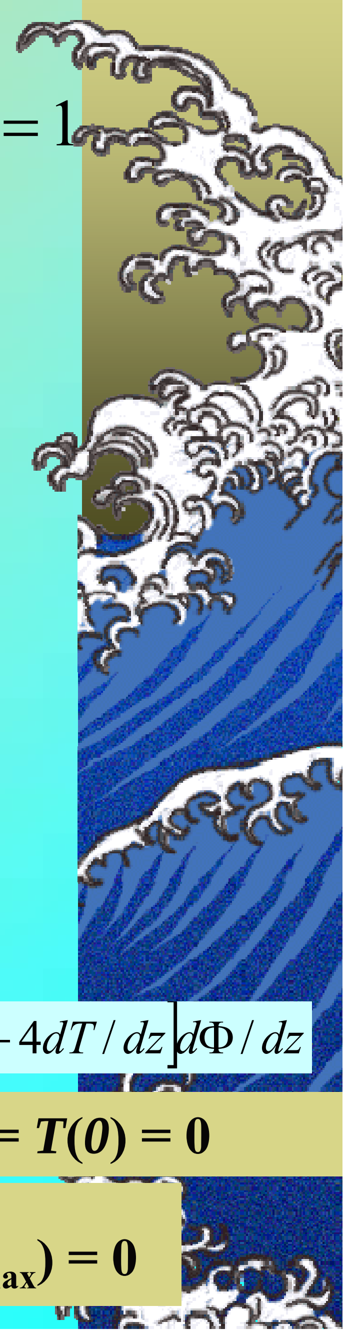
$$\alpha_1=\frac{3c}{2}\,\frac{\int dz\left\{\left[3(dT/dz)-2(d\Phi/dz)^2\right](d\Phi/dz)^2-\alpha^2(d\Phi/dz)^2+\Pi\right\}}{\int (d\Phi/dz)^2\,dz}$$

$$\Pi=\alpha c\left[5(d\Phi/dz)^2-4dT/dz\right]d\Phi/dz$$

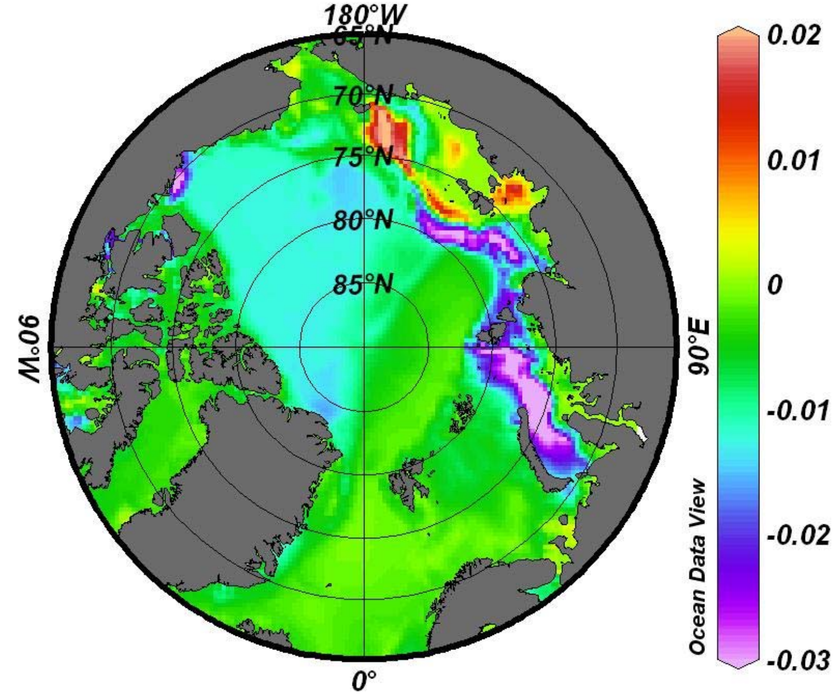
$$\frac{d}{dz}\left[(c-U)^2\frac{dT}{dz}\right]+N^2T=-\alpha\frac{d}{dz}\left[(c-U)\frac{d\Phi}{dz}\right]+\frac{3}{2}\frac{d}{dz}\left[(c-U)^2\left(\frac{d\Phi}{dz}\right)^2\right]$$

$$\boldsymbol{T(-H)=T(0)=0}$$

$$\boldsymbol{T(z_{\max})=0}$$

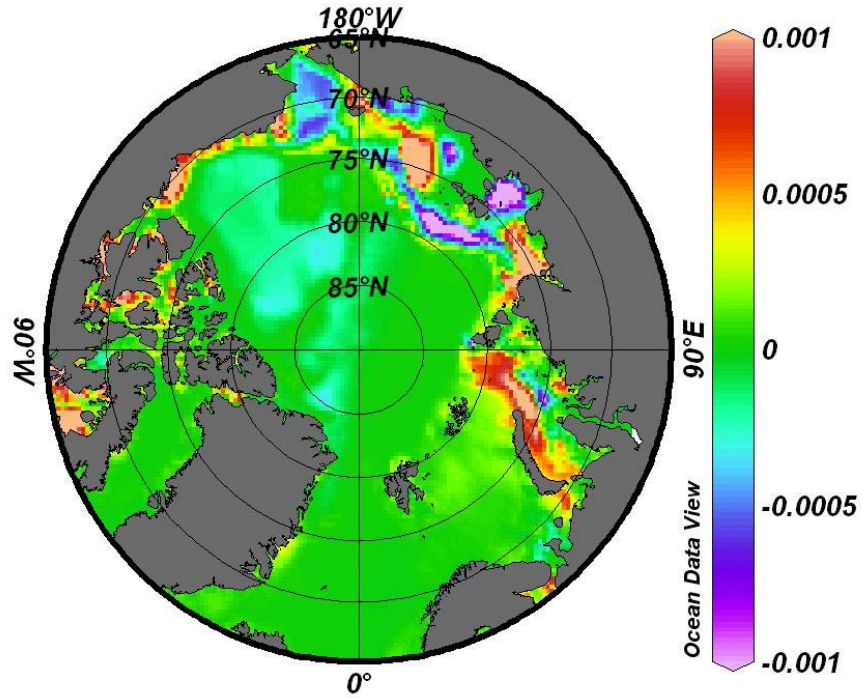


Quadratic nonlinear term, α, c^{-1}



Arctic

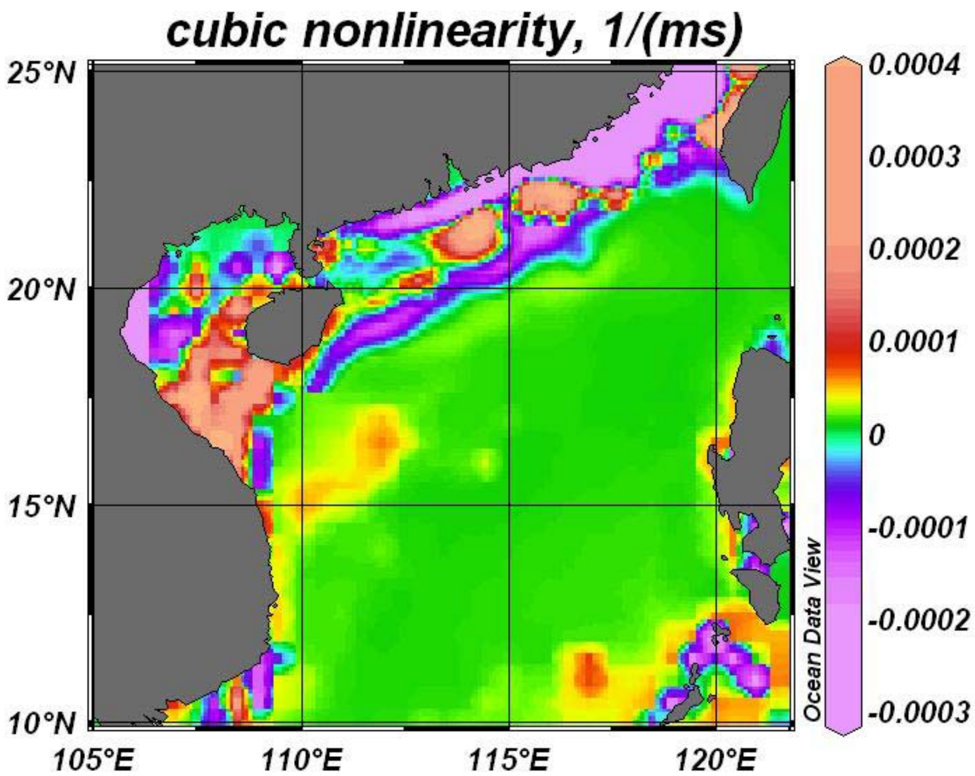
Cubic nonlinear term, α_1, $м^{-1}c^{-1}$

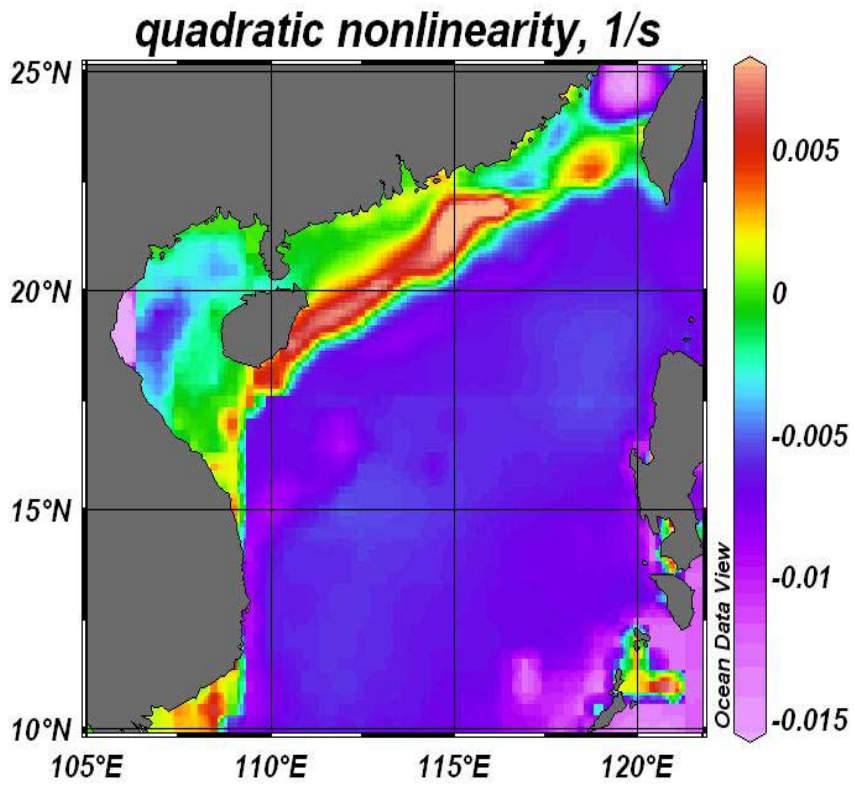


South China Sea

α_1

α

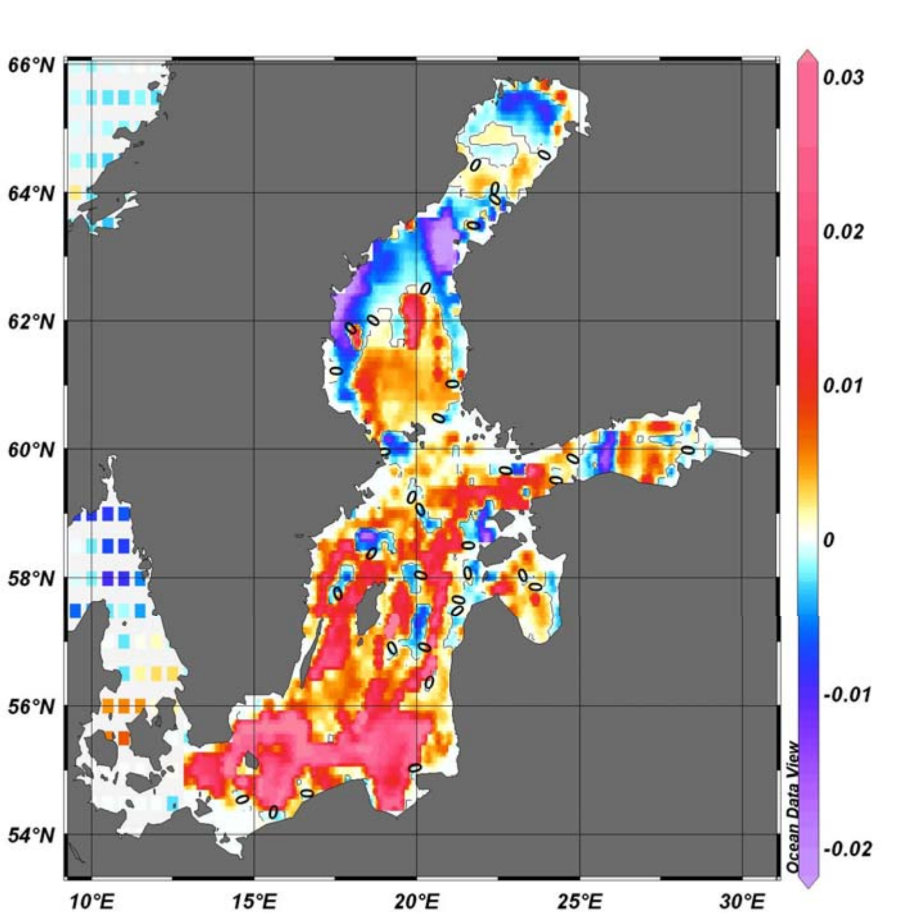


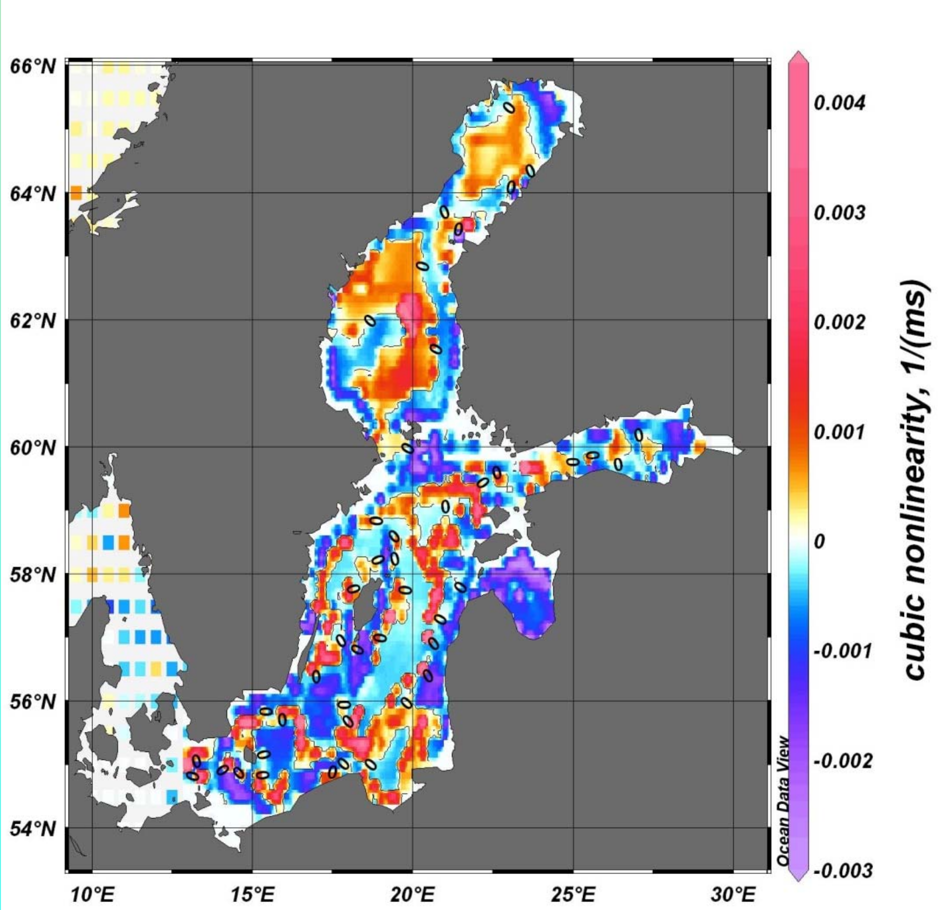


There are large zones of positive cubic coefficients !!!!

Baltik Sea

Cubic nonlinear coefficient





Quadratic nonlinear coefficient

Euler Equations

The nonlinear evolution of the breathers is simulated using Lamb's (Lamb, 1994) model of the Euler equations on a rotating *f* plane for inviscid, incompressible fluid:

$$\vec{V}_t + \left(\vec{V}\nabla\right)\vec{V} - f\vec{V} \times \vec{k} = -\nabla P - \vec{k}\rho g$$

$$\nabla\vec{V} = 0 \qquad \rho = \frac{\rho_f - \rho_0}{\rho_0}$$

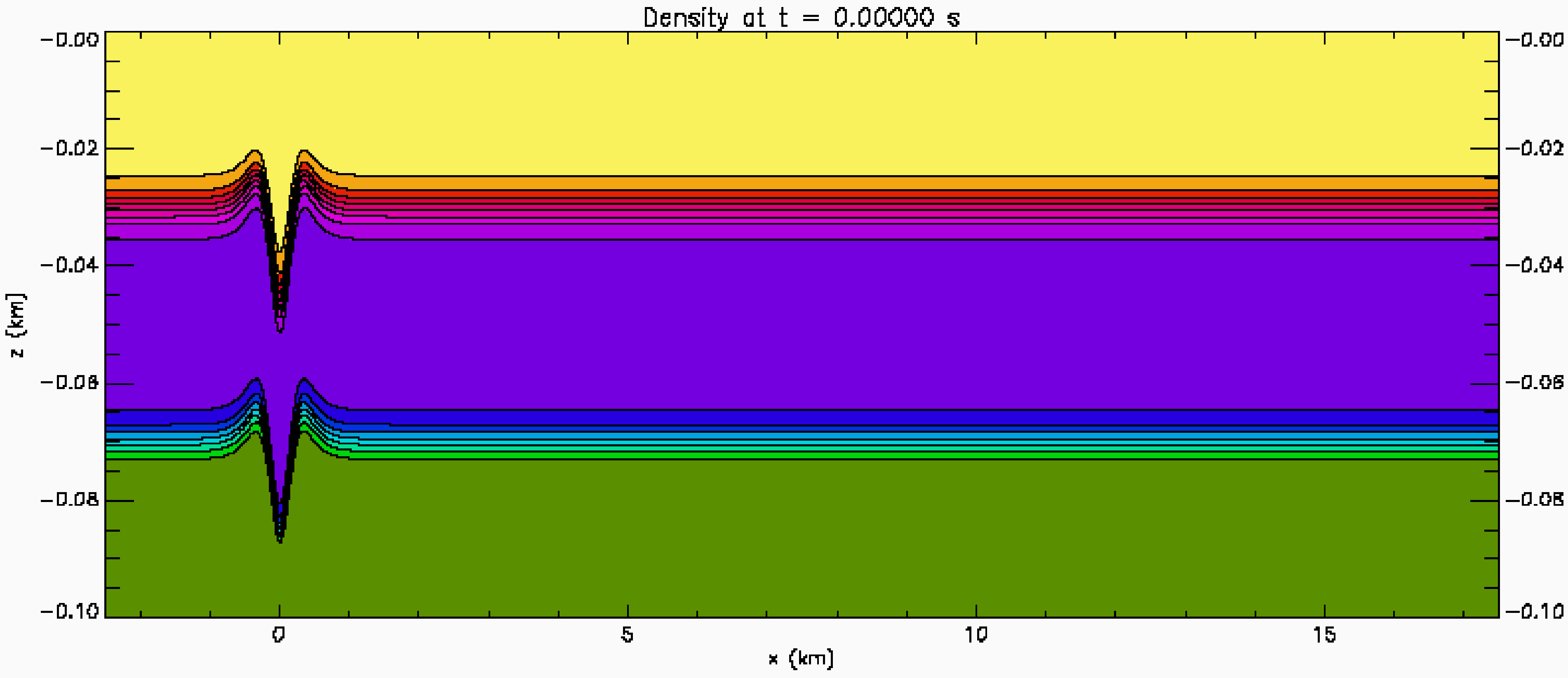
$$\rho_t + \vec{V}\nabla\rho = 0$$

$$\rho_f = \rho_0(1+\rho)$$

$\vec{V}(u,v,w)$ is the velocity vector, ∇ is the three dimensional vector gradient operator, subscript *t* denotes the time derivative,

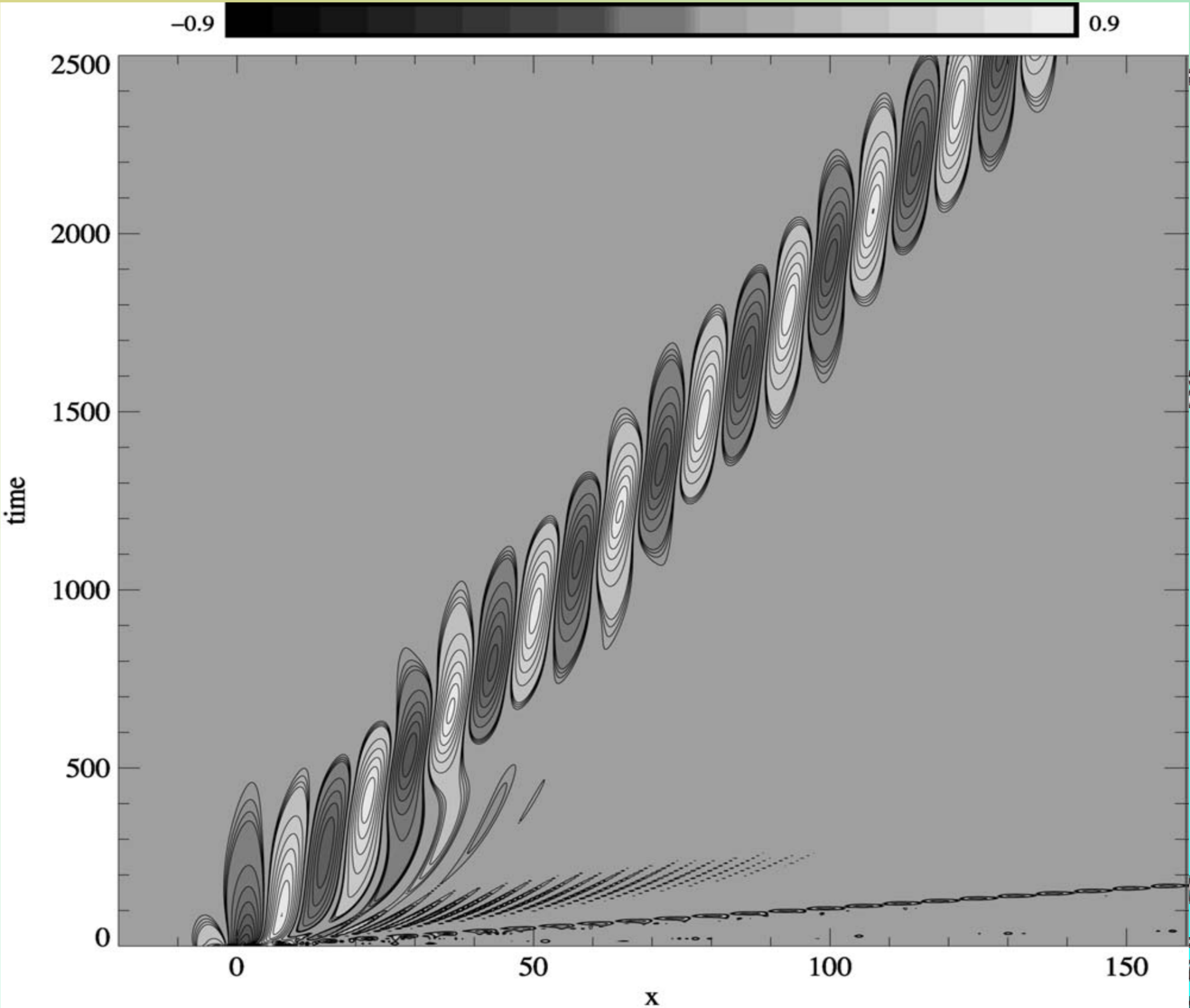
ρ_f – the density of sea water, ρ_0 – the average or characteristic density; ρ is a nondimensional quantity that has a meaning of density anomaly

Numerical (Euler Equations) modeling of breather



K. Lamb, O. Polukhina, T. Talipova, E. Pelinovsky, W. Xiao, A. Kurkin.

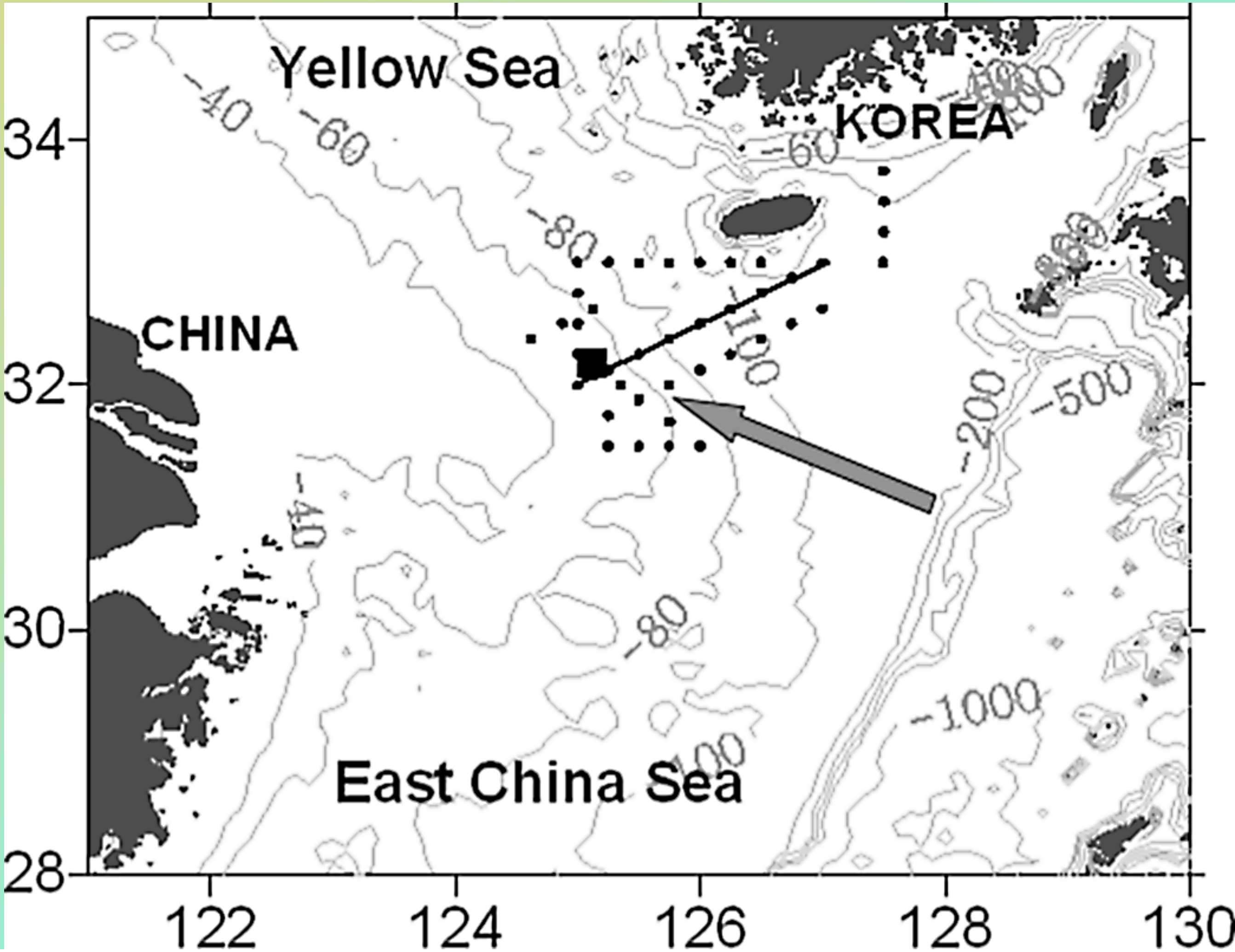
Breather Generation in the Fully Nonlinear Models of a Stratified Fluid. *Physical Rev. E*. 2007, 75, 4, 046306

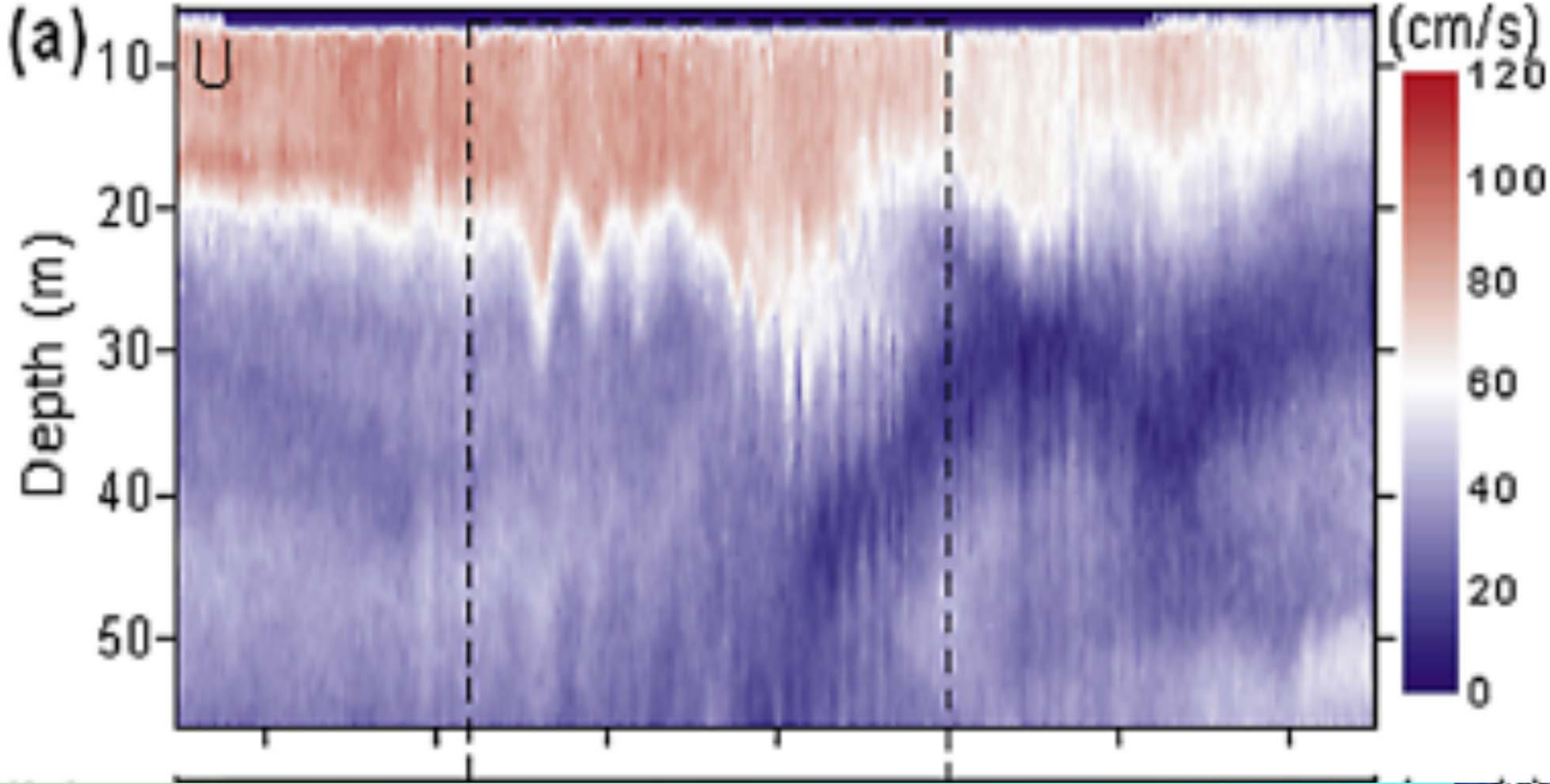


−0.9
0.9
2500
2000
1500
1000
500
0
time
0
50
100
150
x

Motivation

Lee, Lozovatsky et al., 2006





(a)
U
Depth (m)
10
20
30
40
50
(cm/s)
120
100
80
60
40
20
0

Alfred Osborn
"Nonlinear Ocean Waves & the Inverse Scattering Transform", 2010

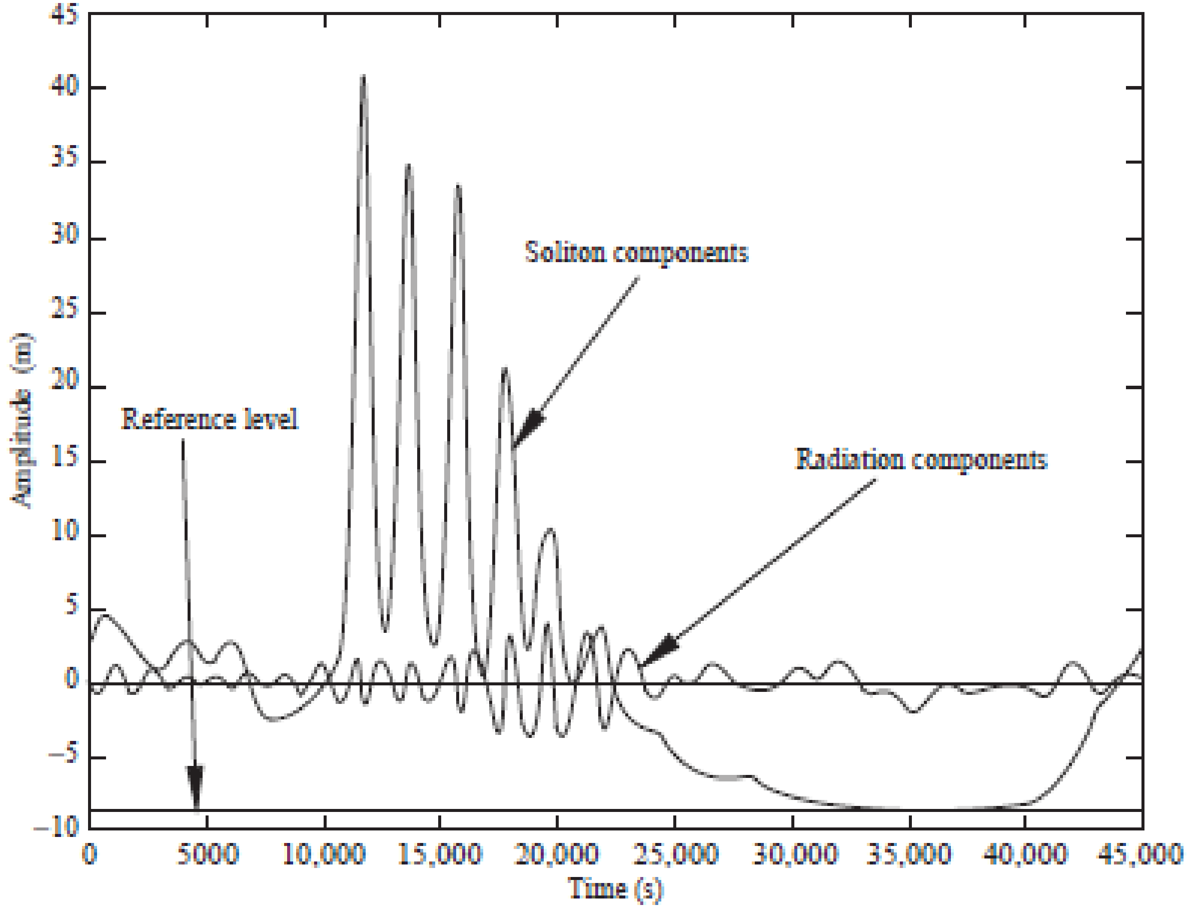
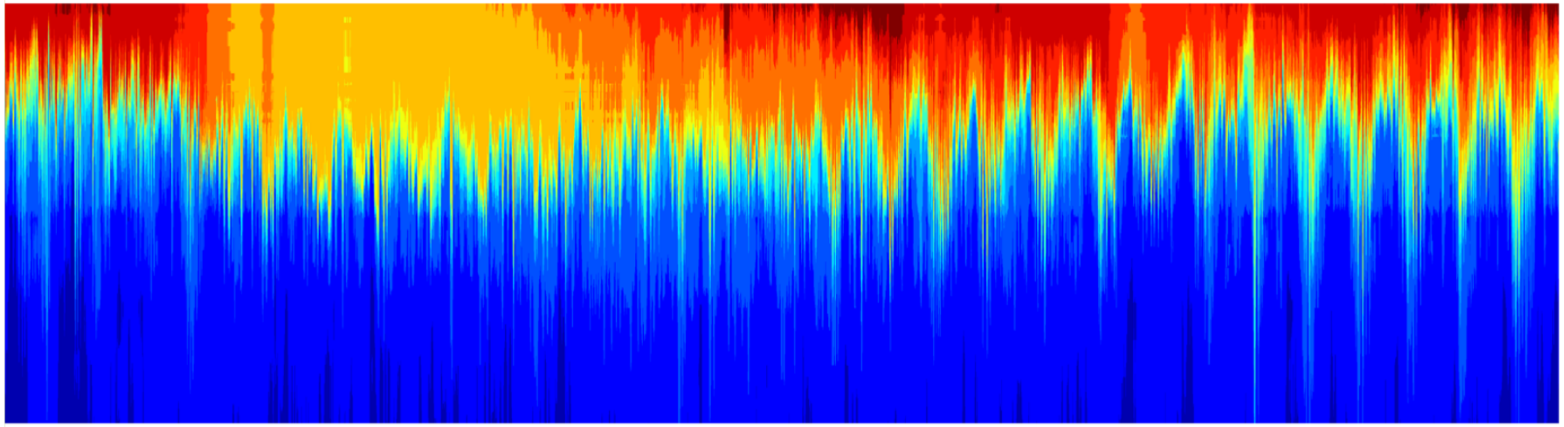
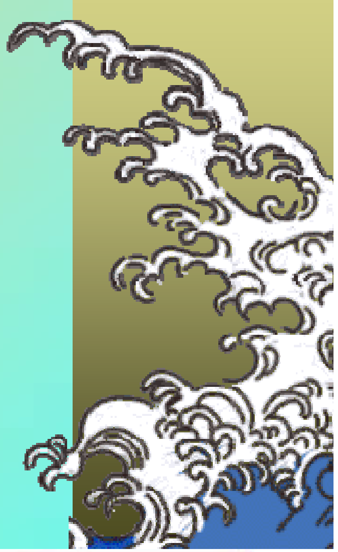


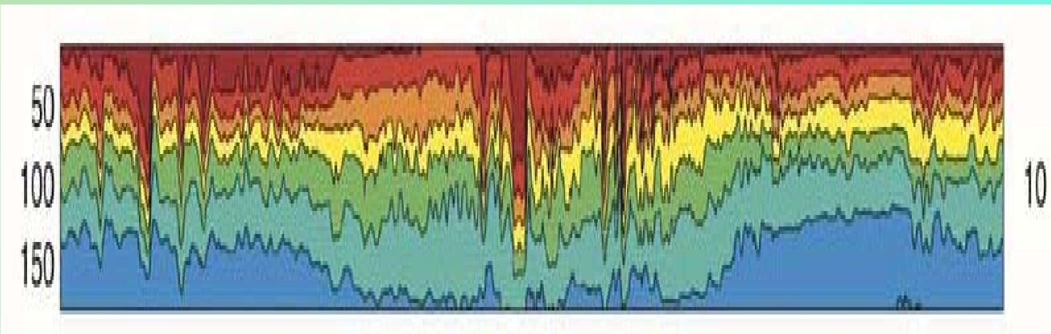
Figure 25.17 The time series of Andaman Sea data in Figure 25.9 has been filtered into soliton and radiation parts.

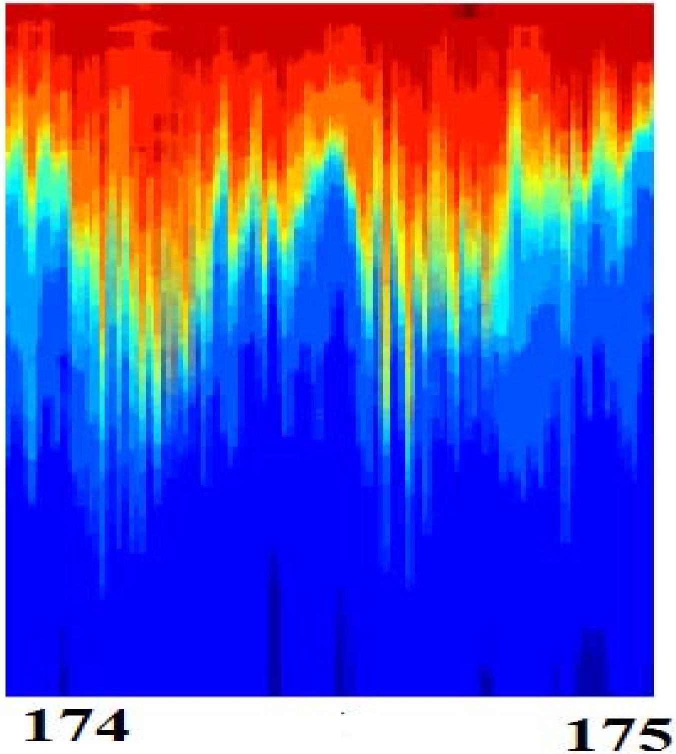
Celtic Sea Vlasenko, Stashchuk



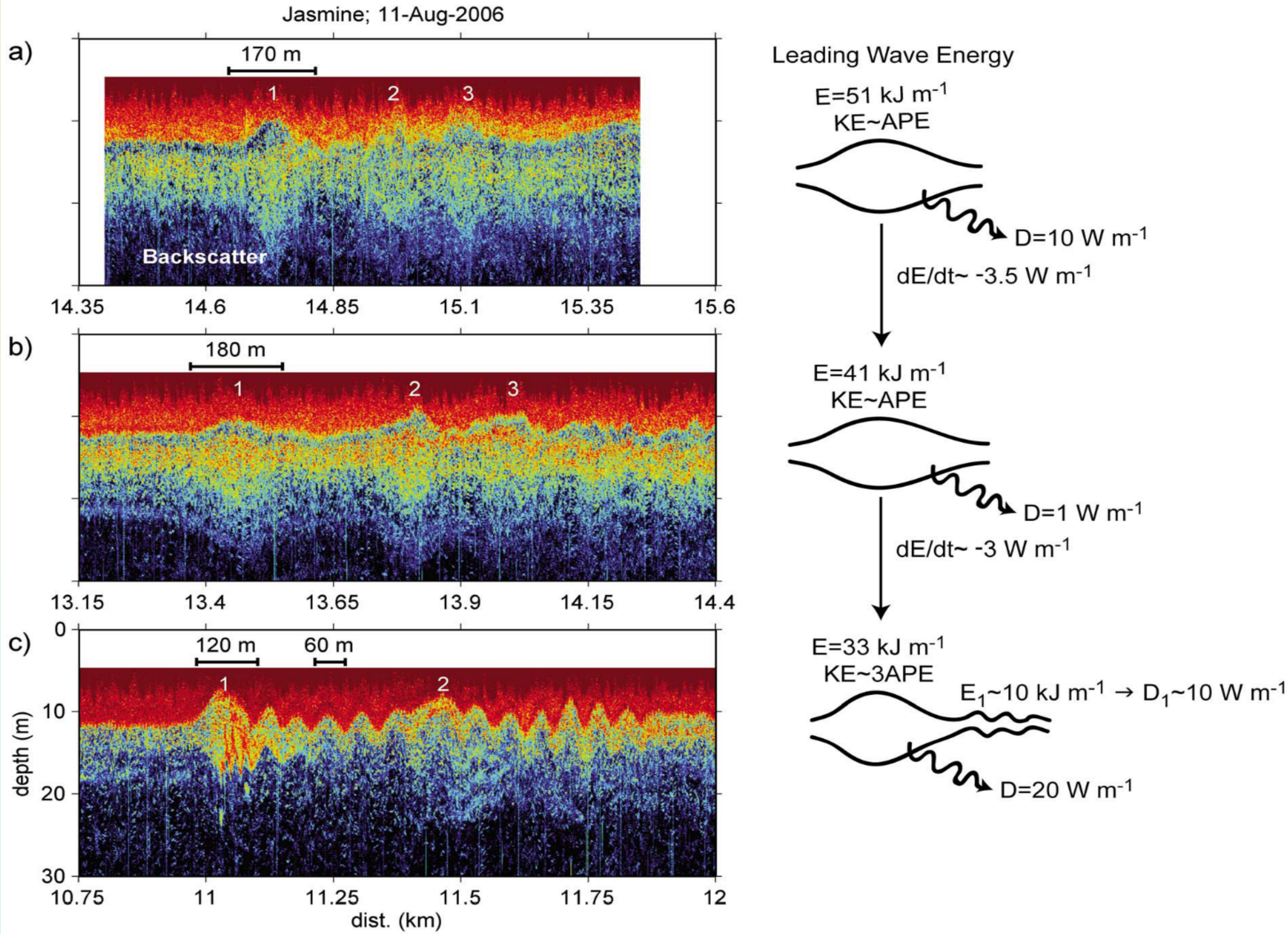


The stratification at the point of observation allows the breather formation





E. L. Shroyer, J. N. Moum and J. D. Nash
JGR, V. 115, C07001, 2010



Inhomogeneous media

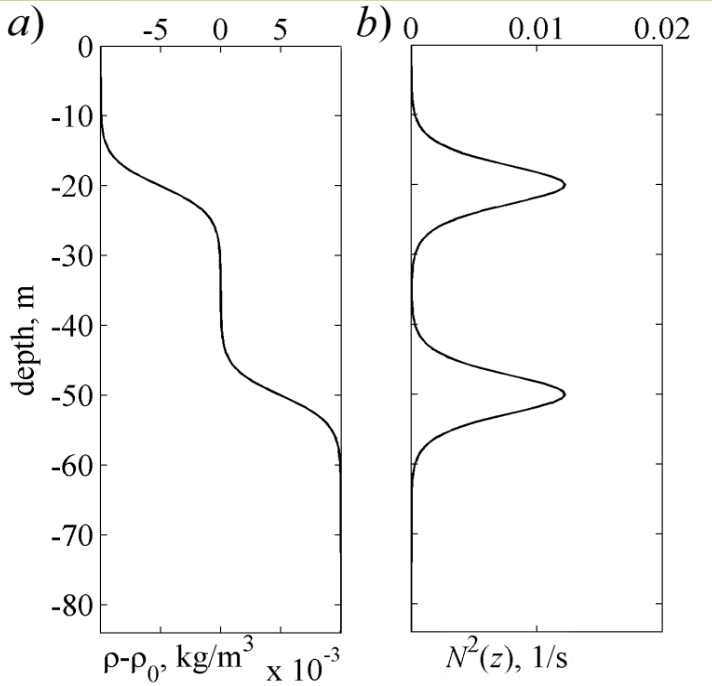
$$\frac{\partial \varsigma}{\partial x} + \left(\frac{\alpha Q}{c^2} \varsigma + \frac{\alpha_1 Q^2}{c^2} \varsigma^2 \right) \frac{\partial \varsigma}{\partial s} + \frac{\beta}{c^4} \frac{\partial^3 \varsigma}{\partial s^3} = \frac{f^2}{2c} \int \varsigma \, ds$$

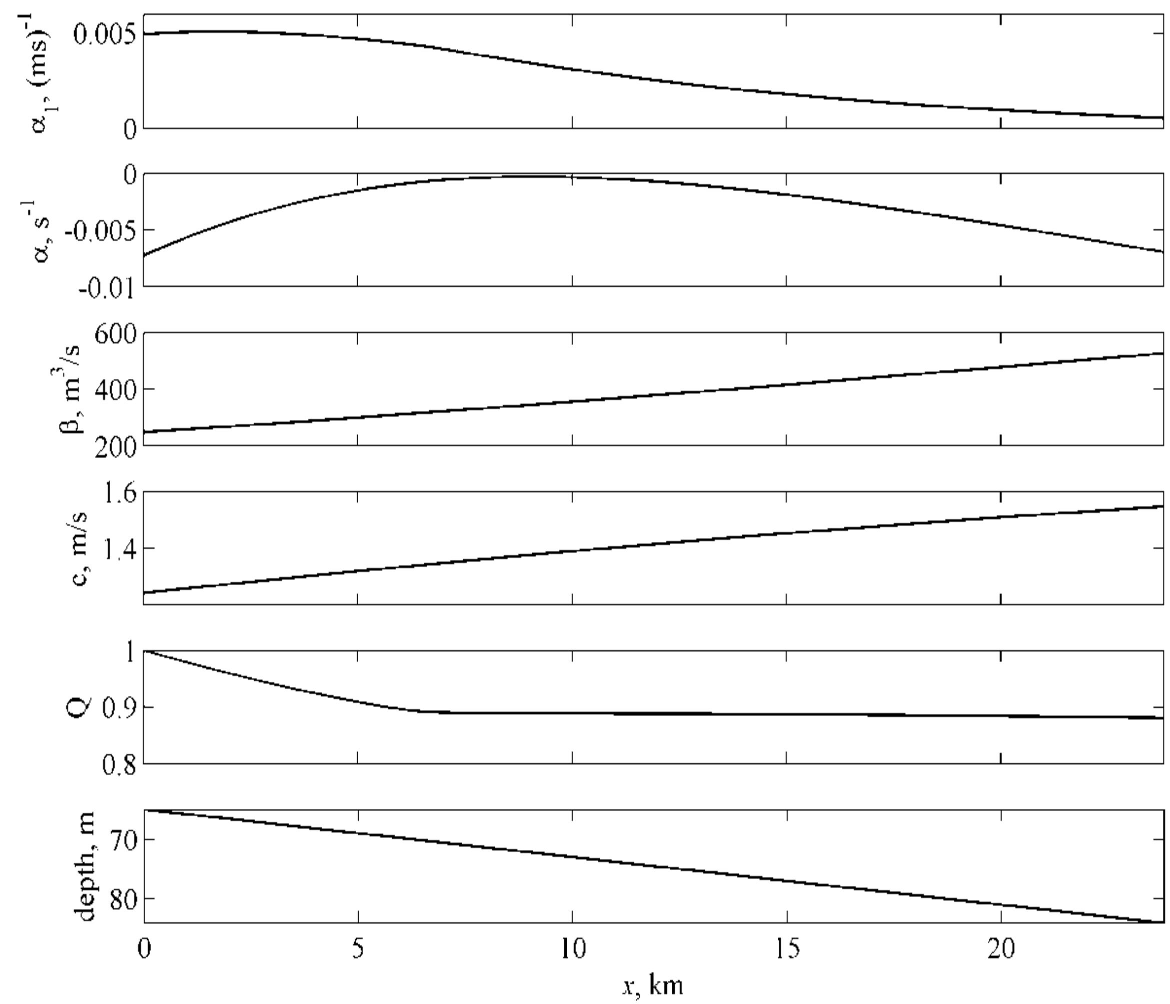
$$s = \int \frac{dx}{c(x)} - t \qquad \varsigma(x, s) = \frac{\eta(x, t)}{Q(x)}$$

f **= (4π/T_e)sinφ, T_e is the Earth's rotation period 24 hr and φ is the geographical latitude**

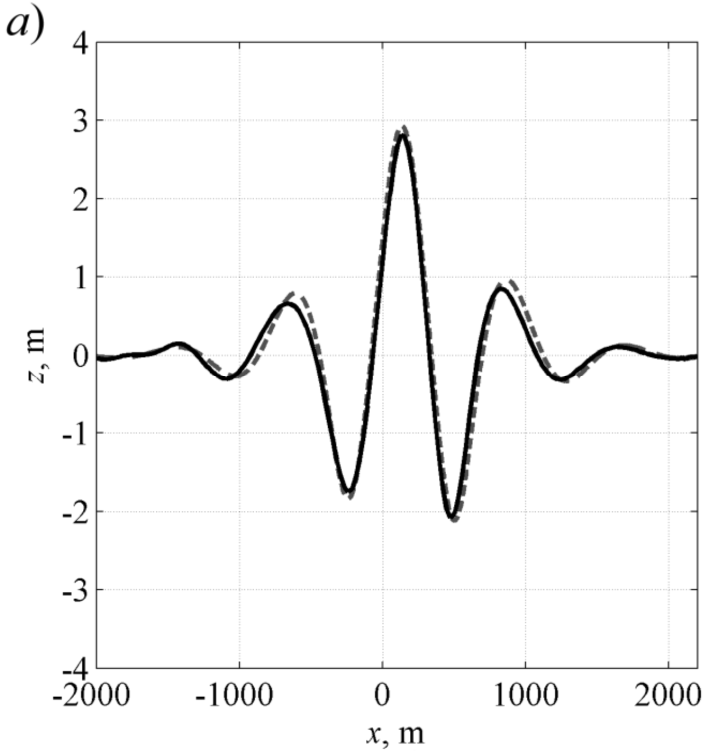
$$Q = \sqrt{\frac{c_0^3}{c^3} \frac{\int (d\Phi_0 / dz)^2 \, dz}{\int (d\Phi / dz)^2 \, dz}}$$

Transformation of breathers on a sloping bottom

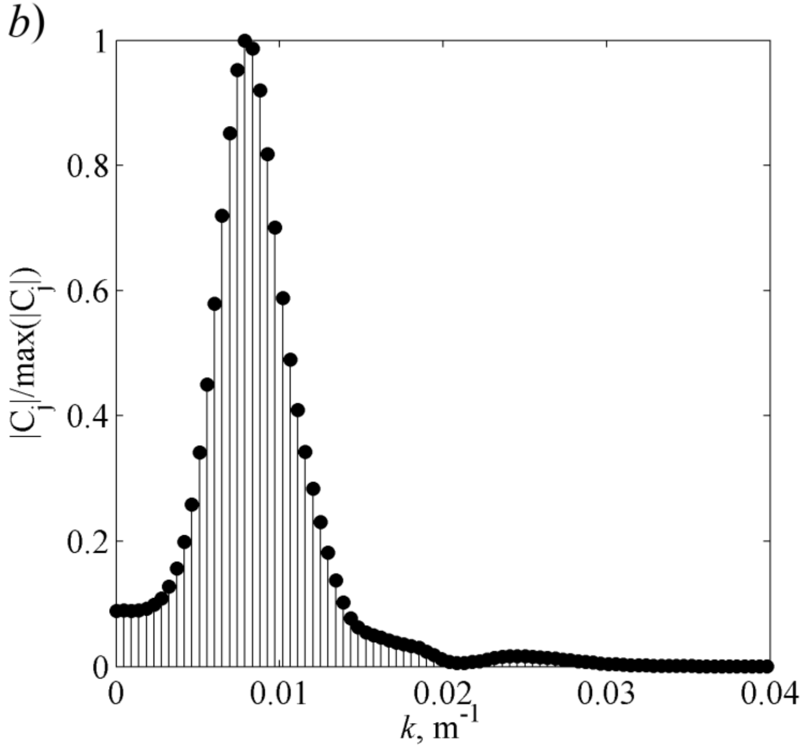




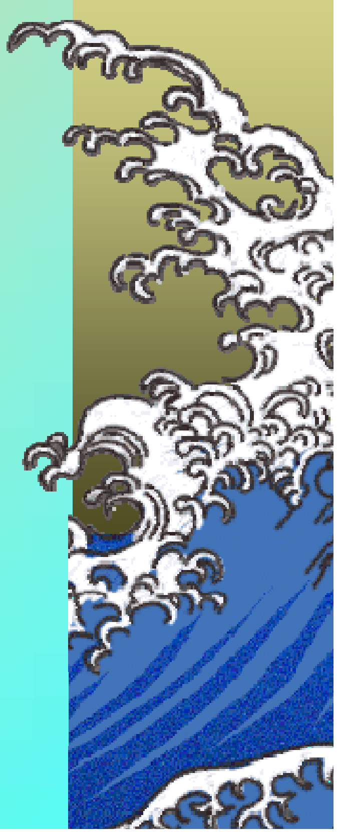
Breather A

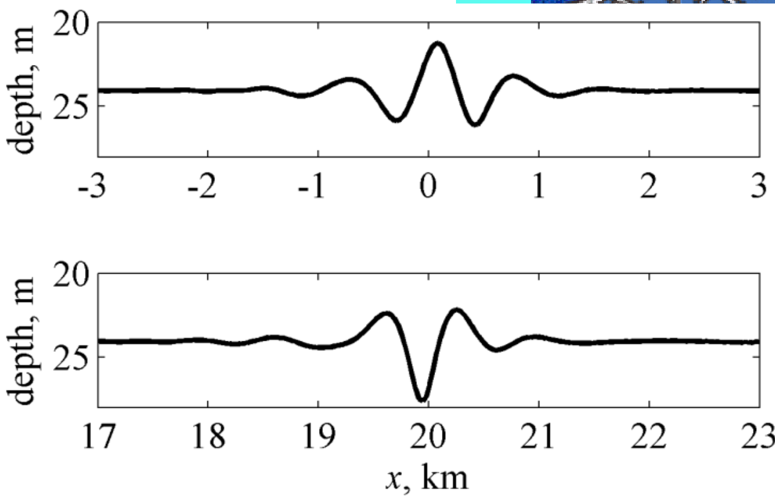


a)
z, m
x, m
4
3
2
1
0
-1
-2
-3
-4
-2000
-1000
0
1000
2000

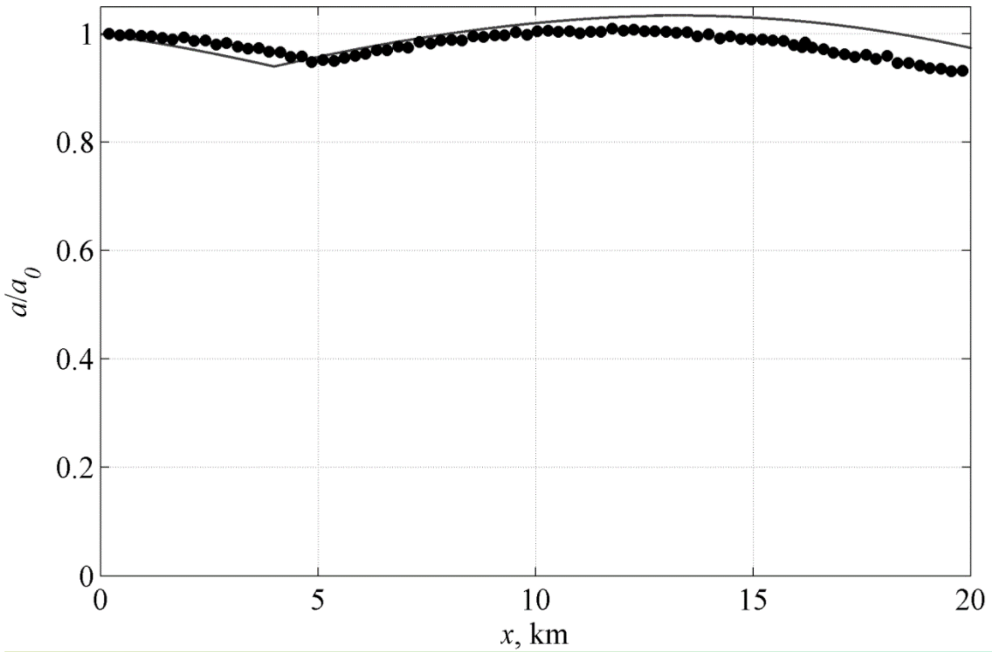


b)
|C_j|/max(|C_j|)
k, m^-1
1
0.8
0.6
0.4
0.2
0
0.01
0.02
0.03
0.04

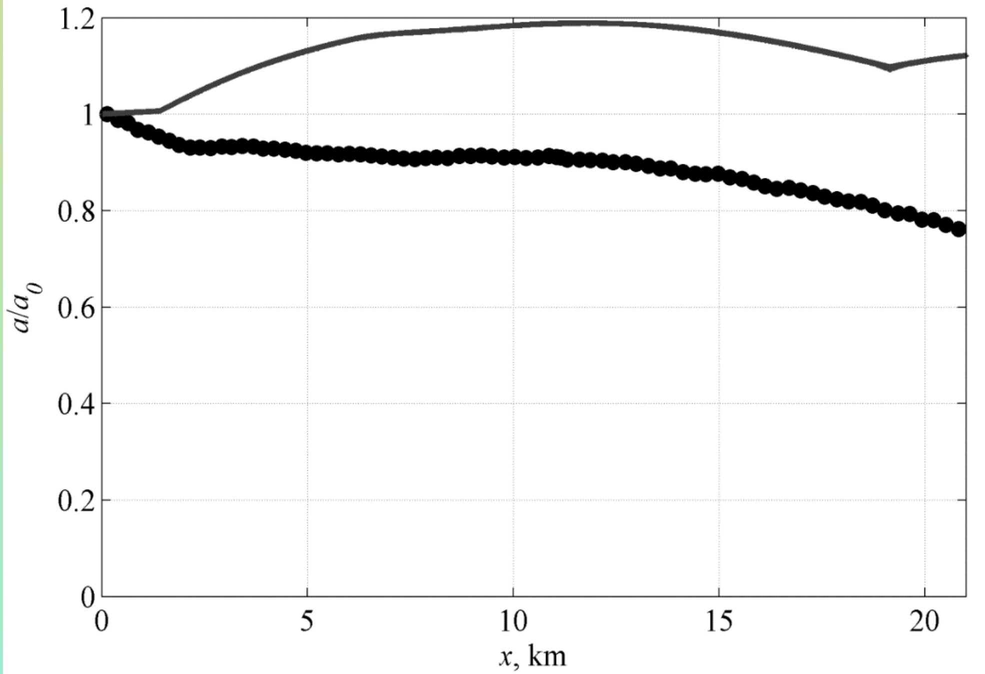




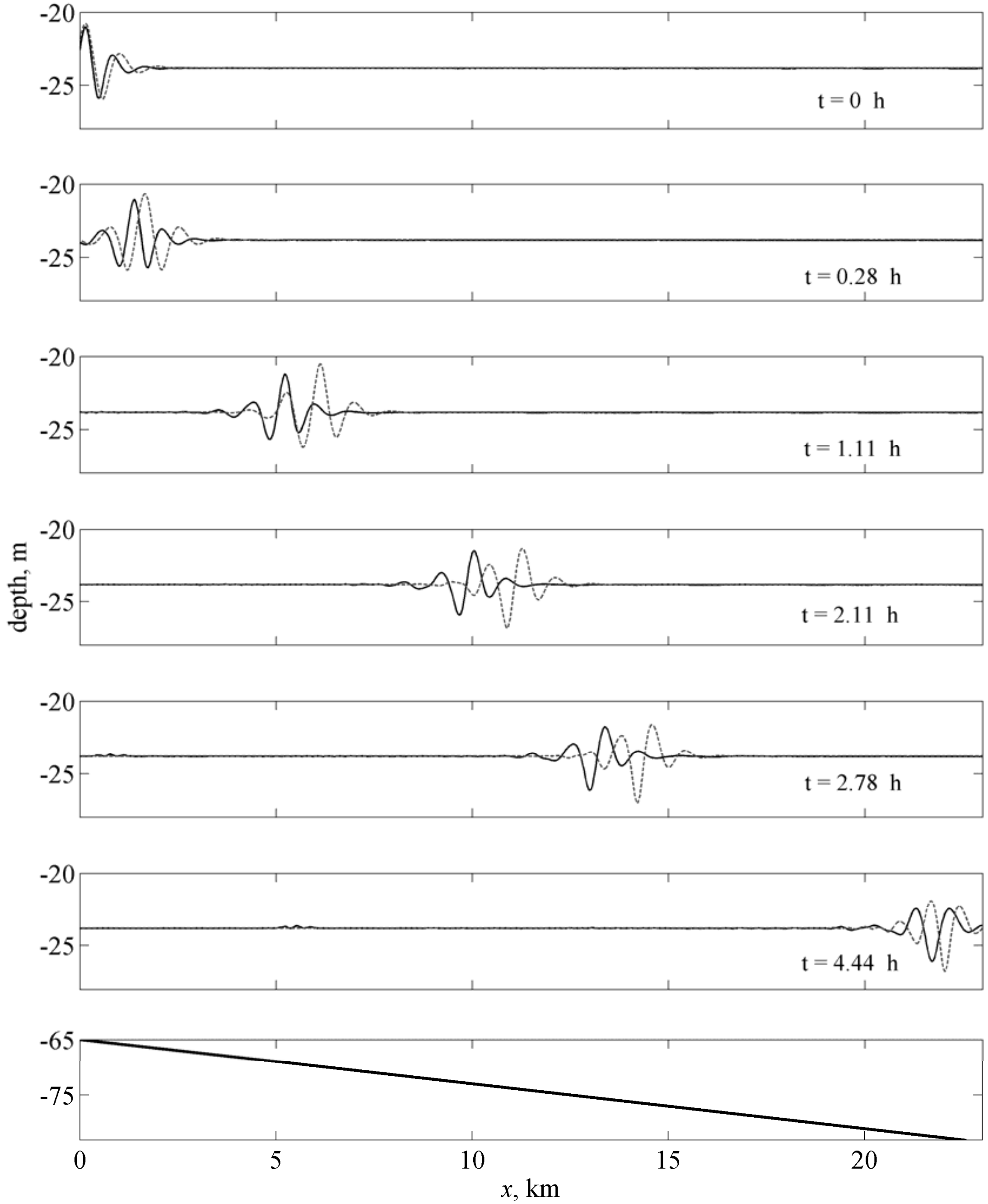
depth, m
20
25
-3
-2
-1
0
1
2
3
depth, m
20
25
17
18
19
20
21
22
23
x, km



Amplitudes of Euler (filled circles) and Gardner (solid line) breathers propagating in a non-rotating basin of constant depth.



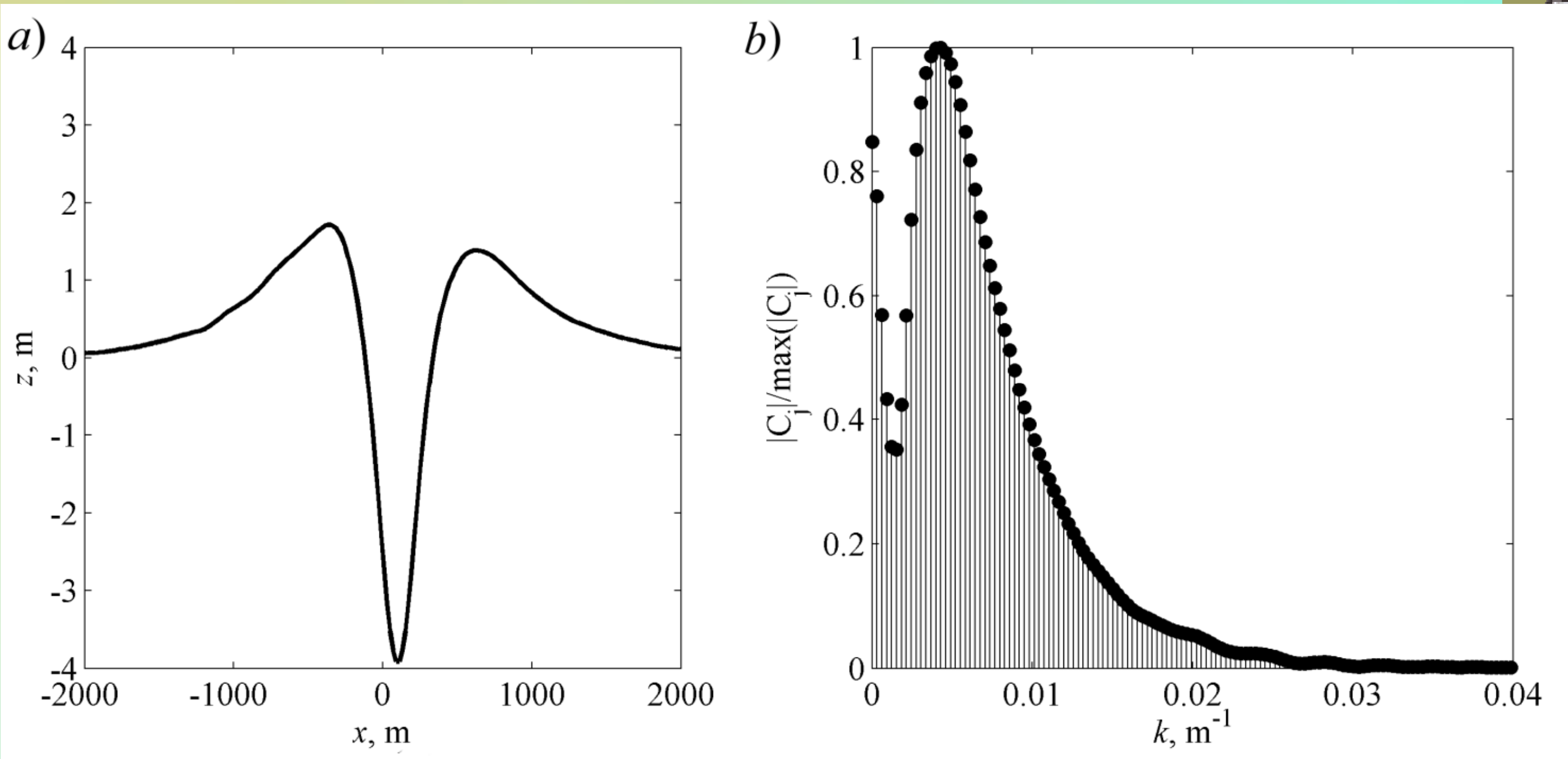
Normalised amplitudes of Euler (filled circles) and Gardner (solid line) breathers propagating over the inclined bottom with a slope of 0.001 in a non-rotating basin.



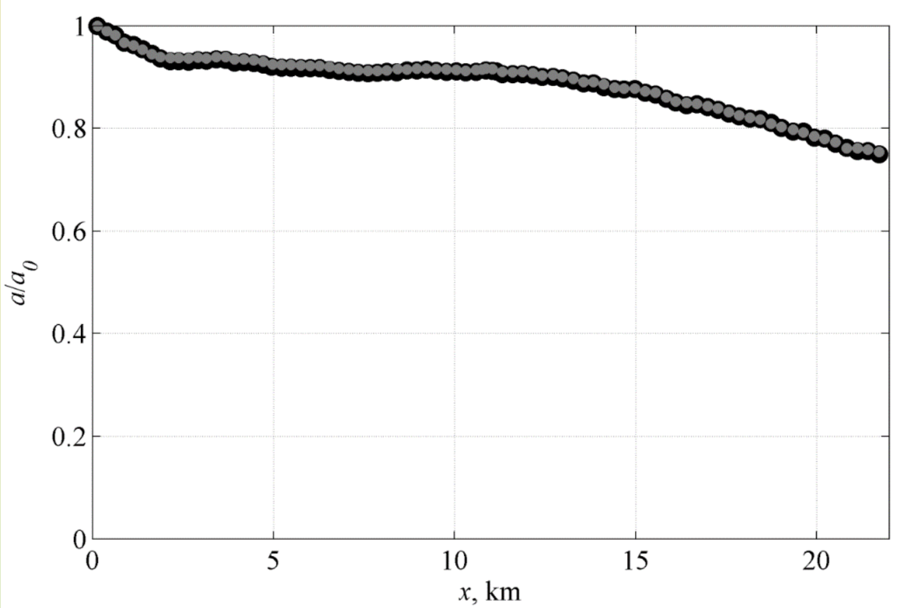
Transformation of breathers on a sloping bottom

Snapshots of the propagation of Euler (solid line) and Gardner (dashed line) breathers over the inclined bottom with a slope of 0.001 in a non-rotating numerical basin of variable depth (depth variations in the bottom panel).

Breather B

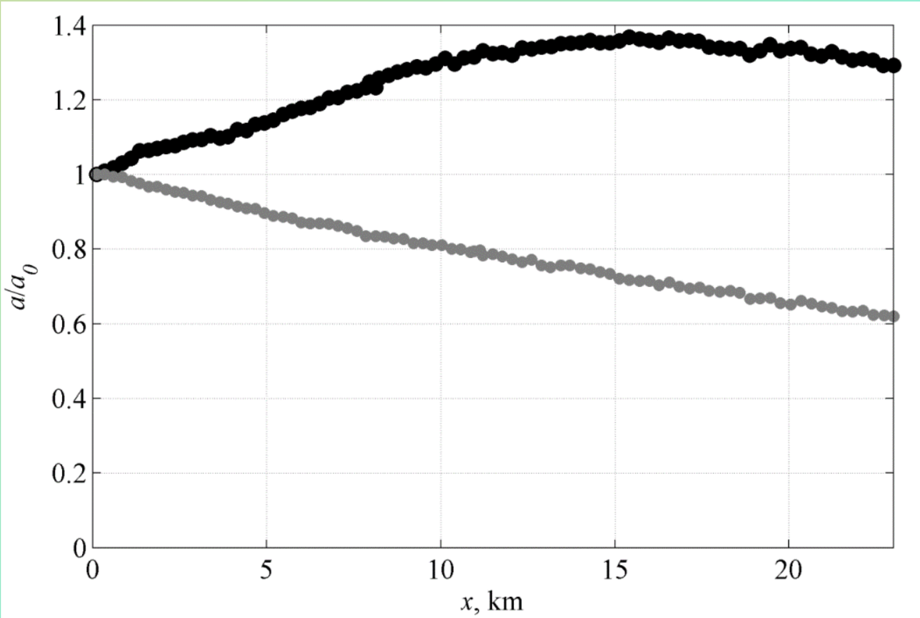


Slope + rotation



$f = 0.00012\ s^{-1}$. This frequency corresponds to the geographical latitude of $\varphi = 56°N$.

Normalised amplitudes of breather A with a narrow amplitude spectrum in fully nonlinear simulations using Euler equations for a rotating (gray circles) and non-rotating (black circles). The initial amplitude was 2.5 m in both cases



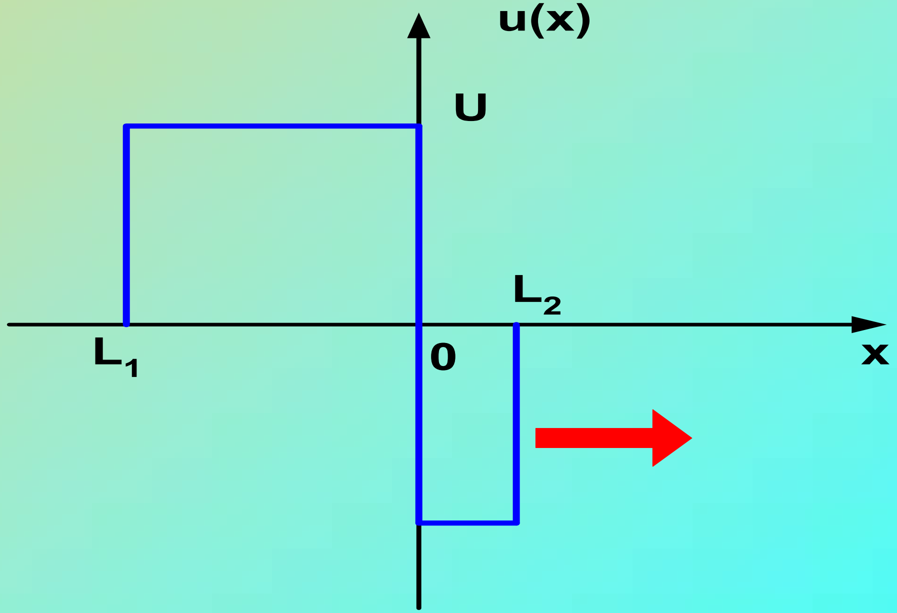
Normalised amplitudes of breather B with a wide amplitude spectrum in fully nonlinear simulations using Euler equations for a rotating (gray circles) and non-rotating (black circles) situation. The initial amplitude was

2.85 m in both cases.

How to generate the breather?

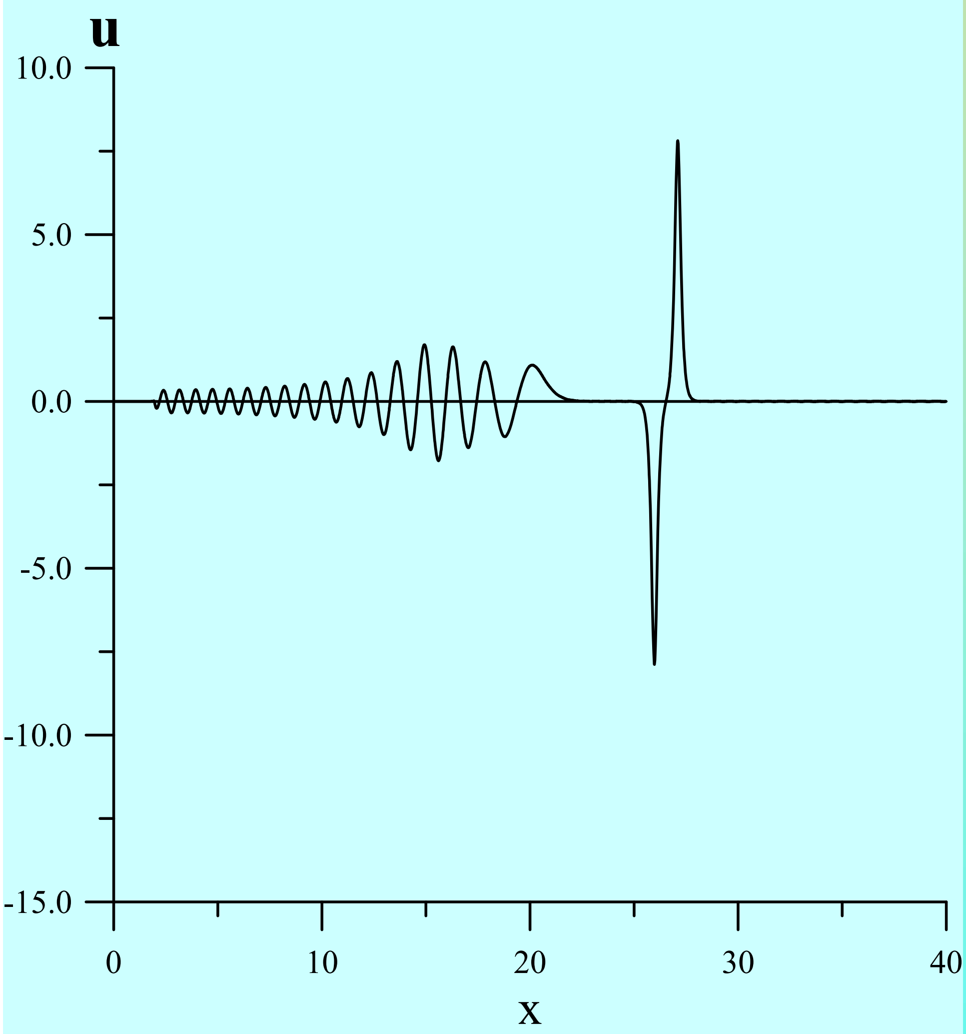
Clarke S., Grimshaw R., Miller P., Pelinovsky E., Talipova T. On the generation of solitons and breathers in the Modified Korteweg - de Vries Equation. 2000, *Chaos*, V. 10, No. 2, 383-392

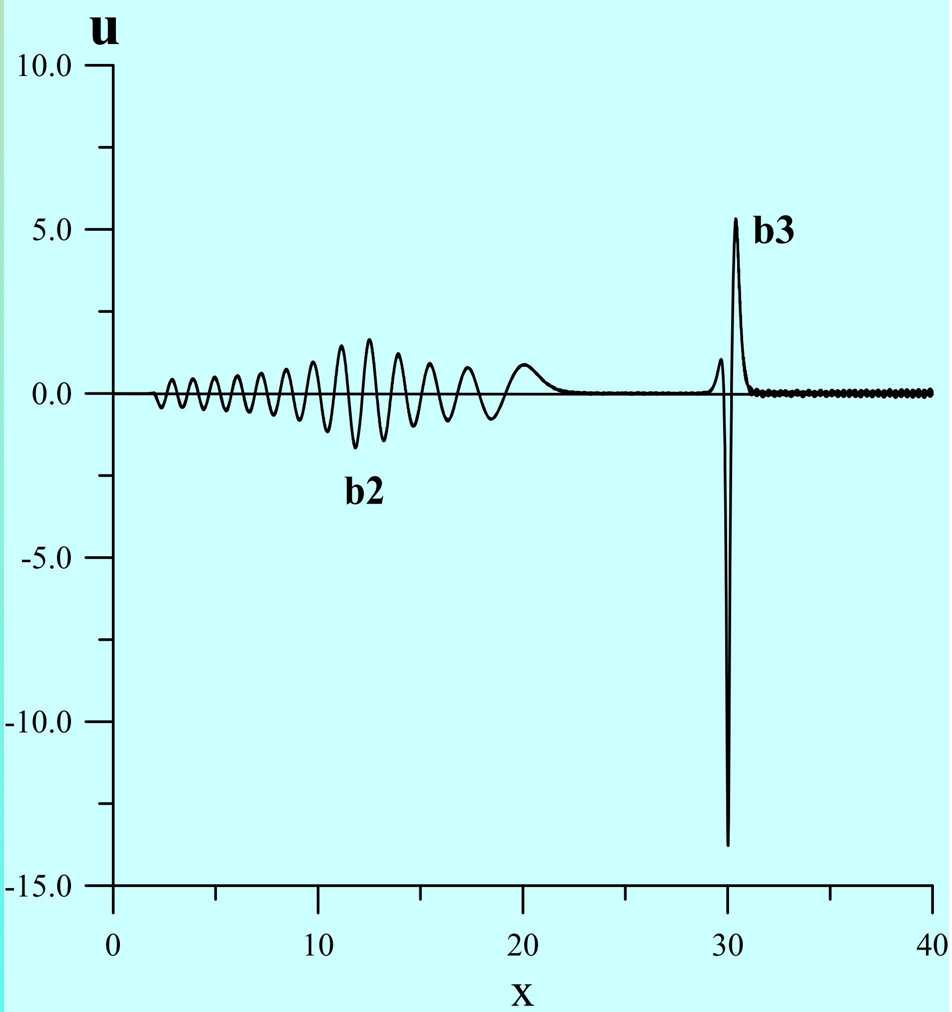
Initial perturbation for the mKdV equation



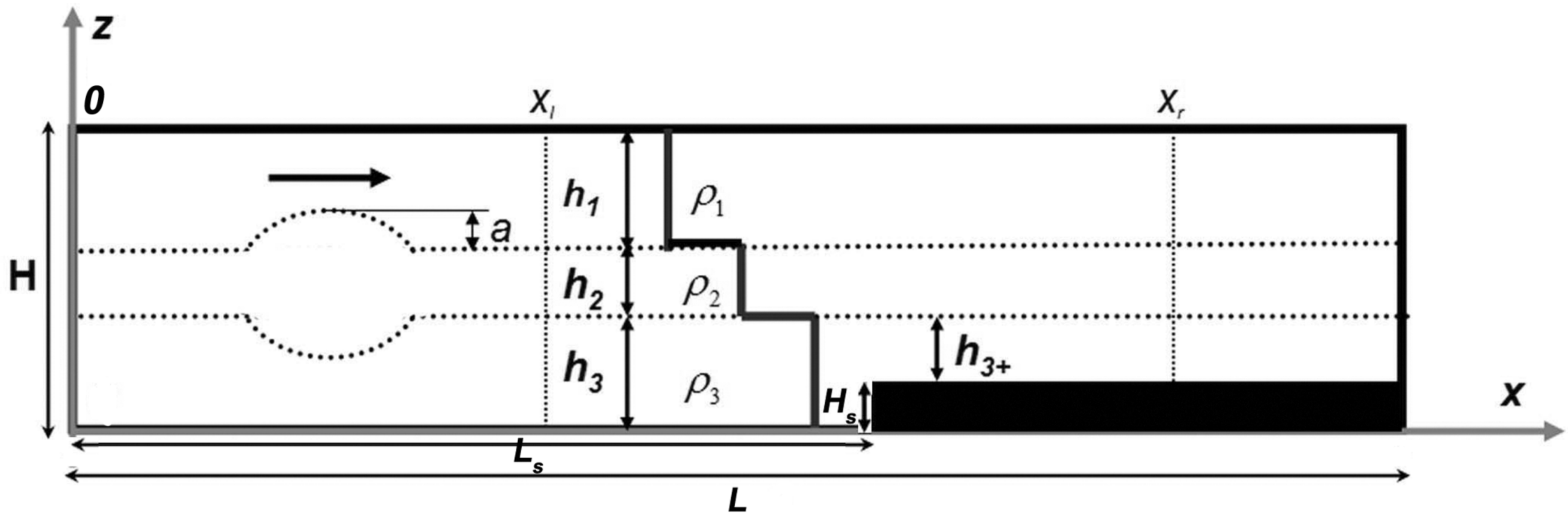
Numerical modelingin mKdV

$U_1 = -U_2 = U = 5, L_1 = L_2$

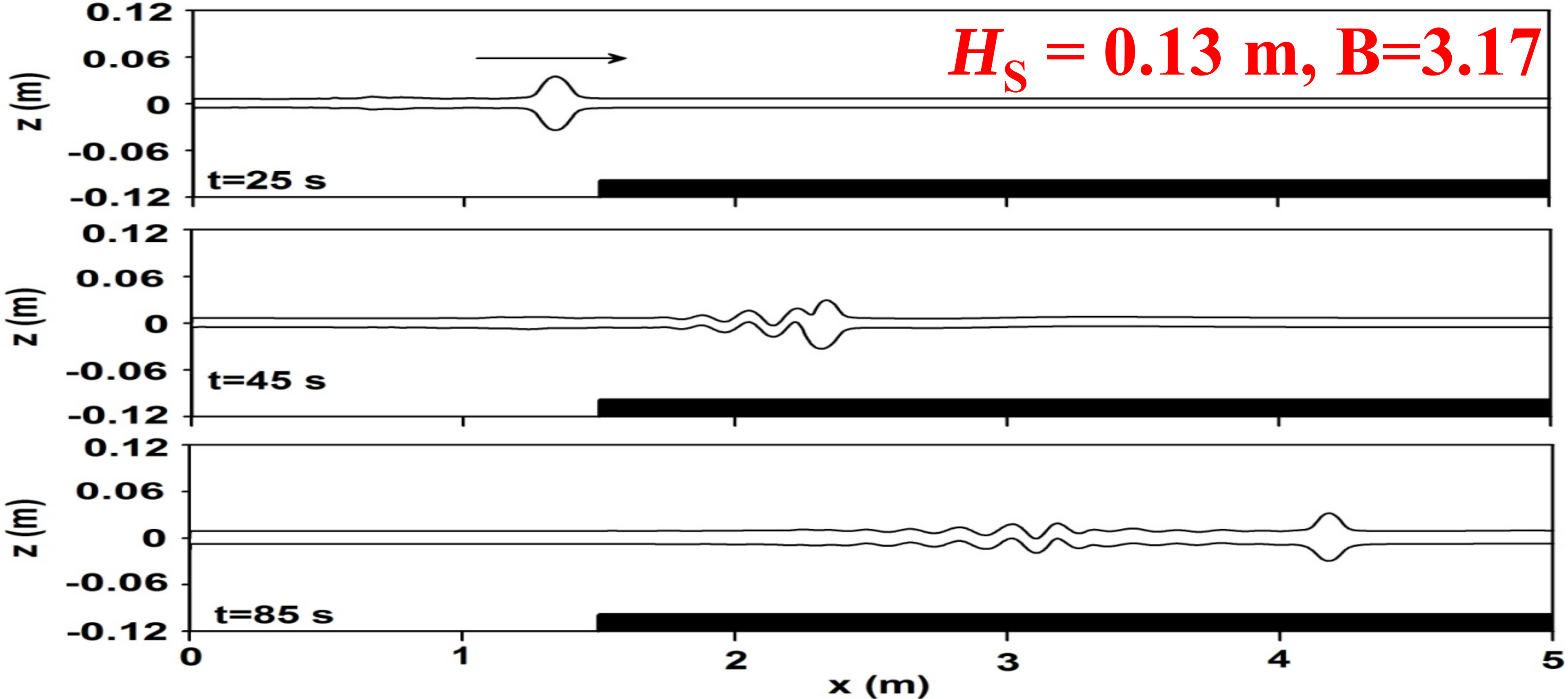




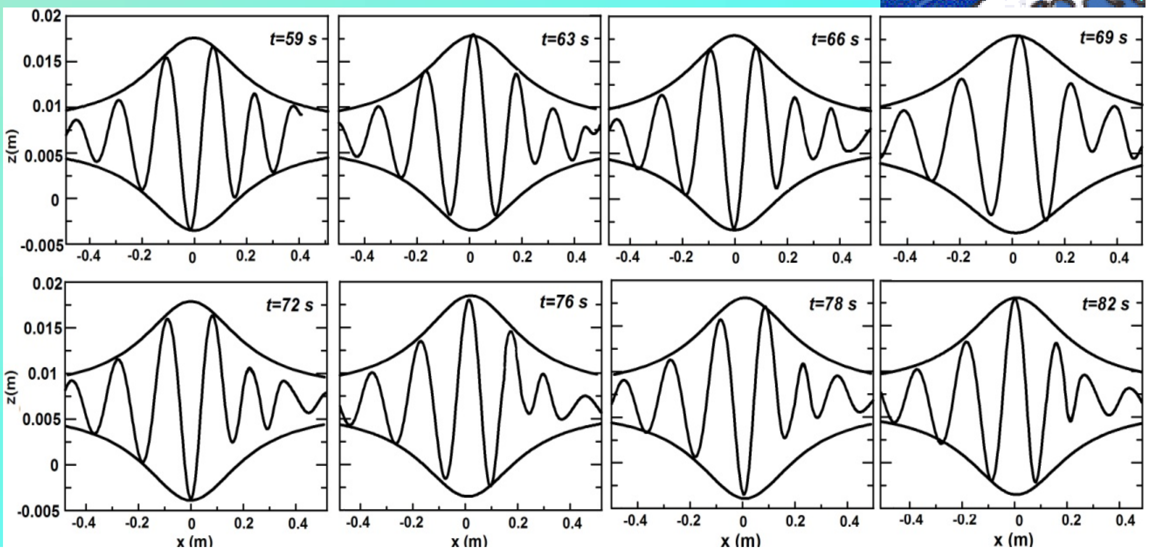
Intermediate wave length
Numerical tank



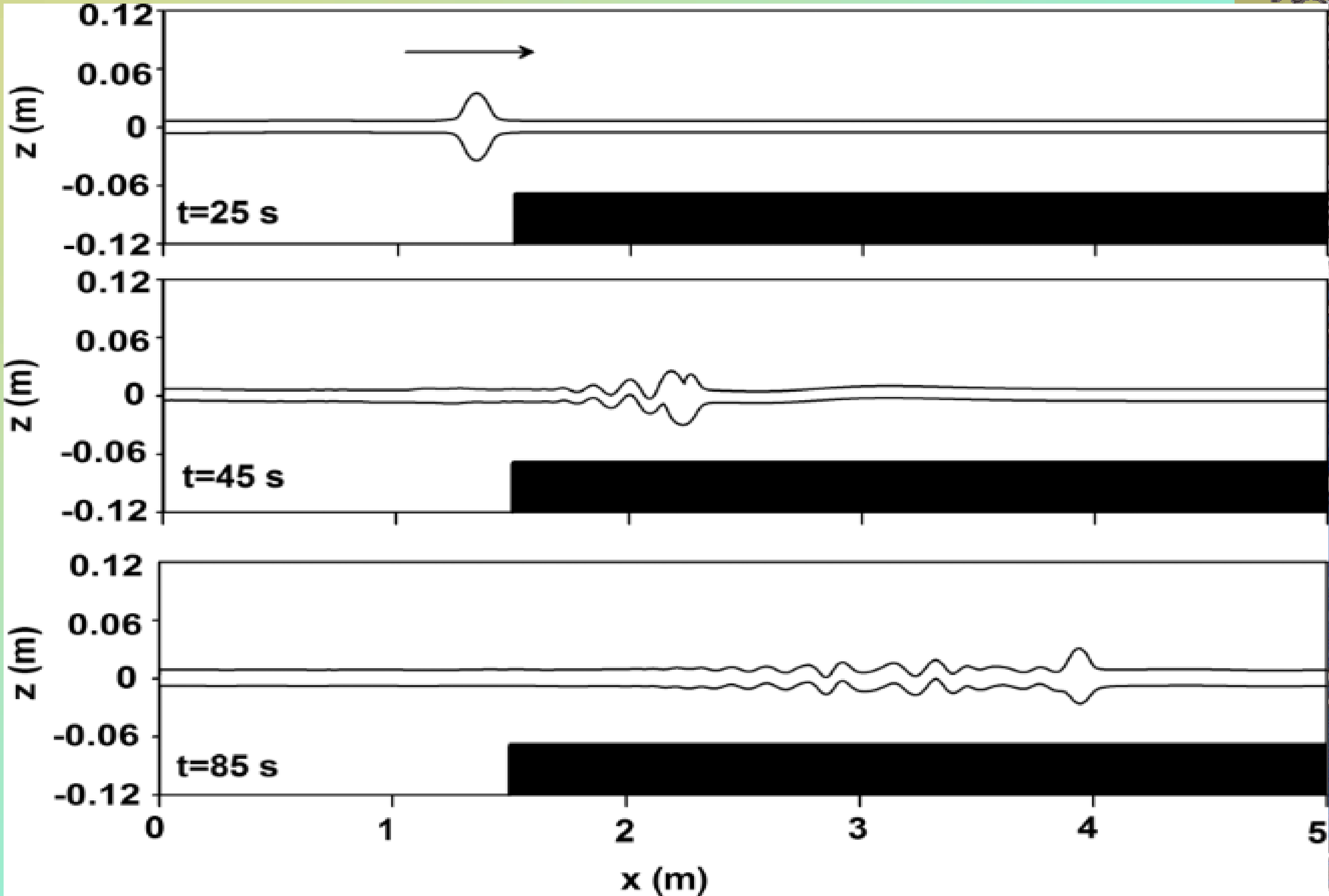
Blocking parameter $B = h_{3+}/|a_i|$

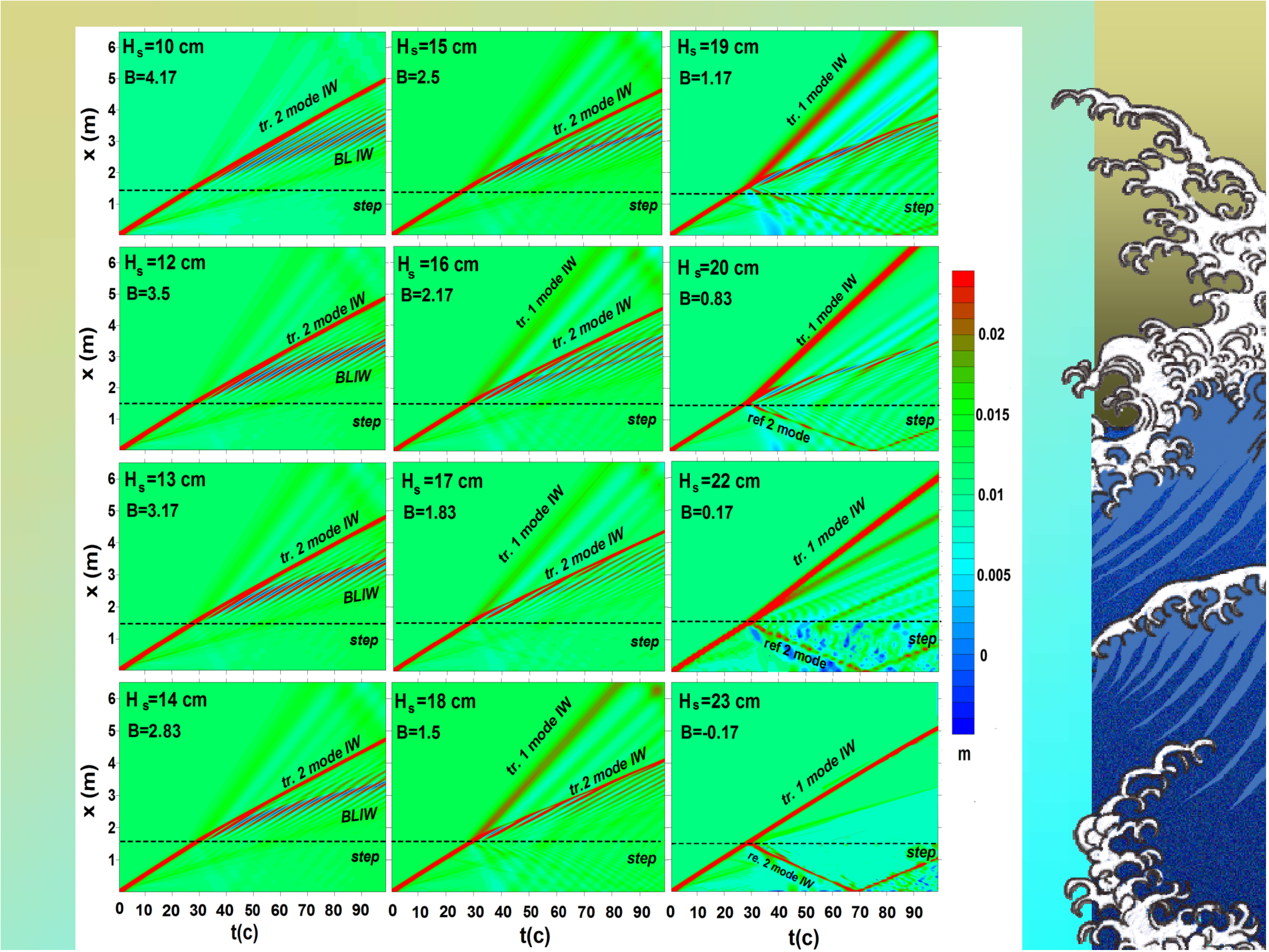


Vertical displacement of upper interface in different time moments. The envelopes obtained using Hilbert transformation.



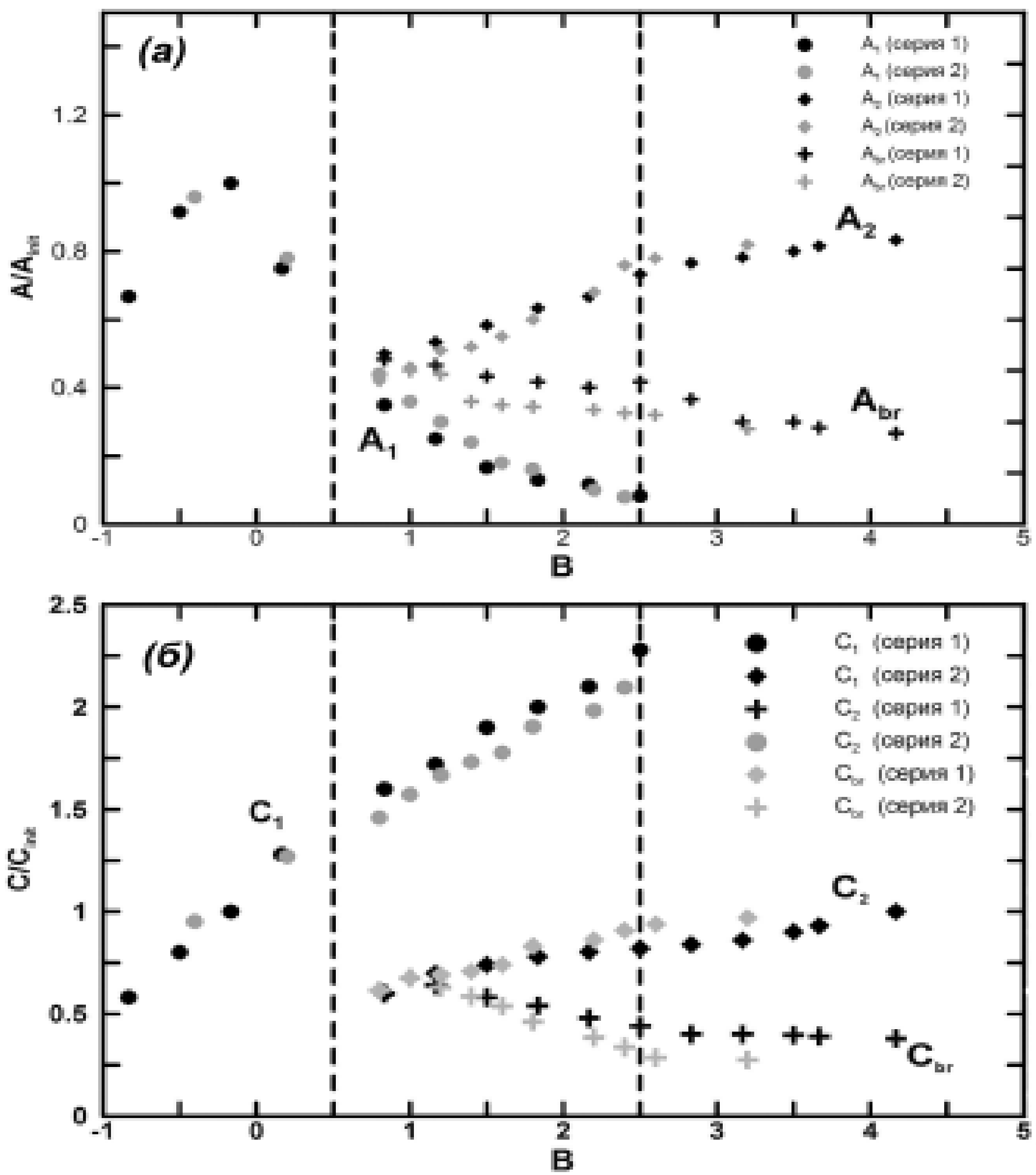
$H_S = 0.16$ m, B=2.17





H_s=10 cm B=4.17 tr. 2 mode IW BL IW step
H_s=15 cm B=2.5 tr. 2 mode IW step
H_s=19 cm B=1.17 tr. 1 mode IW step
H_s=12 cm B=3.5 tr. 2 mode IW BLIW step
H_s=16 cm B=2.17 tr. 1 mode IW tr. 2 mode IW step
H_s=20 cm B=0.83 tr. 1 mode IW ref 2 mode step
H_s=13 cm B=3.17 tr. 2 mode IW BLIW step
H_s=17 cm B=1.83 tr. 1 mode IW tr. 2 mode IW step
H_s=22 cm B=0.17 tr. 1 mode IW ref 2 mode step
H_s=14 cm B=2.83 tr. 2 mode IW BLIW step
H_s=18 cm B=1.5 tr. 1 mode IW tr.2 mode IW step
H_s=23 cm B=-0.17 tr. 1 mode IW re. 2 mode IW step
x (m)
t(c)
0.02
0.015
0.01
0.005
0
m

The amplitudes and nonlinear group speeds for all solitary and breather-like waves of the first and the second modes

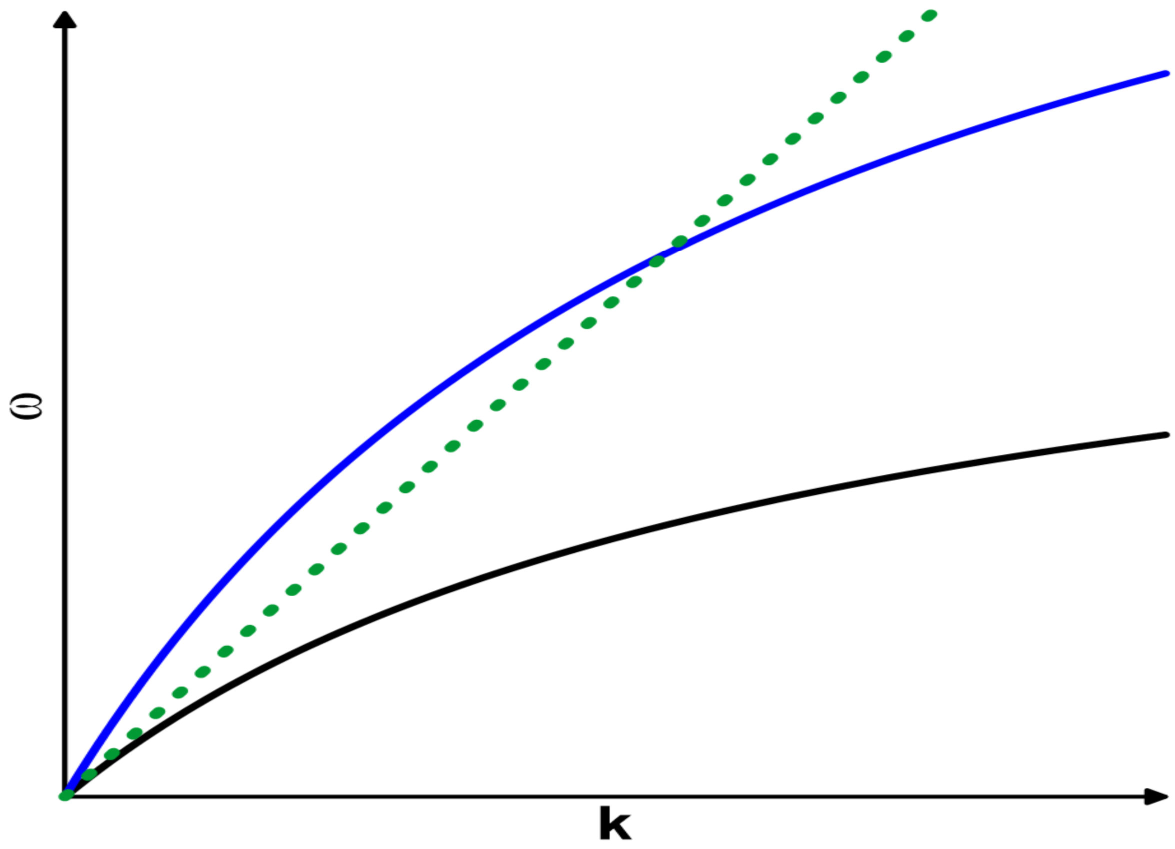


Synchronizm with linear and nonlinear waves

- - breather of the first mode dispersion relation of the first mode

—— dispersion relation of the first mode

——



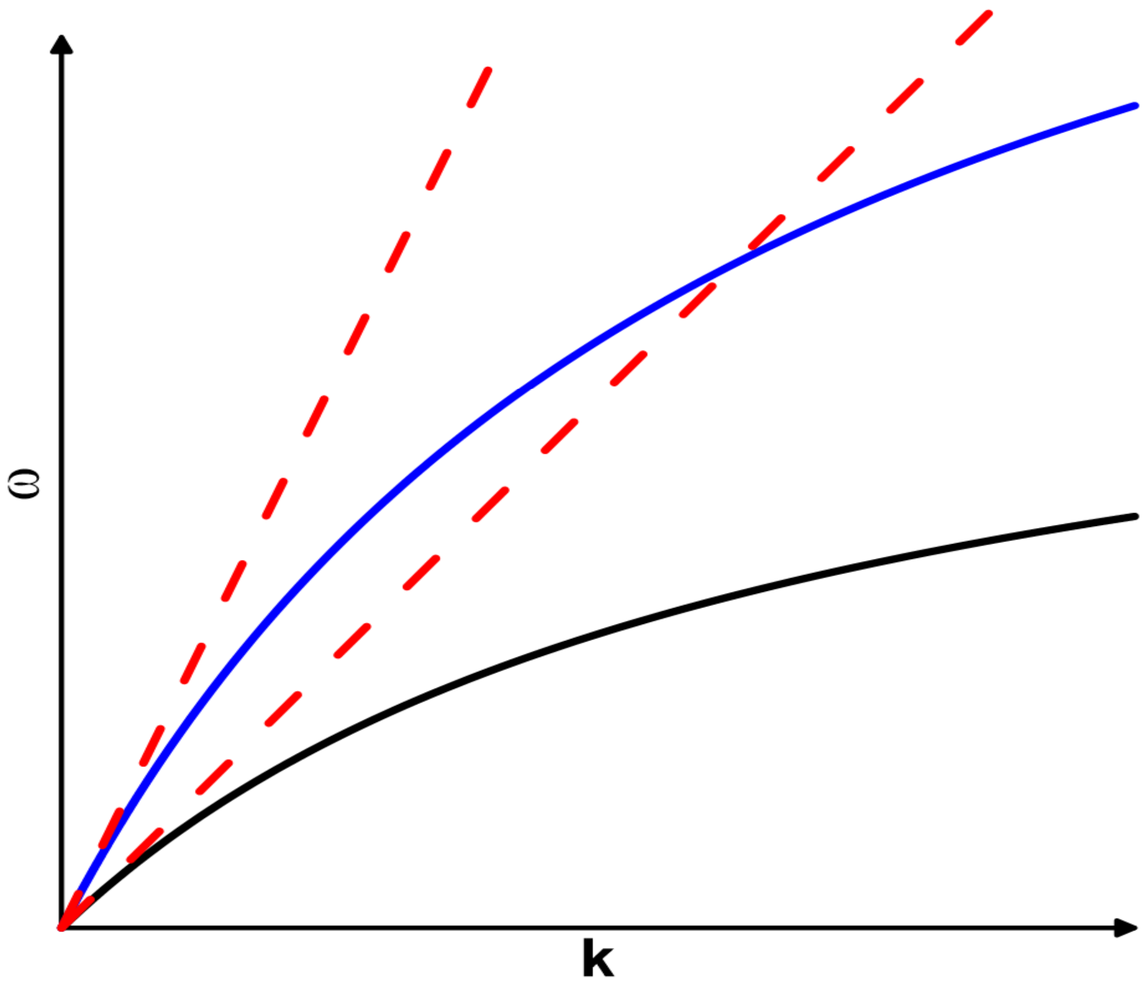
Synchronizm with linear and nonlinear waves

- - breather of the first mode

. . solitary wave of the second mode

—— dispersion relation of the first mode

—— dispersion relation of the first mode



Synchronizm with linear and nonlinear waves

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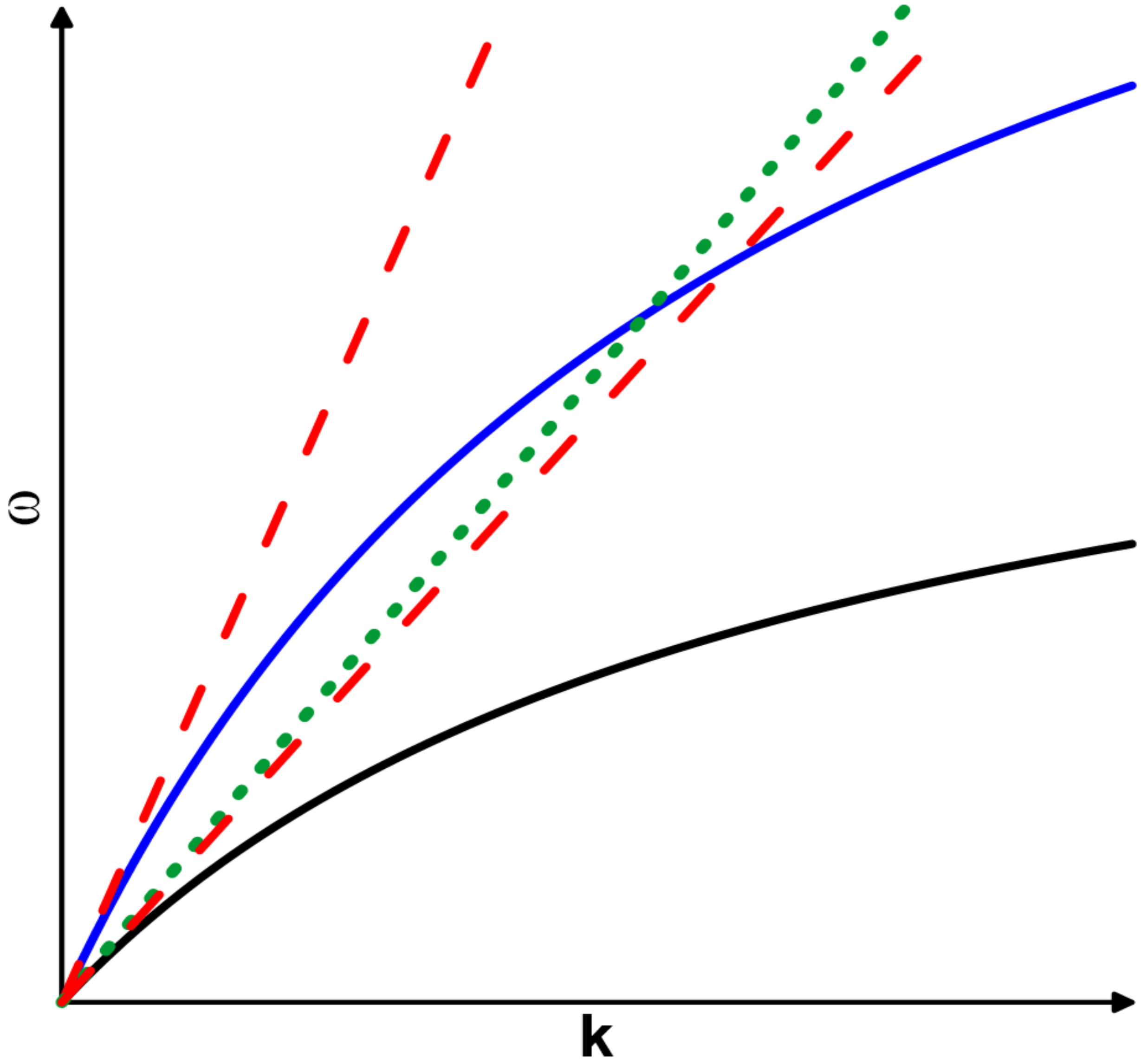
—— dispersion relation of the first mode

—— dispersion relation of the first mode



Synchronizm with linear and nonlinear waves

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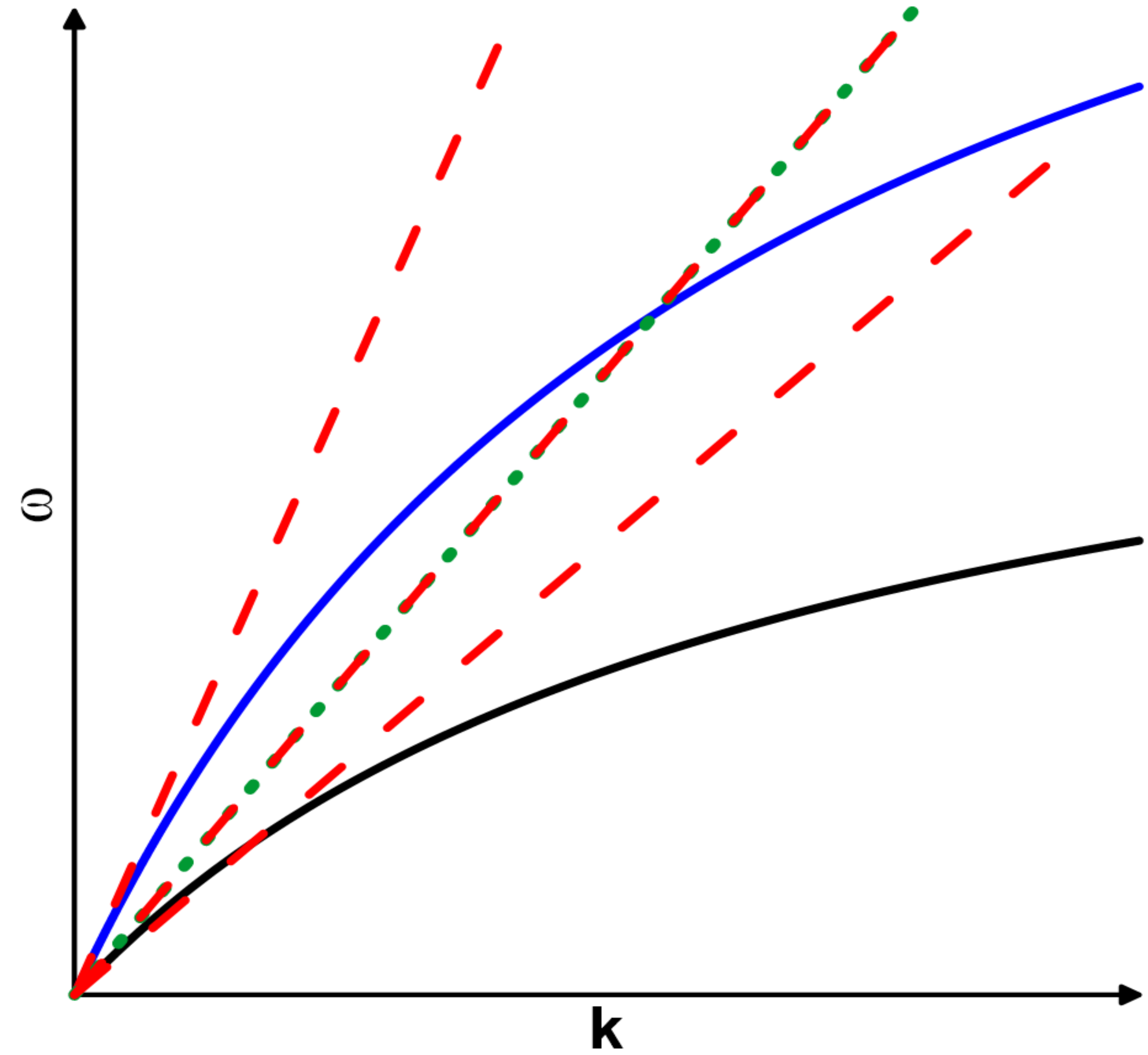
—— dispersion relation of the first mode



Conclusions

- Internal breather wave is the quasi stable-state wave what is found in the numerical experiments
- There are no laboratory experiments with breather-like waves yet
- There are some breather-like wave observations "in situ"