Nonlinear equatorial atmospheric waves coupled to moist convection

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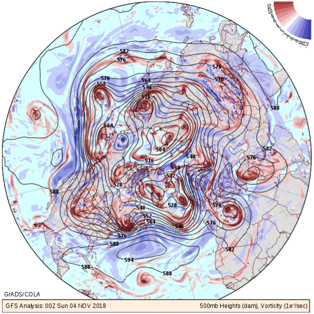
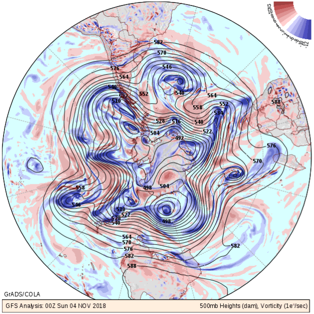
Nonlinear Waves in Oceanography and Beyond,
In honour of R. Grimshaw,
Towoomba, November 2018

Plan

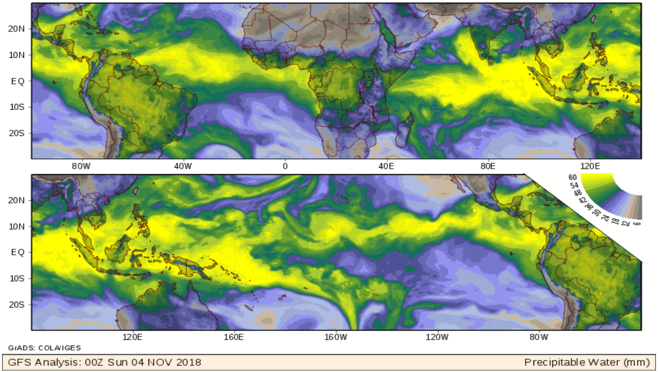
Eddy activity: midlatitude vs tropics





Much lower eddy activity in tropics

Precipitable water



Lots of humidity, clouds and precipitations.

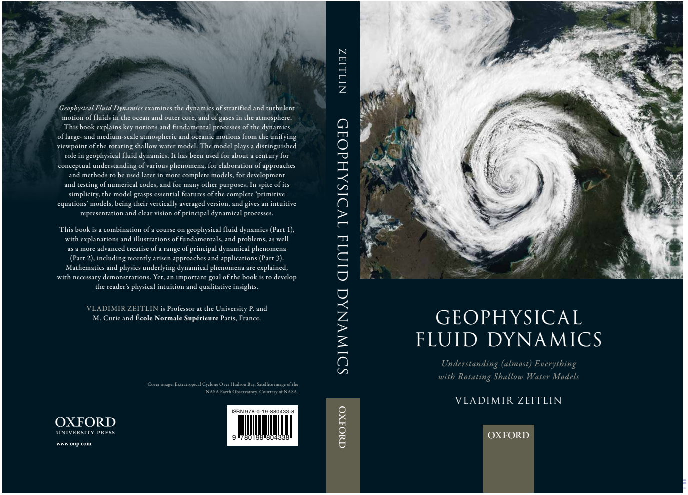
Main modelling problems:

- Enormous range of spatial and time scales to be resolved
- Huge number of parameterisations of "physics" at small spatial scales (cloud microphysics, air-sea and soil-air exchanges, small-scale turbulence etc)
- Complexity of thermodynamics of the moist air

Main idea:

Small is beautiful: full complexity of primitive equations (and high-resolution DNS of prohibitive cost) not necessary for (quantitative) understanding of the big picture. Simplified rotating shallow water (RSW) models obtained by vertical averaging of primitive equations: optimal between simplicity and fidelity of representation.

Shallow-water modelling of the atmosphere and ocean



Dynamical core of the adiabatic Primitive Equations

Use of pressure as vertical coordinate and hydrostatics, valid for large-scale motions, on the tangent plane $\rightarrow$

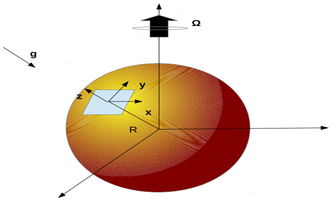
$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}_h + f(y)\hat{\boldsymbol{z}} \times \boldsymbol{v}_h = -\nabla_h \phi,$$

$$-g\frac{\theta}{\theta_0} + \frac{\partial \phi}{\partial \bar{z}} = 0,$$

$$\frac{d}{dt}\theta \equiv \frac{\partial \theta}{\partial t} + \boldsymbol{v} \cdot \nabla \theta = 0; \quad \nabla \cdot \boldsymbol{v} = 0.$$

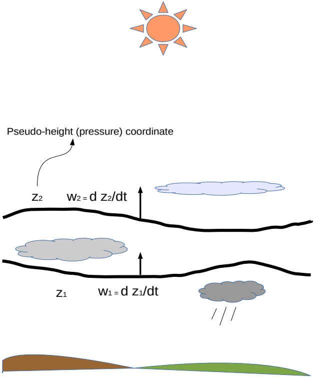
θ - potential temperature (entropy), ϕ - geopotential, p - pressure, $\boldsymbol{v}_h$ - horizontal velocity, $\boldsymbol{v} = (\boldsymbol{v}_h, w)$, Coordinates: x, y, $\bar{z}$, pseudo-height: $\bar{z} = z_0 \left(1 - \left(\frac{p}{p_s}\right)^{\frac{R}{c_p}}\right)$, $\nabla = (\nabla_h, \partial_{\bar{z}})$.

Tangent-plane approximation



Coriolis parameter: $f = f(y) = f_0 + \beta y$ on the tangent plane

Averaging over layers between material surfaces



Classical RSW models

1-layer RSW

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + f(y)\, \hat{\mathbf{z}} \times \mathbf{v} + g \nabla h = 0\,,$$

$$\partial_t h + \nabla \cdot (\mathbf{v} h) = 0\,,$$

2-layer RSW

$$\begin{cases} \partial_t \mathbf{v}_1 + (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1 + f(y)\hat{\mathbf{z}} \times \mathbf{v}_1 = -g\nabla(h_1 + h_2), \\ \partial_t \mathbf{v}_2 + (\mathbf{v}_2 \cdot \nabla)\mathbf{v}_2 + f(y)\hat{\mathbf{z}} \times \mathbf{v}_2 = -g\nabla(h_1 + \alpha h_2) \end{cases}$$

$$\begin{cases} \partial_t h_1 + \nabla \cdot (h_1 \mathbf{v}_1) = 0, \\ \partial_t h_2 + \nabla \cdot (h_2 \mathbf{v}_2) = 0, \end{cases}$$

$h_{1,2}$ - layer thicknesses, $\alpha = \frac{\theta_2}{\theta_1} \geq 1$ - stratification.

Incorporating passive scalars

Advection of any quantity q in primitive equations $\frac{d}{dt}q = 0 \Rightarrow$ conservation law for the bulk amount of q in the air column $Q_i = \int_{z_{i-1}}^{z_i} q\, dz$:

$$\partial_t Q_i + \nabla \cdot (Q_i \mathbf{v}_i) = 0.$$

"Dry" system: pot. temperature θ and moisture q are advected:

$$\frac{d}{dt}\theta = 0, \quad \frac{d}{dt}q = 0.$$

Moist-convective system: moist enthalpy is advected:

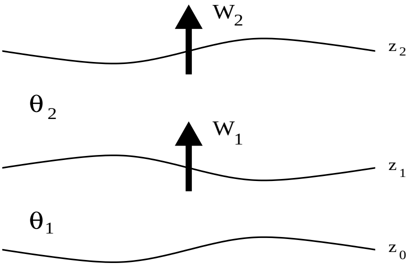
$$\frac{d}{dt}\left(\theta + \frac{L}{c_p}q\right) = 0,$$

L - latent heat of condensation, c_p - specific heat of the air.

Adding convective fluxes

$w = \frac{dz}{dt} \to w = \frac{dz}{dt} + \mathcal{W}$, vertical velocity due to convective flux $\mathcal{W}$ to be defined and linked to other variables $\to$ convective shallow water

$$w_0 = \frac{dz_0}{dt}, \quad w_1 = \frac{dz_1}{dt} + \mathcal{W}_1, \quad w_2 = \frac{dz_2}{dt} + \mathcal{W}_2.$$



Linking convection to condensation: moist-convective RSW

Condensation sink in bulk humidity $Q_i = \int_{z_{i-1}}^{z_i} q\, dz$, q - specific humidity :

$$\partial_t Q_i + \nabla \cdot (Q_i \boldsymbol{v}_i) = -P_i,$$

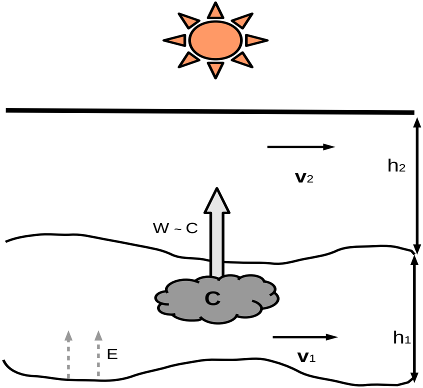
Conservation of moist enthalpy in pseudo-height coordinates $\rightarrow$

$$\theta_{i+1} = \theta(z_i) + \frac{L}{c_p} q(z_i) \approx \theta_i + \frac{L}{c_p} q(z_i) > \theta_i,$$

Averaging over the layer $\rightarrow$

$$\mathcal{W}_i = \beta_i P_i, \quad \beta_i = \frac{L}{c_p(\theta_{i+1} - \theta_i)} > 0.$$

Schematics of the moist-convective (mcRSW) model



2-layer mcRSW model with moist lower layer

Upper surface isobaric: $z_2 = \mathrm{const}$, $\phi(z_0) = \mathrm{const}$ (ground), $Q_2 = 0$, $Q_1 = Q$,

$$
\begin{cases}
\partial_t \boldsymbol{v}_1 + (\boldsymbol{v}_1 \cdot \boldsymbol{\nabla})\boldsymbol{v}_1 + f(y)\hat{\boldsymbol{z}} \times \boldsymbol{v}_1 = -g\boldsymbol{\nabla}(h_1 + h_2), \\
\partial_t \boldsymbol{v}_2 + (\boldsymbol{v}_2 \cdot \boldsymbol{\nabla})\boldsymbol{v}_2 + f(y)\hat{\boldsymbol{z}} \times \boldsymbol{v}_2 = -g\boldsymbol{\nabla}(h_1 + \alpha h_2) + \frac{\boldsymbol{v}_1 - \boldsymbol{v}_2}{h_2}\beta P, \\
\partial_t h_1 + \boldsymbol{\nabla} \cdot (h_1 \boldsymbol{v}_1) = -\beta P, \\
\partial_t h_2 + \boldsymbol{\nabla} \cdot (h_2 \boldsymbol{v}_2) = +\beta P, \\
\partial_t Q + \boldsymbol{\nabla} \cdot (Q \boldsymbol{v}_1) = -P + E,
\end{cases}
$$

Condensation: relaxational parametrisation with relaxation time τ and saturation threshold Q^s:

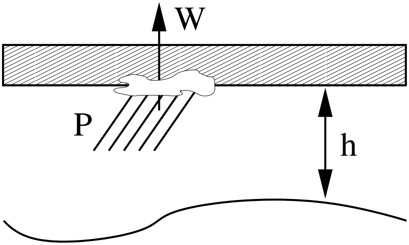
$$
P = \frac{Q - Q^s}{\tau}\mathcal{H}(Q - Q^s)
$$

Surface evaporation - standard: $E = \propto |\boldsymbol{v}_1|(Q^s - Q)\mathcal{H}(Q^s - Q)$.

One-layer reduction

In the limit of infinitely deep upper layer a simplified one-layer model arises (condensation = precipitation):

$$\begin{cases} \partial_t \boldsymbol{v}_1 + (\boldsymbol{v}_1 \cdot \boldsymbol{\nabla})\boldsymbol{v}_1 + f(y)\hat{\boldsymbol{z}} \times \boldsymbol{v}_1 = -g\boldsymbol{\nabla} h_1, \\ \partial_t h_1 + \boldsymbol{\nabla} \cdot (\boldsymbol{v}_1 h_1) = -\beta P, \\ \partial_t Q + \boldsymbol{\nabla} \cdot (Q\boldsymbol{v}_1) = -P + E. \end{cases}$$



Improving mcRSW

mcRSW: condensation $\equiv$ precipitation. Advection of liquid water $\Rightarrow$ conservation of bulk amount of water $W(x, y, t)$ in the air column.

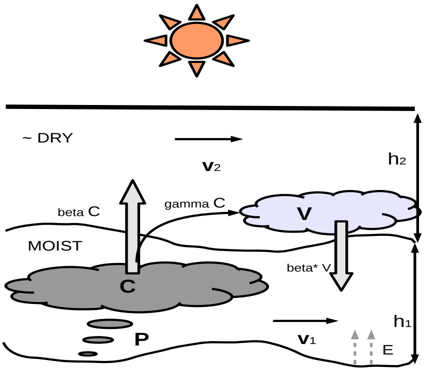
$$\partial_t W + \nabla.(W\mathbf{v}) = +C - V - P$$

C - condensation source, V - vaporisation sink, P - precipitation sink. Vaporisation$\Rightarrow$ cooling $\Rightarrow$ downward convective flux. $\mathcal{W} = -\beta^* V$. Precipitation/vaporisation sinks:

$$P = \frac{W - W_p}{\tau_p}\mathcal{H}(W - W_p), \quad V \propto (Q^s - Q)\mathcal{H}(Q^s - Q).$$

Entrainment: part of the liquid water goes up convective updrafts: $C \to (1 - \gamma)C$, γ - entrainment coefficient.

Schematics of the "improved" mcRSW model



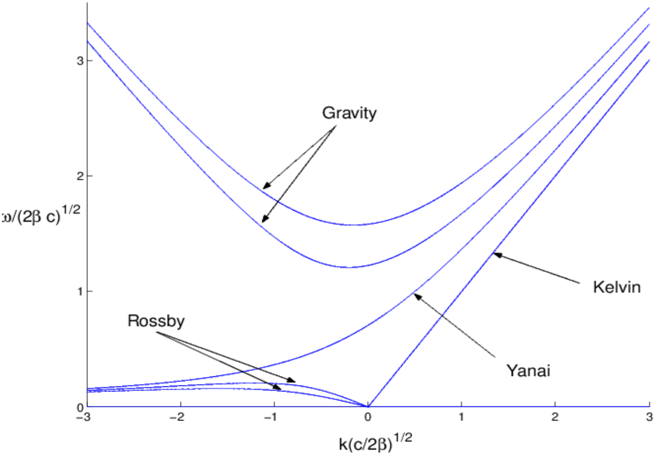
RSW equations on the equatorial tangent plane

Momentum and mass conservation in the 1-layer “dry“ RSW:

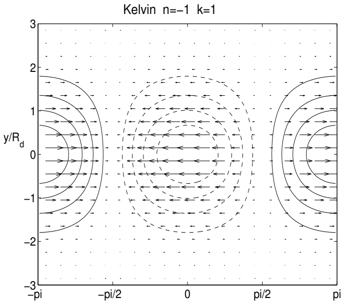
$$\begin{cases} \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \beta y\, \hat{\boldsymbol{z}} \times \boldsymbol{v} = -g \boldsymbol{\nabla} h, \\ \partial_t h + \boldsymbol{\nabla} \cdot (\boldsymbol{v} h) = 0. \end{cases}$$

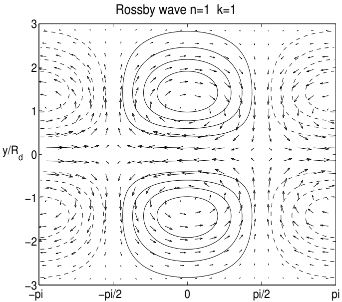
Coriolis parameter f has no constant part: $f = \beta y$.
Linearisation about the state of rest gives a rich spectrum of trapped equatorial waves.

Dispersion diagram for equatorial waves



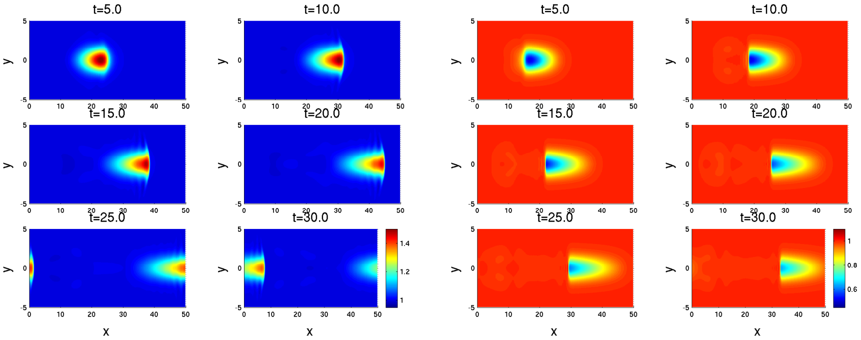
Long unidirectional equatorial waves



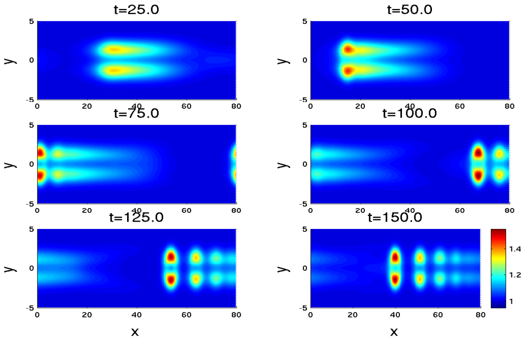


Kelvin: non-dispersive - breaking;
Rossby: weakly dispersive - KdV solitons.

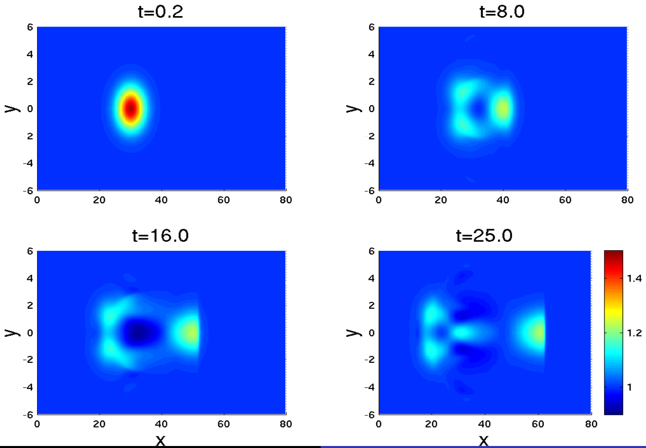
Breaking of Kelvin waves



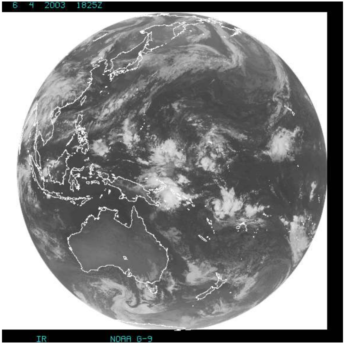
Rossby solitons

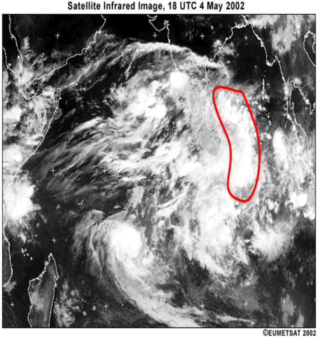


Typical evolution of a large-scale pressure anomaly in tropics: generation of Kelvin and Rossby waves



Observations of atmospheric equatorial waves

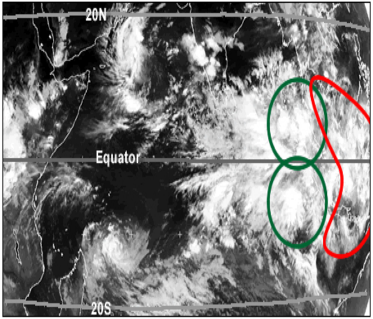




Left panel: Rossby wave. *Right panel:* Kelvin wave.

Observed coupling between Rossby and Kelvin waves in the vicinity of the warmpool

Satellite Infrared Image, 18 UTC 7 May 2002



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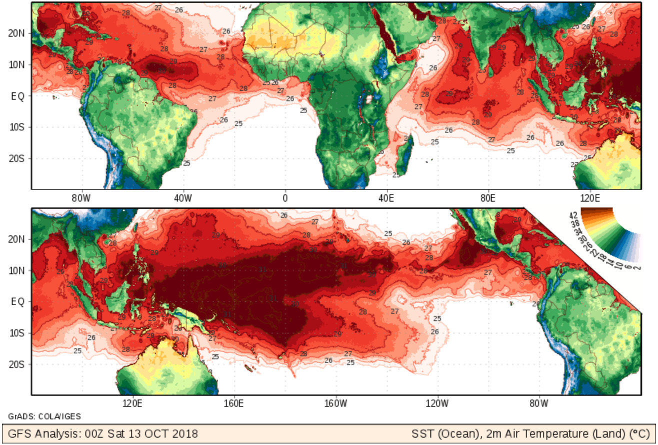
Maritime continent and warm-pool

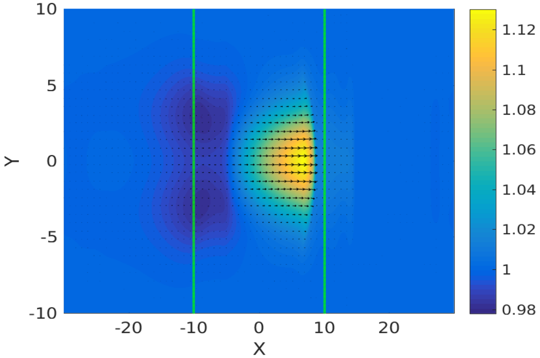


Workflow

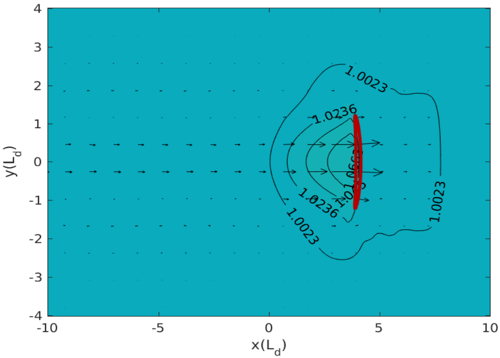
- Initialise the model (1-layer, to begin) with a Kelvin-wave packet
- Represent the warm-pool as a region of higher humidity
- Incorporate topography:
  - idealised Sumatra, in a form of oblique bump with realistic scales
  - full topography of the Maritime Continent
- Introduce diurnal heating and radiative damping
- Look what happens

Generation of a Rossby wave by a Kelvin wave entering warm-pool, no topography

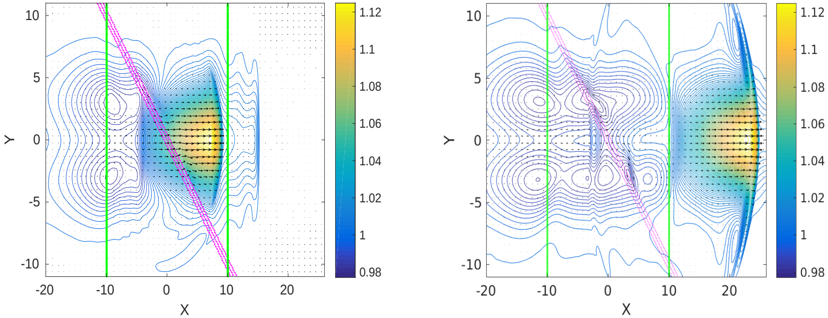
A snapshot of pressure:



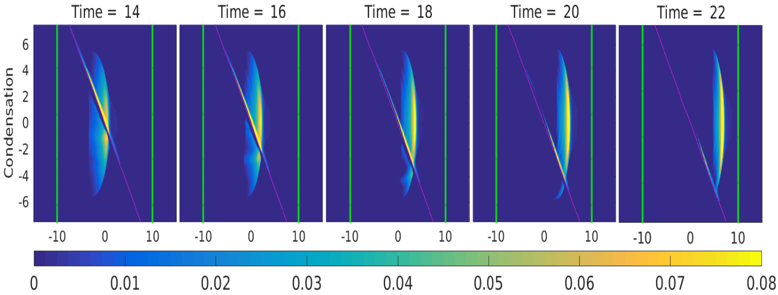
Precipitation at the front of the Kelvin wave



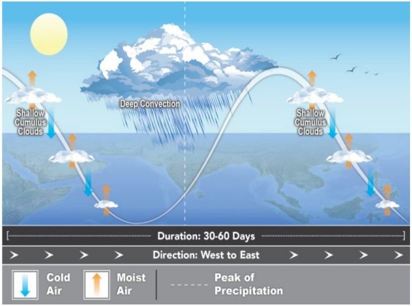
Kelvin wave passing through warm-pool with idealised topography. Evolution of pressure

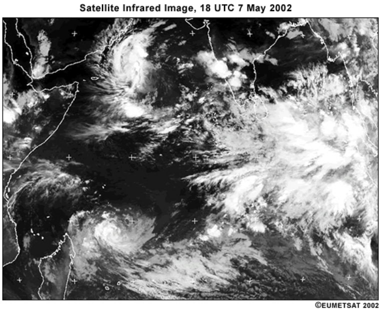


Kelvin wave passing through warm-pool with idealised topography. Evolution of condensation/precipitation



Madden-Julian Oscillation (MJO)

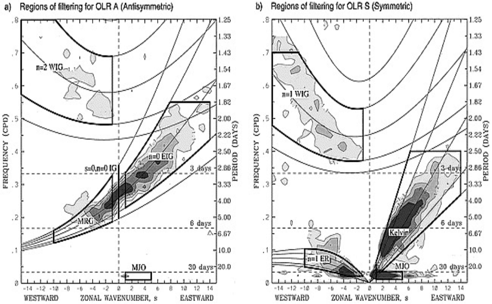




Periodic moving slowly eastward over Indo-Pacific warm-pool, dying out in the Pacific.

MJO enigma

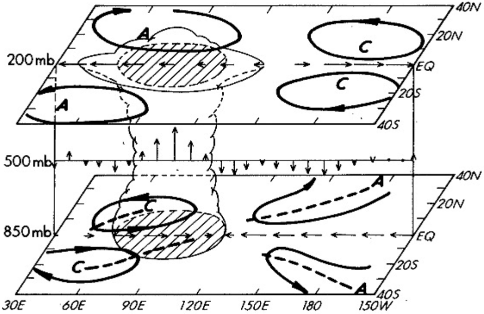
Outgoing long-wave radiation (OLR) and dispersion diagram of equatorial waves (Wheeler & Kiladis, 1999)



MJO is much slower than Kelvin waves, the only eastward-moving ones !

Dynamical structure of MJO

Synopsis of observations:



Twin cyclones in the lower layer (Zhang, 2005) $\rightarrow$ idea: MJO $\leftrightarrow$ a modon (first Yano & Tribbia, 2017, at planetary scale on the sphere)

Charney's (1963) balance for tropical motions

Observation: pressure variations in tropics are weak.
Quasi-barotropic scaling: Pressure perturbation parameter λ:
$h = H(1 + \lambda\eta)$, $H = const$
"Vortex" scaling (single velocity and single spatial scales):
$(x, y) \sim L,\ (u, v) \sim V,\ t \sim \frac{L}{V} \Rightarrow$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \overline{\beta} y \hat{\mathbf{z}} \wedge \mathbf{v} + \frac{gH\lambda}{V^2} \nabla \eta = 0\,, \quad \overline{\beta} = \beta L^2 / V.$$

Assumption

$$\lambda \to 0,\ \text{and}\ \frac{gH\lambda}{V^2} = \mathcal{O}(1) \Rightarrow V << \sqrt{gH},$$

i.e. the typical velocity is much smaller than the phase velocity of the Kelvin waves $c_K = \sqrt{gH}$.

Vorticity equation

Rescaled equations

$$
\begin{aligned}
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \overline{\beta}\, y\, \hat{\mathbf{z}} \wedge \mathbf{v} + \nabla \eta &= 0\,, \\
\lambda(\partial_t \eta + \mathbf{v} \cdot \nabla \eta) + (1 + \lambda \eta)\, \nabla \cdot \mathbf{v} &= 0.
\end{aligned}
$$

Leading order in λ: $\nabla \cdot \mathbf{v}_0 = 0 \Rightarrow$ streamfunction ψ for $(u,\, v)$.

Vorticity equation

Cross-differentiation of momentum equations:

$$
\nabla^2 \psi_t + \mathcal{J}(\psi, \nabla^2 \psi) + \overline{\beta} \psi_x = 0.
$$

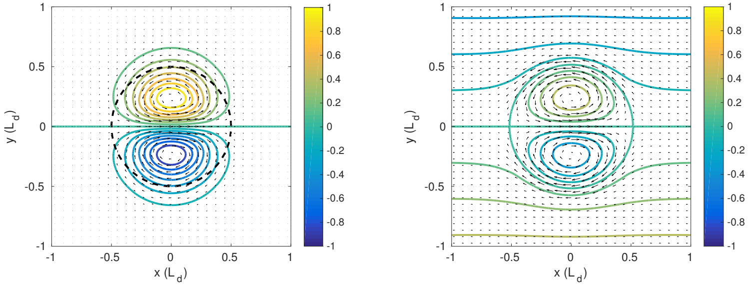
Steady-moving modon solutions of asymptotic vorticity equation

Matched *external* and *internal* solutions with respect to a circle $r = a$ in polar coordinates (r, θ):

$$\begin{cases} \psi_{ext} = -\frac{Ua}{K_1(pa)} K_1(pr) \sin\theta, \quad r > a, \\ \psi_{int} = \left[\frac{Up^2}{k^2 J_1(ka)} J_1(kr) - \frac{r}{k^2}(1 + U + Uk^2)\right] \sin\theta, \quad r < a, \end{cases}$$

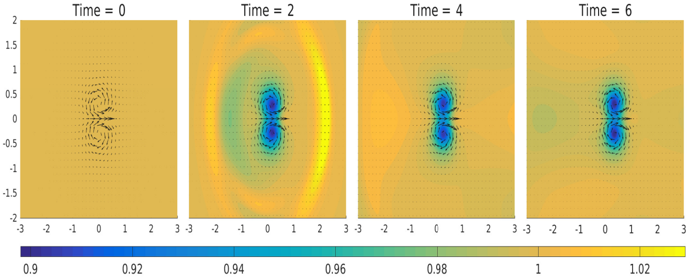
J_1 and K_1 - Bessel functions, p is real, and $p^2 = \overline{\beta}/U$, so $U > 0$, and the motion is *eastward*. Each pair $(a, p) \rightarrow$ series of eigenvalues k arising from matching conditions, the lowest corresponds to a dipole.

Phase portrait of an asymptotic modon



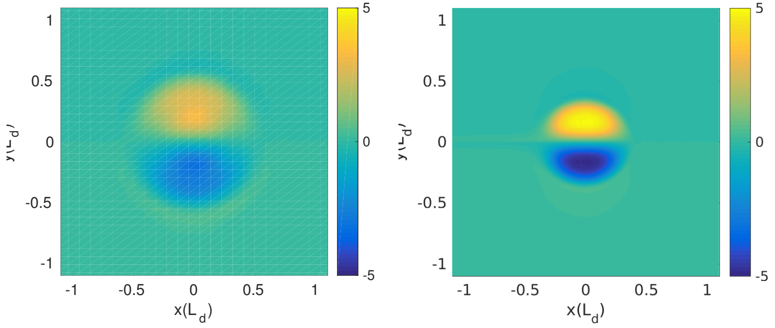
Streamlines and velocity field of an asymptotic modon in stationary (*left*) and co-moving (*right*) frames.
Dashed line: circular separatrix of radius *a*.

Adjustment of the asymtotic modon



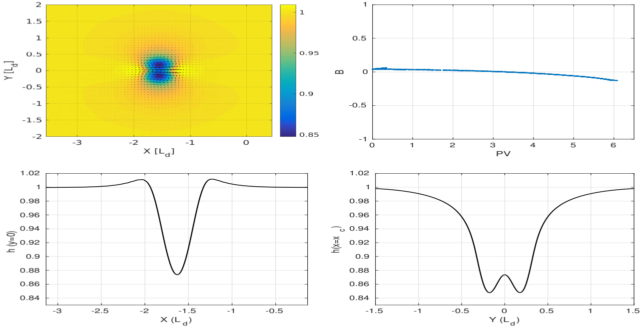
Thickness and velocity fields given by numerical simulation initialised with the velocity field of an asymptotic modon.

Comparison of asymptotic and "exact modons



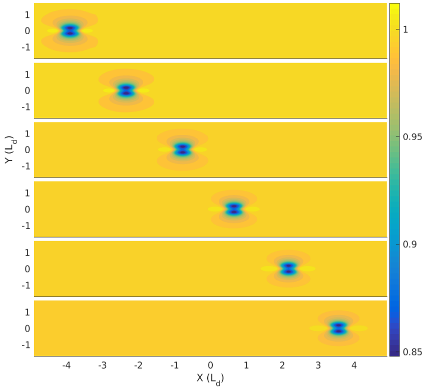
Vorticity of the asymptotic (left) and corresponding "exact" (right) modons.

Coherence of the adjusted modon



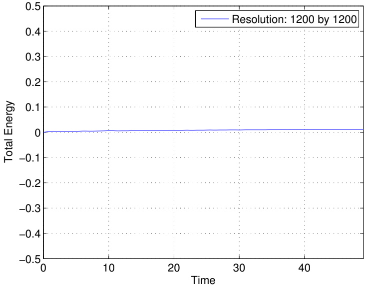
Top to bottom, left to right: thickness h of the "exact" modon at $t = 15\,[1/\beta L_d]$, Bernoulli function in the co-moving frame vs potential vorticity, zonal and meridional sections of the modon.

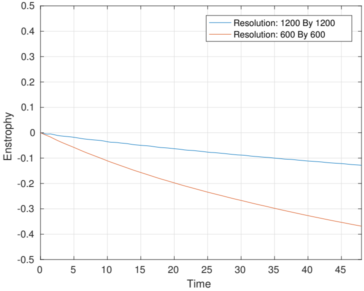
Motion of an "exact" modon



Snapshots of h for eastward-moving equatorial modon at $t = 0, 10, 20, 30, 40, 50\ [1/(\beta L_d)]$, $a = 0.5$, $U = 0.2$, $\beta = 1$

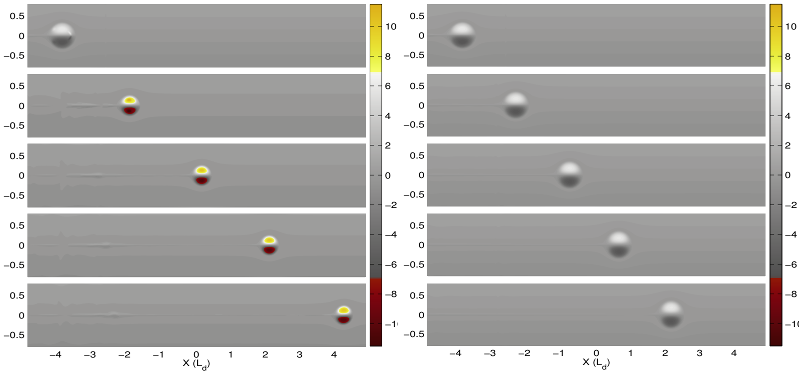
Energy and enstrophy evolution





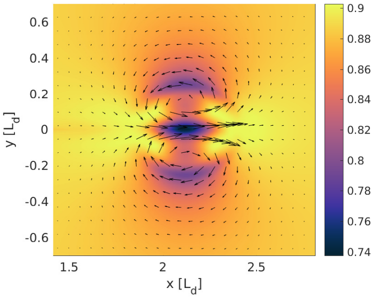
Evolution of normalized deviations from the initial values of energy (left) and enstrophy (right) of the modon at different resolutions

Moist-convective vs "dry" modons



Evolution of moist-convective (left) vs adiabatic modons with the same initial conditions. $t = 0, 10, 20, 30, 40\ [1/(\beta L_d)]$;

Moisture and convective patterns of the equatorial modon



Distribution of velocity and water vapor (left) and superposition of pressure and condensation (red lines) isolines (right) at $t = 30\ [1/(\beta L_d)]$

Overall conclusion

Equatorial waves coupled to moist convection are at the origin of essential dynamical features of the tropical atmosphere over the warm-pool/Maritime Continent

What's next?

- Using two or three-layer model, to include vertical structure
- Using improved mcRSW to include liquid water and precipitation
- Including horizontal temperature gradients and influence of sea-surface temperature$\rightarrow$ Thermal Shallow Water

References

- mcRSW model:
  Lambaerts J., Lapeyre G., Zeitlin V., and Bouchut F. "Simplified two-layer models of precipitating atmosphere and their properties" Phys. Fluids **23**, 046603, 2011.
- improved mcRSW model:
  Rostami, M., and Zeitlin, V. "Modelling instabilities of hurricane-like vortices with improved moist-convective rotating shallow water", Q.J.Roy. Met. Soc. https://doi.org/10.1002/qg.3292.
- application to hurricanes:
  Lahaye N. , and Zeitlin, V. "Understanding instabilities of tropical cyclones and their evolution with moist-convective rotating shallow water model", J. Atmos. Sci. **73**, 505-523, 2016.
- equatorial modons Rostami, M., and Zeitlin, V. “Eastward-moving equatorial modons: a missing chain-link in the dynamics of tropical atmosphere?”, Geoph. Res. Lett. *submitted*