Interaction of multi-lumps within the Kadomtsev-Petviashvili equation

Zhi-Ming Lu   Wencheng Hu

Shanghai Institute of Applied Mathematics & Mechanics,

Shanghai University, China

In Honor of Roger Grimshaw

Outline

1. Introduction
2. Lump solution to KPI
3. Interaction of multi-lumps
4. Conclusions

Kadomtsev–Petviashvili(KP)equation

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)+3\sigma^2\frac{\partial^2 u}{\partial y^2}=0$$

$$\sigma^2=-1,\quad KP1;\qquad \sigma^2=1,\quad KP2$$

- Obtained firstly in plasma (Kadomtsev B B, Petviashvili V I 1970)

It's classified as the KP1/KP2 equation for wave process in media with positive/negative dispersion when the parameter $\sigma^2=-1$ or $\sigma^2=1$ respectively.

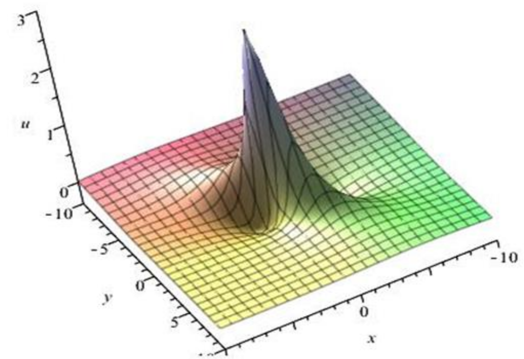
The applicability of KP1 and KP2

- Surface and internal water waves
- Plasma waves
- Elastic waves on thin plates
- Waves in solids
- Optical waves
- Bose-Einstein condensates

…

Introduction

KP1 admits lump-type solution localized in all directions and falling off as x^{-2}, y^{-2}

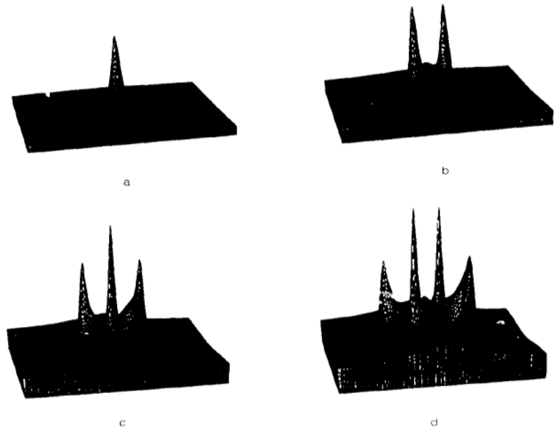


- Single-lump (numerically) (Petviashvili V I 1976)

- Single and multi-lump (analytical)
  (Manakov S V, Zakharov V E, Bordag L A,1977,
  Ablowitz M J, Satsuma J. 1978)

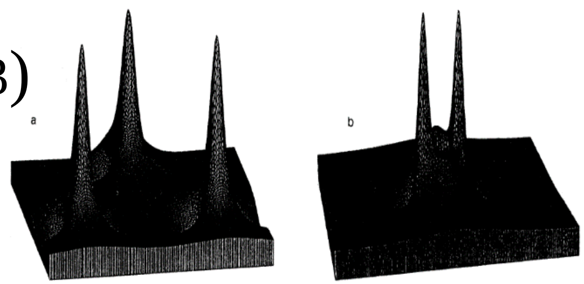
➢Symmetric stationary multi-lump solutions without free parameters

(Pelinovskiĭ, D. E, Stepanyants, Yu. A 1993)

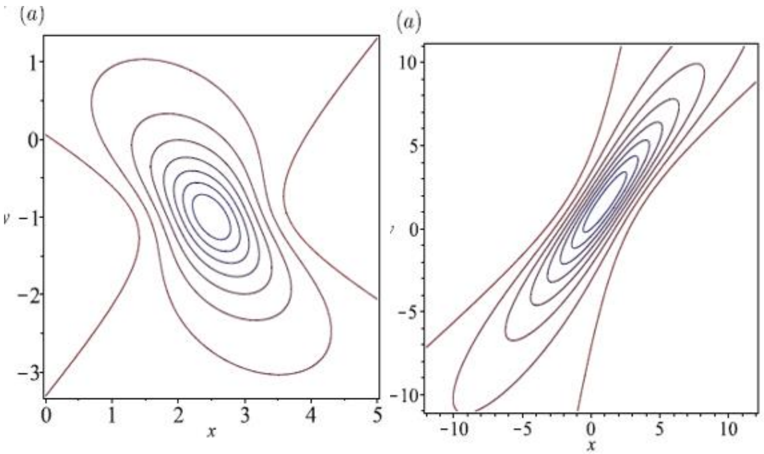


➢Symmetric stationary multi-lump solutions with free parameters

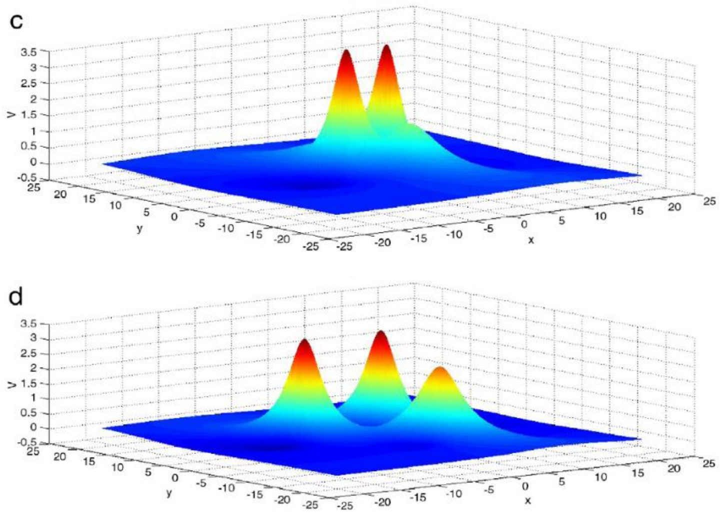
(Gorshkov K A, Pelinovskii D E, Stepanyants Y A 1993)



➢Skew lump solution
(Wen-Xiu Ma 2015)



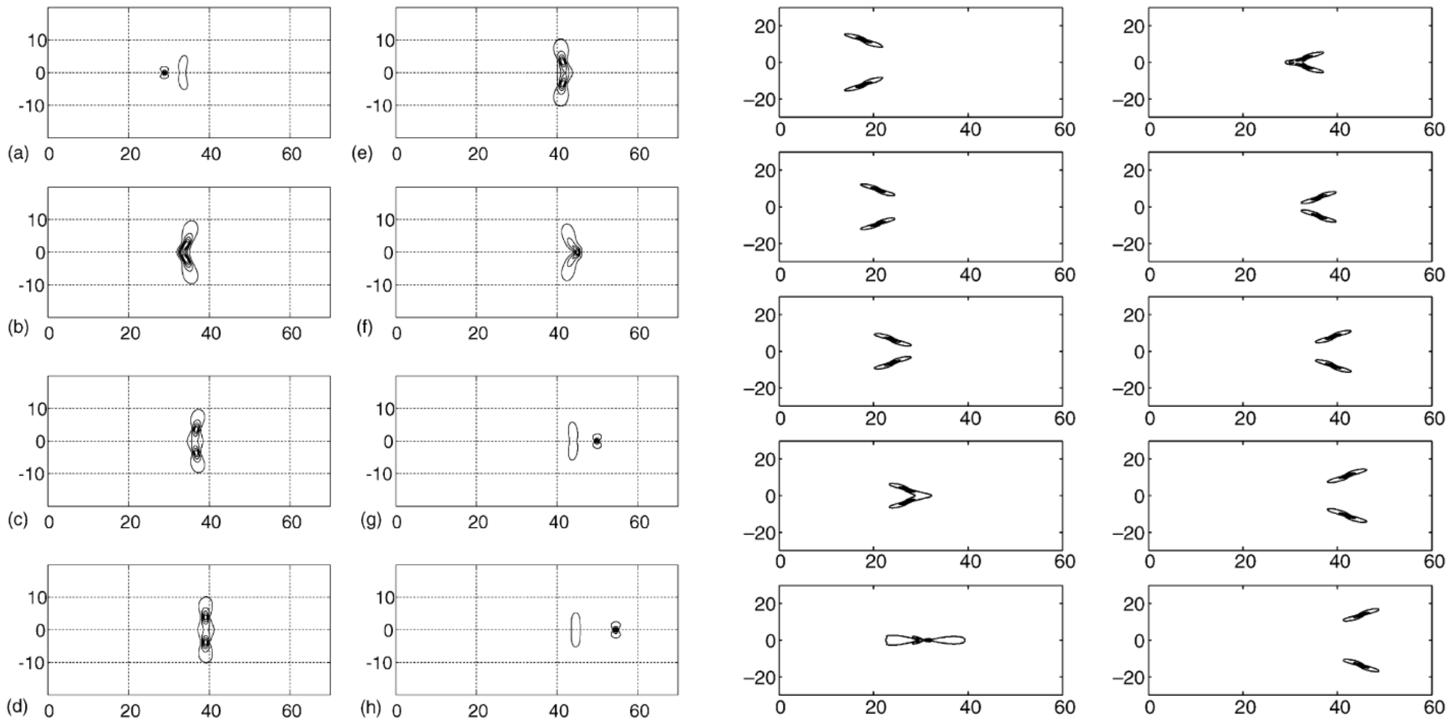
➢Skew stationary multi-lump solution
(Singh N, Stepanyants Y 2016)



Introduction

- Interaction of single-lump soliton

  (Lu Z, Tian E M, Grimshaw R， 2004)



elastic or Non-elastic collision

Some questions remained about the lump solution

- How detailed structures depending on the free parameters of the stationary multi-lump solutions?
- Interaction of multi-lump soliton is not clear yet so for. Does the interaction between multi-lumps behave like the interaction between the single-lump ones?
- The generation mechanism of multi-lumps and how to generate multi-lumps in a laboratory. etc.

The aim of the work

(i) Obtain more generalized stationary skew multi-lump solution of the KPI by the Hirota bilinear method.

(ii) Show detailed structures of the analytical solution for some different free parameters

(iii) Study the interaction of the lumps numerically.

Lump solution to KPI

Lump solution to KPI

Hirota bilinear method

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)-3\frac{\partial^2 u}{\partial y^2}=0$$

$$\xi = x - V_x t, \quad \eta = y - V_y t$$

$$(-V_x)u_{\xi\xi} - u_{\eta\eta} - V_y u_{\xi\eta} + 3(u^2)_{\xi\xi} + u_{\xi\xi\xi\xi} = 0$$

Use the transformation

$$\xi = \frac{1}{\sqrt{V_x}} X; \quad \eta = \frac{1}{V_x} Y; \quad \mathrm{u} = \frac{V_x}{3} U$$

Lump solution to KPI

$$U_{XX} + U_{YY} - \mu U_{XY} - (U^2)_{XX} - U_{XXXX} = 0; \quad \mu = \frac{V_y \sqrt{V_x}}{V_x}$$

Hirota form: $$U = 6\ln(f)_{XX}$$

$$(D_X{}^2 + D_Y{}^2 - \mu D_X D_Y - D_X{}^4) f \cdot f =$$
$$f(f_{XX} + f_{YY} - \mu f_{XY}) - (f_X)^2 - (f_Y)^2 + \mu f_X f_Y - 3(f_{XX})^2 - ff_{XXXX} + 4 f_X f_{XXX}$$

Single-lump solution

$$f = X^2 + a_2 Y^2 + a_3 XY + a_4 X + a_5 Y + a_6$$

$$2a_2 - a_3^2 + \mu a_3 - 2 = 0,$$

$$2a_2 - a_3^2 - 2a_2^2 + a_2 \mu a_3 = 0,$$

$$-2a_4 + 2a_2 a_4 + 2\mu a_5 - 2a_3 a_5 = 0,$$

$$2a_5 - 2a_3 a_4 - 2a_2 a_5 + 2\mu a_2 a_4 = 0,$$

$$-2a_3 - 2a_2 a_3 + 4\mu a_2 = 0,$$

$$-\mu a_3 a_6 - a_5^2 - a_4^2 + 2a_6 + \mu a_4 a_5 + 2a_2 a_6 - 12 = 0.$$

The corresponding solution

$$f = X^2 + Y^2 + \mu XY + \frac{12}{4-\mu^2}$$

Corresponding solution in terms of $U(X,Y)$

$$U(X,Y) = (4-\mu^2)\frac{1+\frac{4-\mu^2}{12}[\frac{4-\mu^2}{4}Y^2 - (X+\frac{\mu}{2}Y)^2]}{\{1+\frac{4-\mu^2}{12}[\frac{4-\mu^2}{4}Y^2 + (X+\frac{\mu}{2}Y)^2]\}^2}$$

Lump solution to KPI

The lump moves with $V = (V_x,\ V_y)$ and amplitude

$$U_{\max} = 4 - \mu^2 = 4 - \frac{V_y{}^2}{3V_x}$$

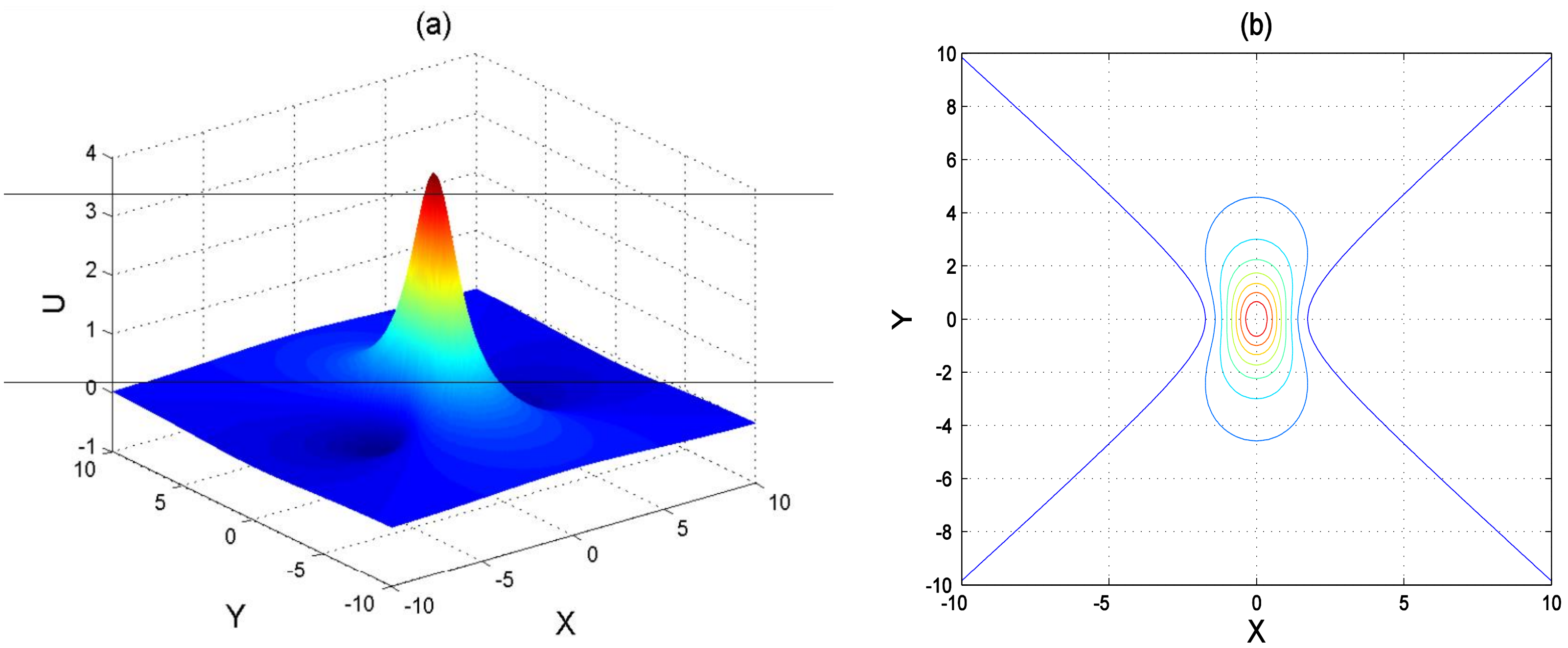
The solution is non-singular when μ varies in the range

$$-2 < \mu < 2$$

The restriction on the possible velocity components

$$V_y{}^2 < 12V_x$$

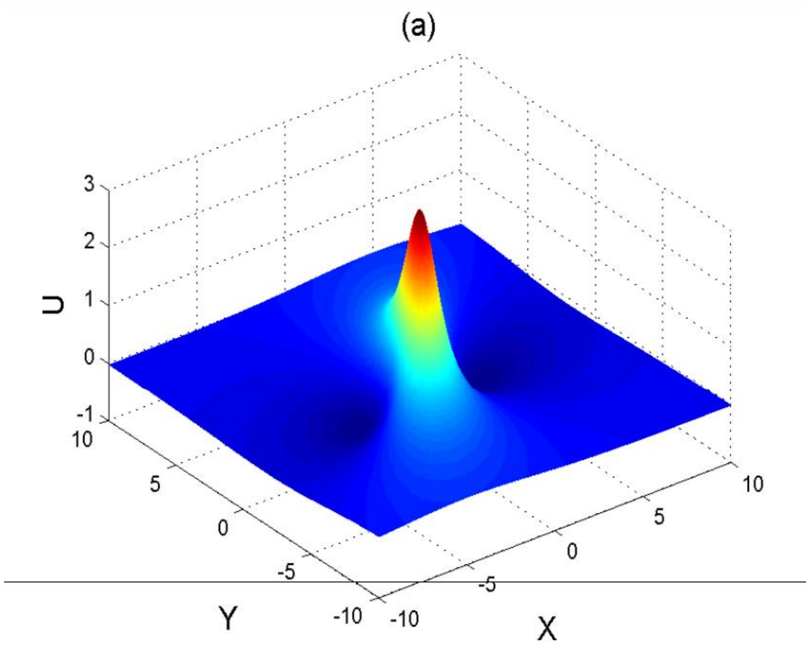
Lump solution to KPI

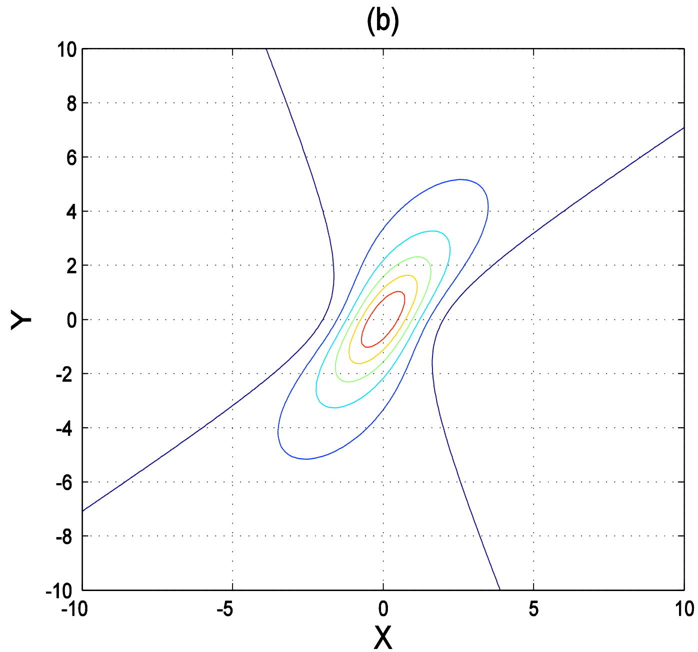


(a) The 3D plot of the symmetrical lump with $\mu = 0$.

(b) The corresponding contour plot.

Lump solution to KPI





(a) The 3D plot of the symmetrical lump with $\mu = -1$.

(b) The corresponding contour plot.

Lump solution to KPI

Three-lump solution $M = \frac{N(N+1)}{2}$

$$f = X^6 + a_2 X^5 Y + \cdots + a_{28}$$

66 equation for 27 parameters.

When we keep two free parameters $a, \ b$

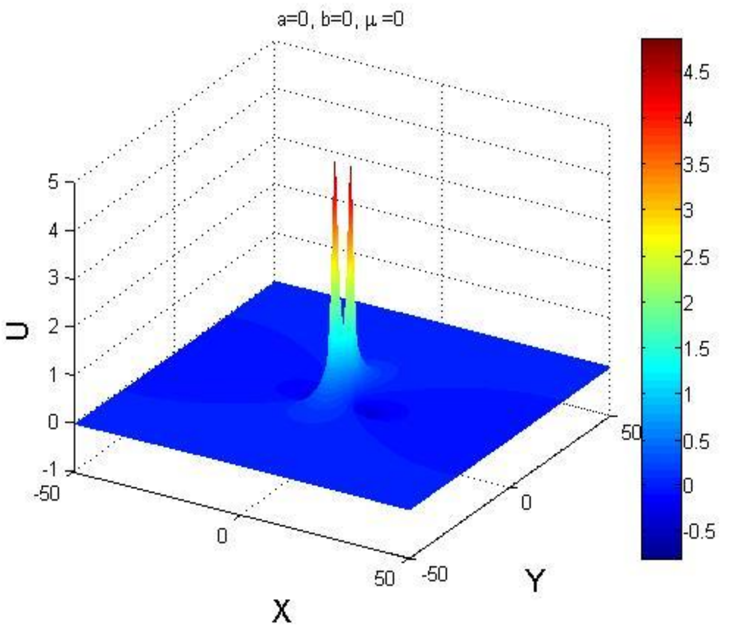
We obtain the analytical formula as below

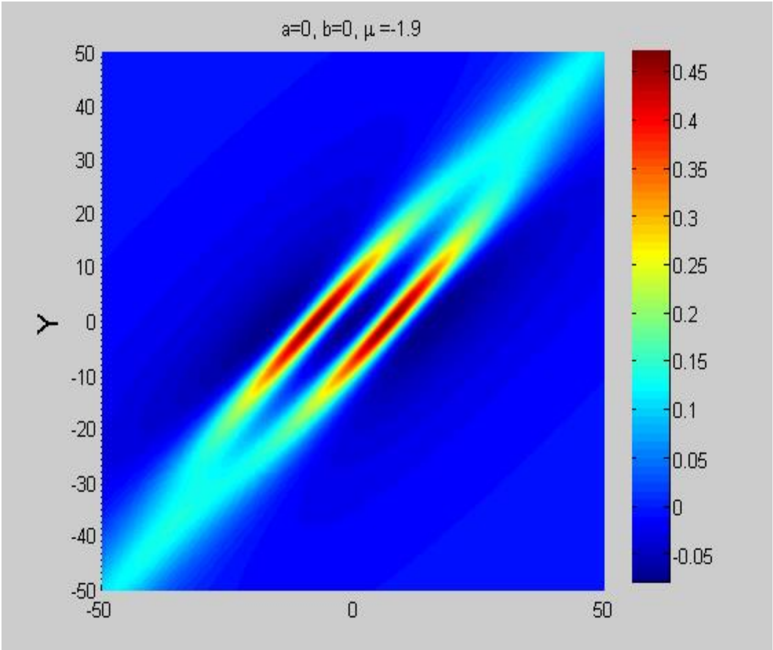
3-Lump solution to KPI

$$
\begin{aligned}
f(X,Y) = {} & X^6 + Y^6 + 3\mu X^5 Y + (3\mu^2+3)X^4Y^2 + \mu(\mu^2+6)X^3Y^3 \\
& +(3\mu^2+3)X^2Y^4 + 3\mu XY^5 - \frac{100X^4}{\mu^2-4} - \frac{200\mu X^3 Y}{\mu^2-4} - \frac{60(\mu^2+6)X^2Y^2}{\mu^2-4} \\
& + \frac{40\mu(\mu^2-9)XY^3}{\mu^2-4} + \frac{4(3\mu^2-17)(\mu^2+1)Y^4}{\mu^2-4} + \frac{a\mu-b}{3}X^3 + aX^2Y \\
& + bXY^2 + \frac{b\mu-a}{3}Y^3 - \frac{2000X^2}{(\mu^2-4)^2} - \frac{2000\mu XY}{(\mu^2-4)^2} \\
& - \frac{400(6\mu^2-19)Y^2}{(\mu^2-4)^2} + \frac{4}{3}\frac{(a\mu-b)X}{\mu^2-4} + \frac{4}{3}\frac{(3a\mu^2-3b\mu-5a)Y}{\mu^2-4} \\
& - \frac{1}{9}\frac{a^2+b^2-ab\mu}{\mu^2-4} - \frac{120000}{(\mu^2-4)^3}
\end{aligned}
$$

Lump solution to KPI

Bi-lump

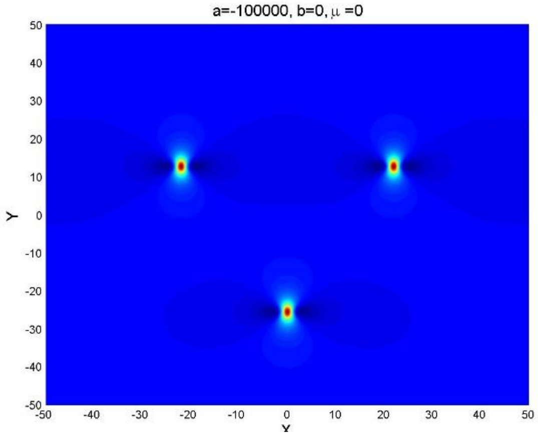


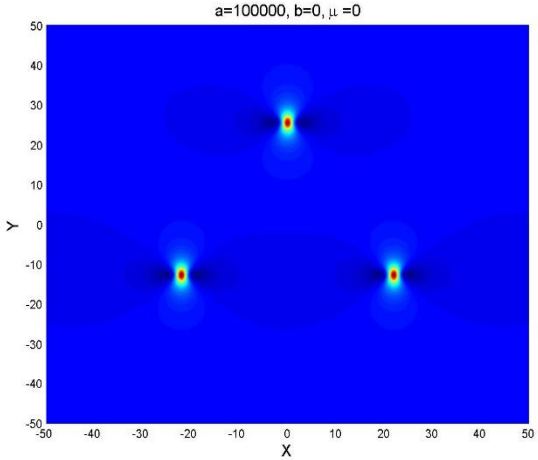


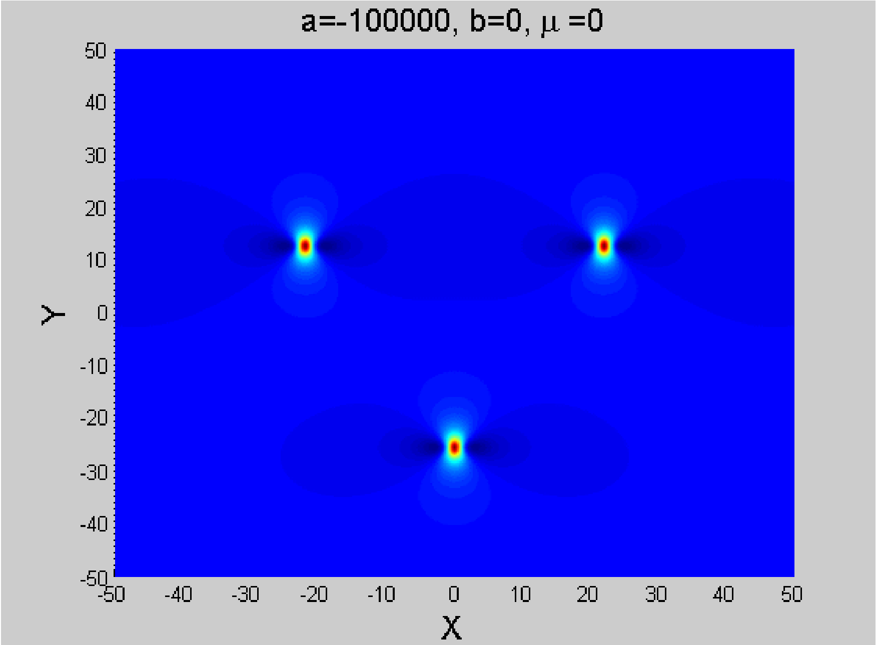
Plots of solution *U* with *a=b=0* and different values of μ

Lump solution to KPI

Triangular



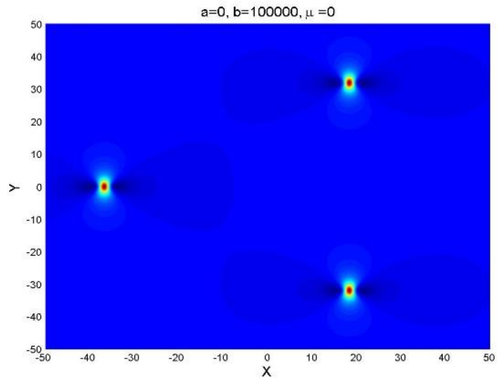


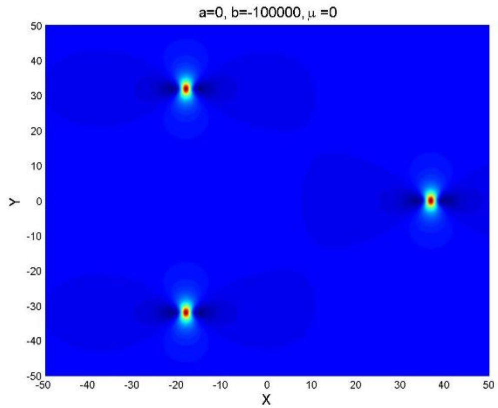


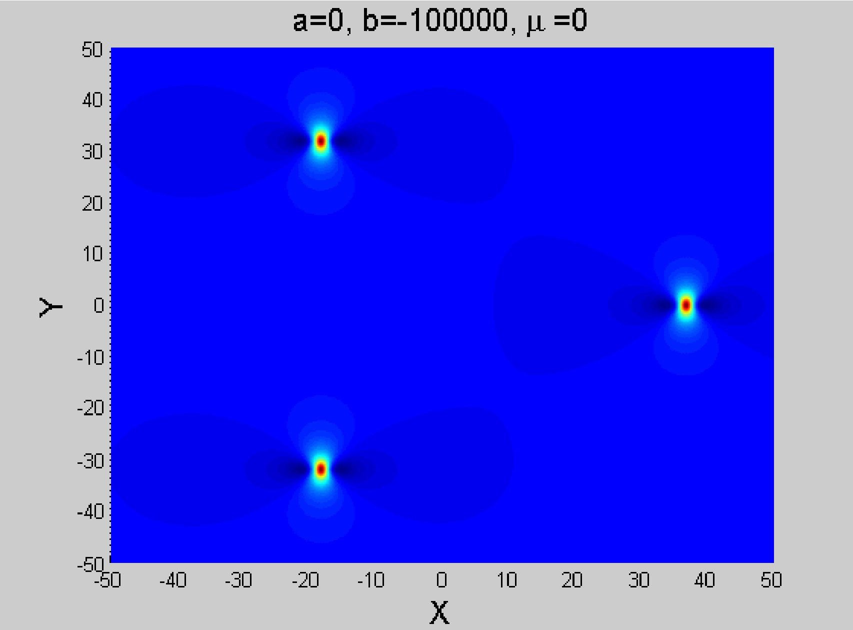
Plots of solution U with μ $=b=0$ and different values of a

Lump solution to KPI

Triangular



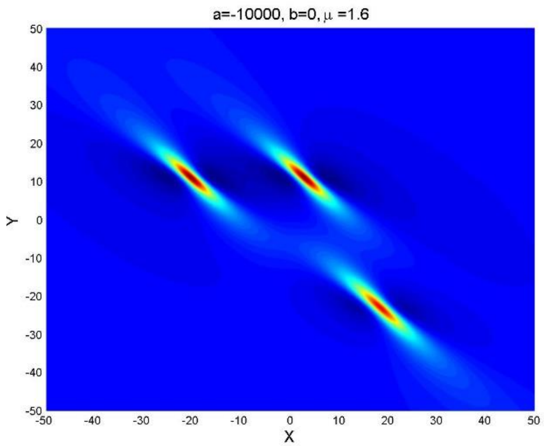


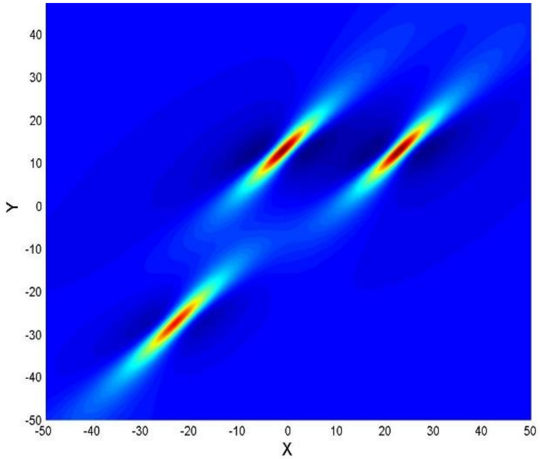


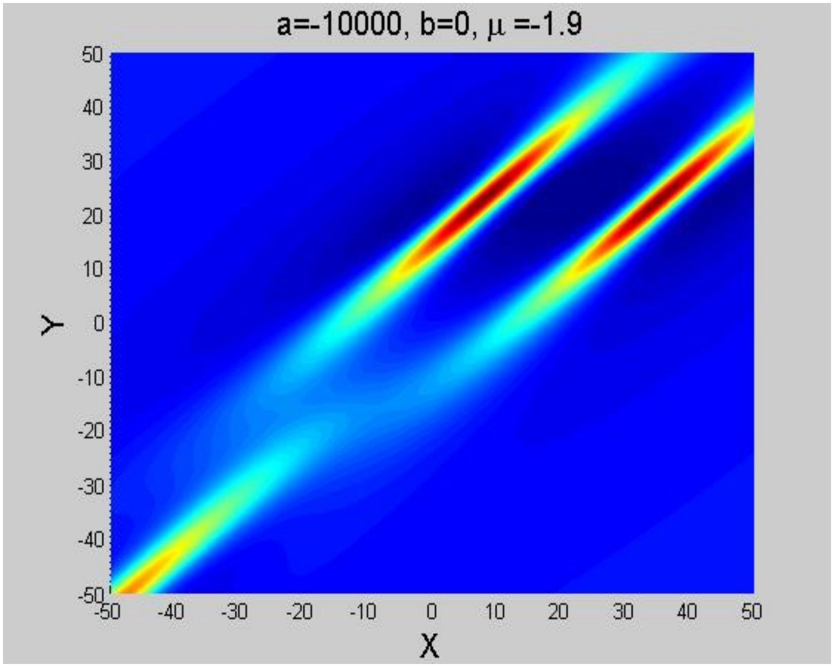
Plots of solution U with $\mu=a=0$ and different values of b

Lump solution to KPI

Skew triangular

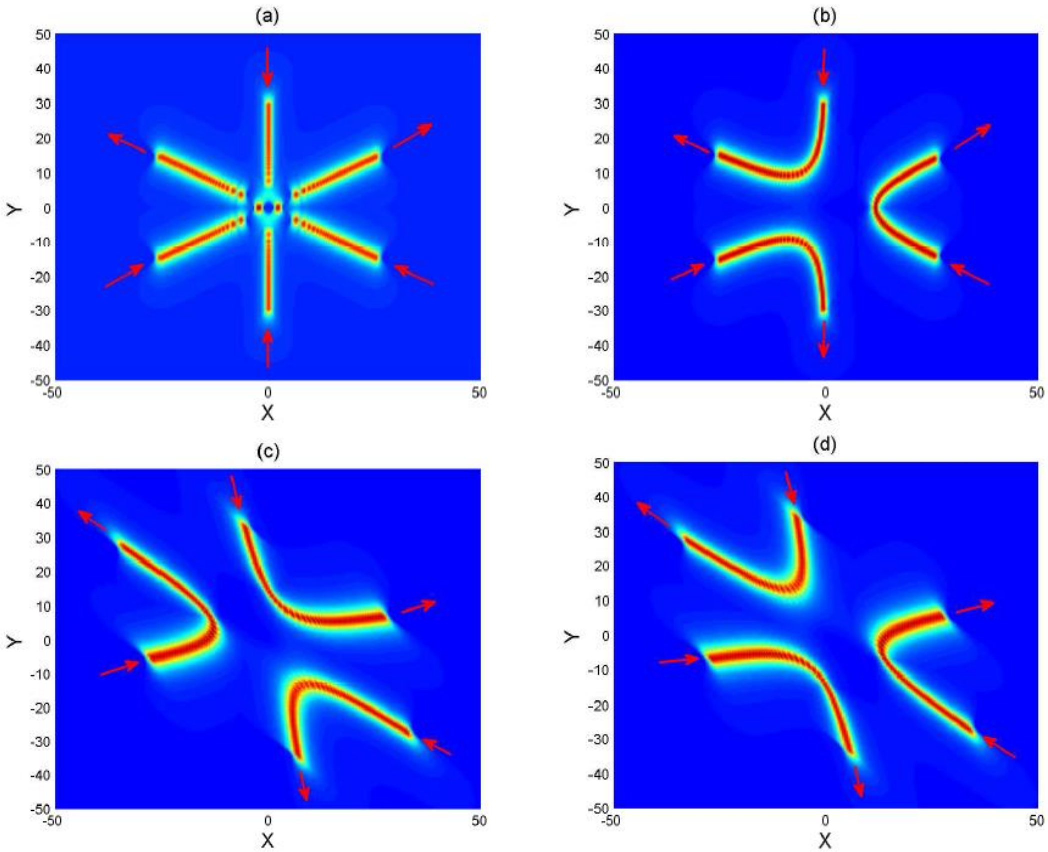






Plots of solution U with $a=-10000, b=0$ and different values of μ

Lump solution to KPI

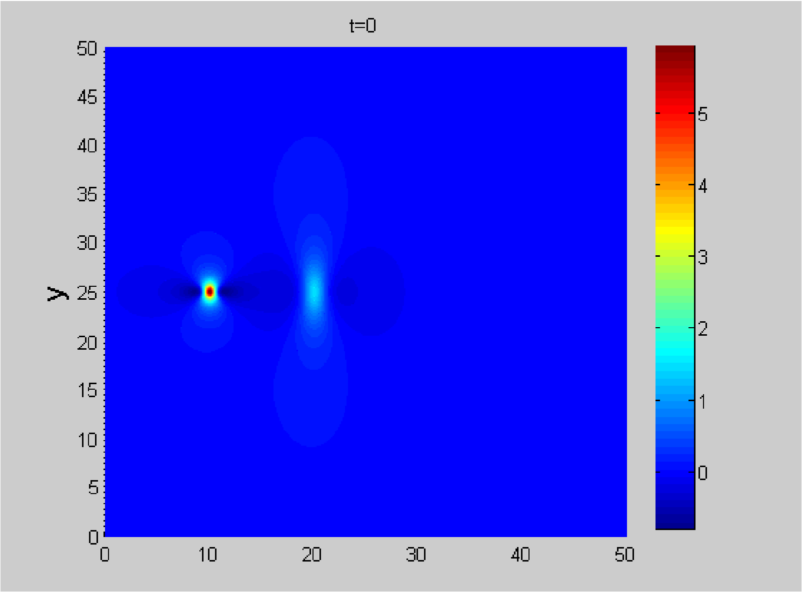


The trajectories of lump peaks when a varies in the interval[-50000; 50000]. Frame (a) b = 0; μ = 0; frame (b) b = 10000; μ = 0; frame (c) b = -10000; μ = 1; and frame (d) b = 10000; μ = 1. The red arrows indicate starting and ending values of a.

Interaction of lumps

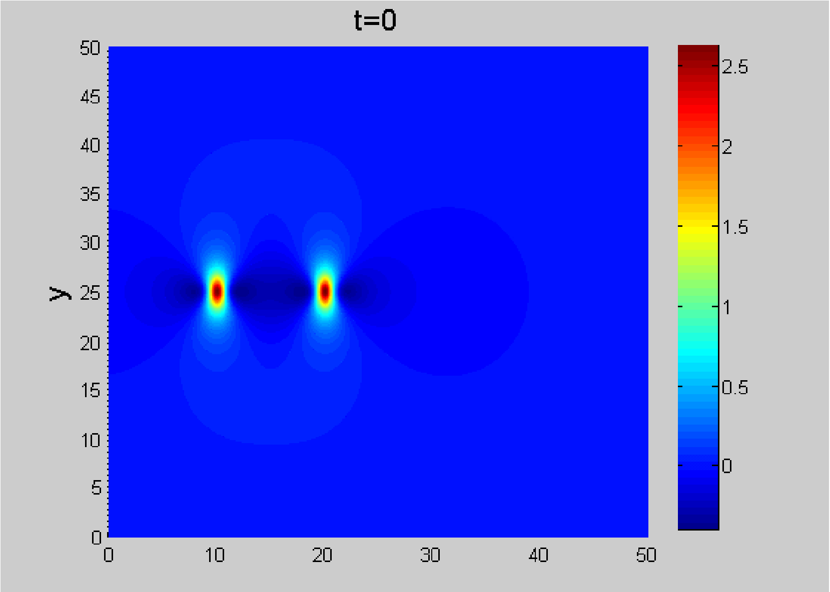
Direct interaction of two single-lumps

Normal scattering



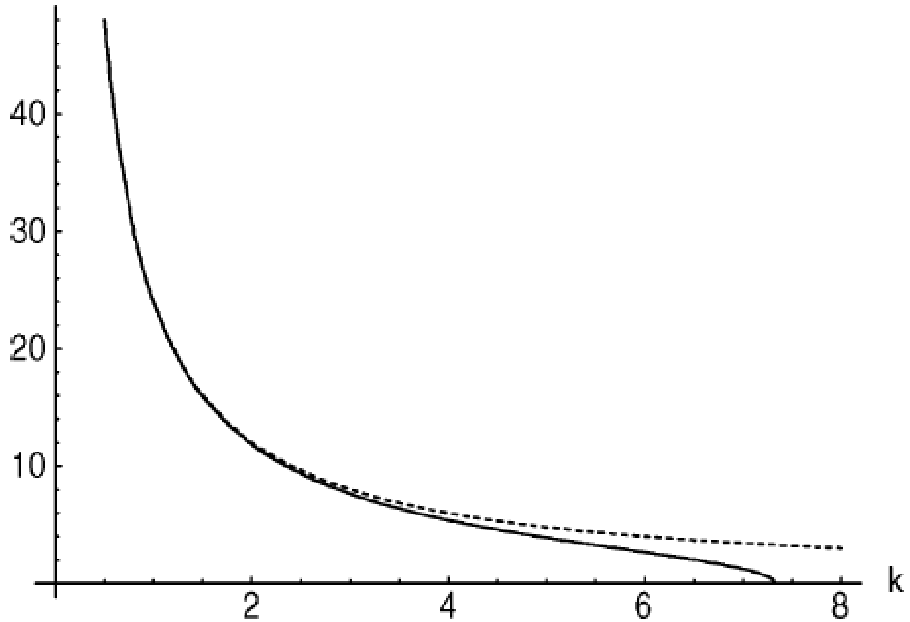
Line interaction of symmetric lumps with different speeds

Anomalous scattering



Line interaction of symmetric lumps with the same speeds

Interaction of lumps



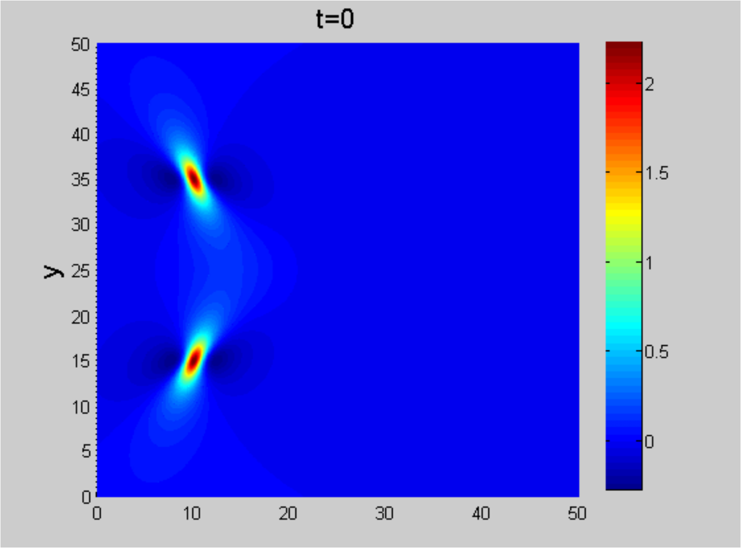
Maximum separation distance in the y-direction, depends on the initial relative velocity difference k $\left(k = \frac{V_1 - V_2}{V_2}\right)$.

scarcely depends on the initial phase

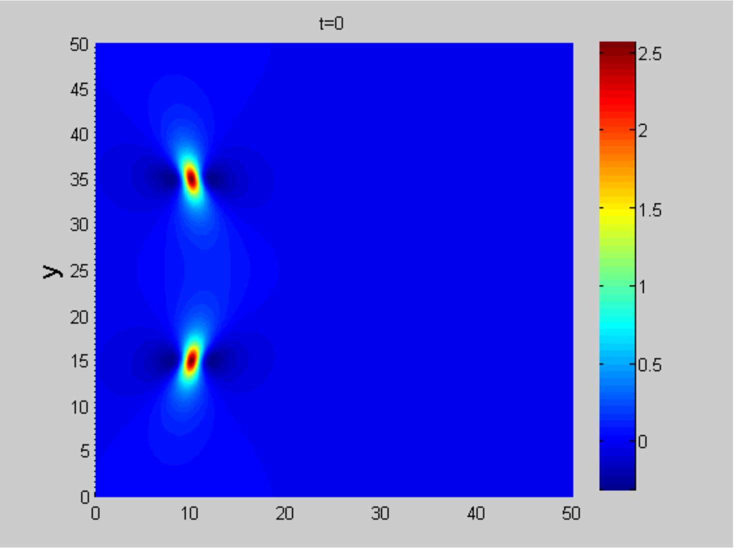
(Lu Z, Tian E M, Grimshaw R，2004)

Oblique interaction of two single-lump

Normal scattering



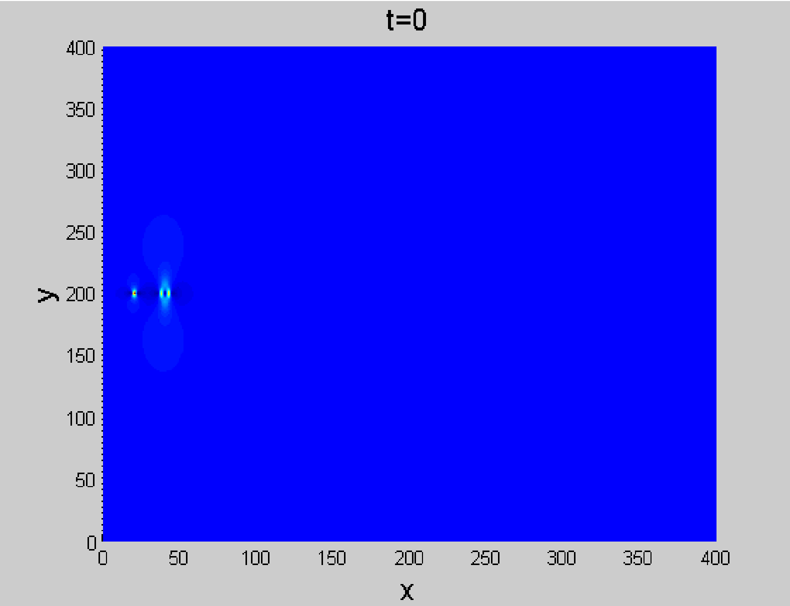
Anomalous scattering



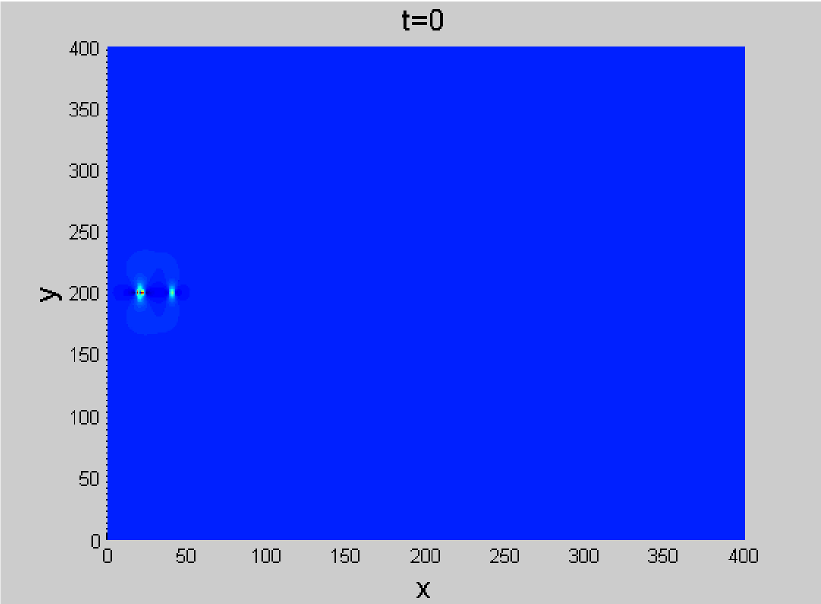
Oblique interaction of skew lumps with big μ (small amplitudes)

Oblique interaction of skew lumps with small μ (big amplitudes)

Direct interaction of a single and bi-lumps



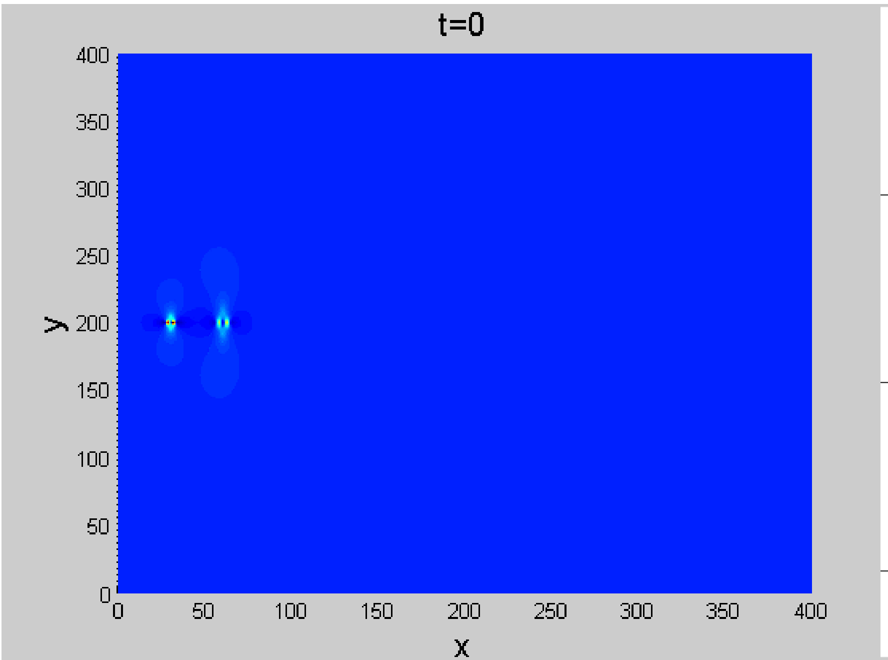
Line interaction of a lump and a bi-lump



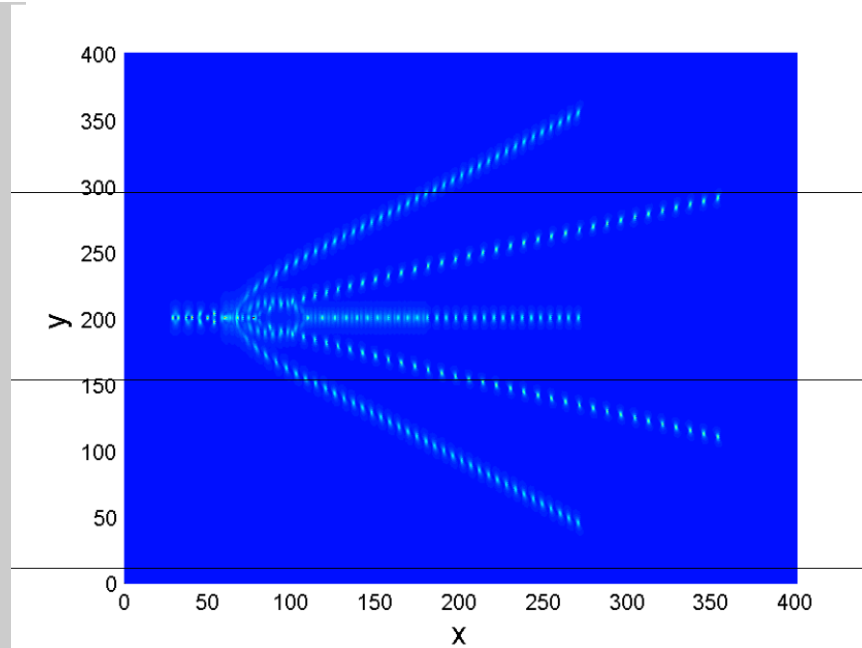
Line interaction of a bi-lump and a lump

Interaction of lumps

Direct interaction of two bi-lumps

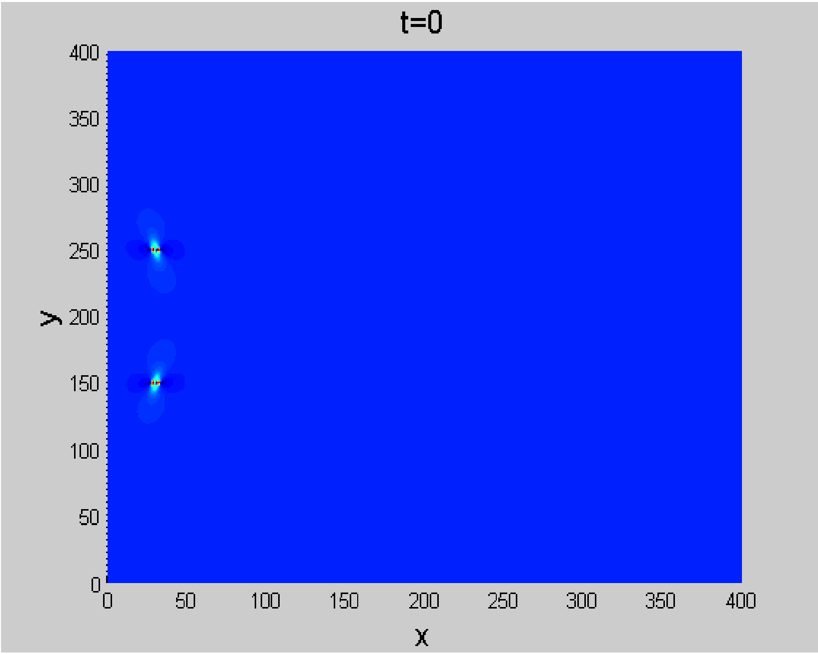


Line interaction of two bi-lumps

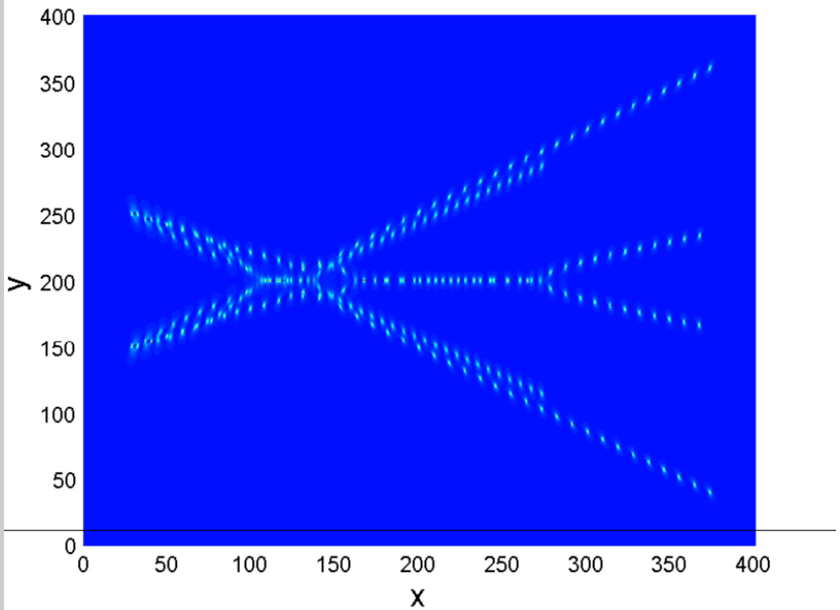


The trajectories of lumps emerging in the process of line interaction of two symmetrical bi-lumps

Oblique interaction of two bi-lump solitons

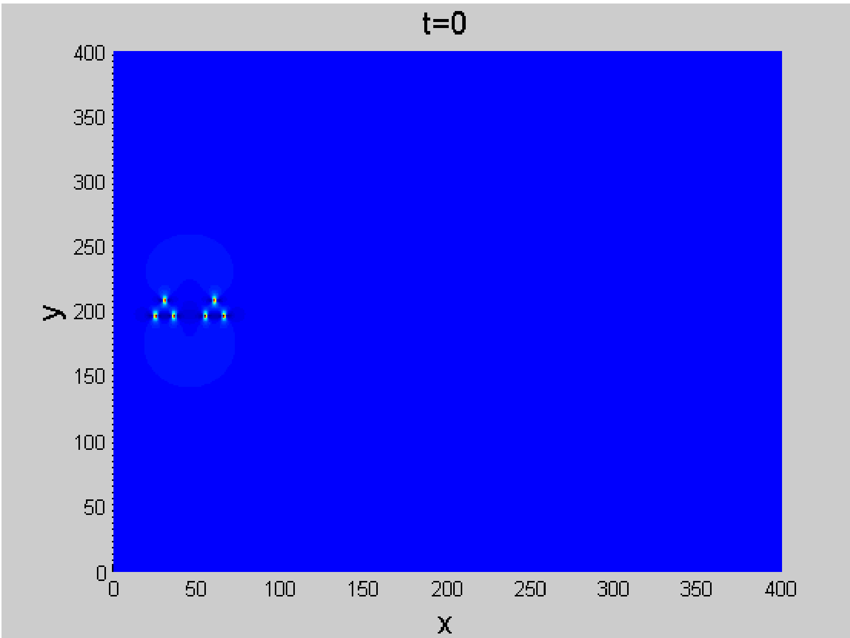


Oblique interaction of two skew lumps

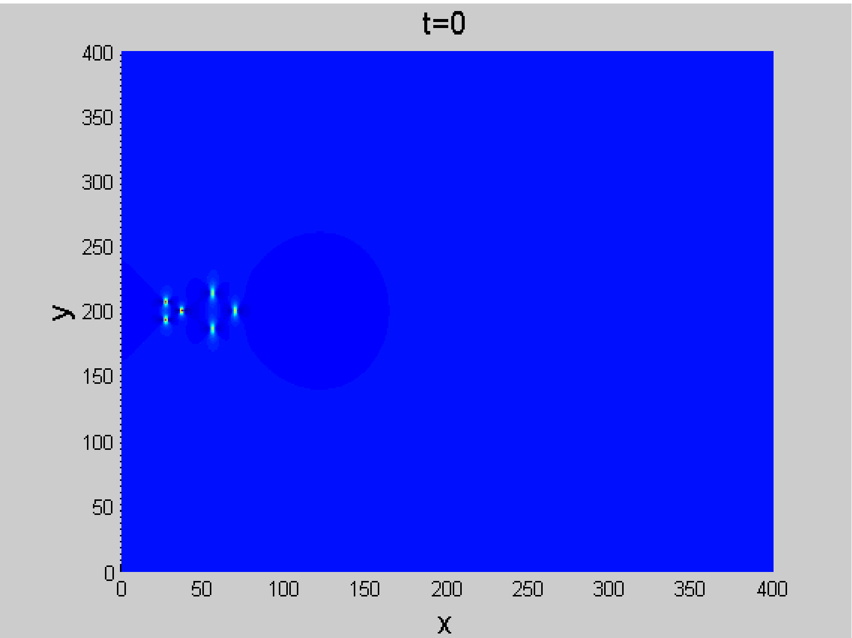


The trajectories of lumps emerging in the process of oblique interaction of two skew bi-lumps

Direct interaction of two triple-lumps

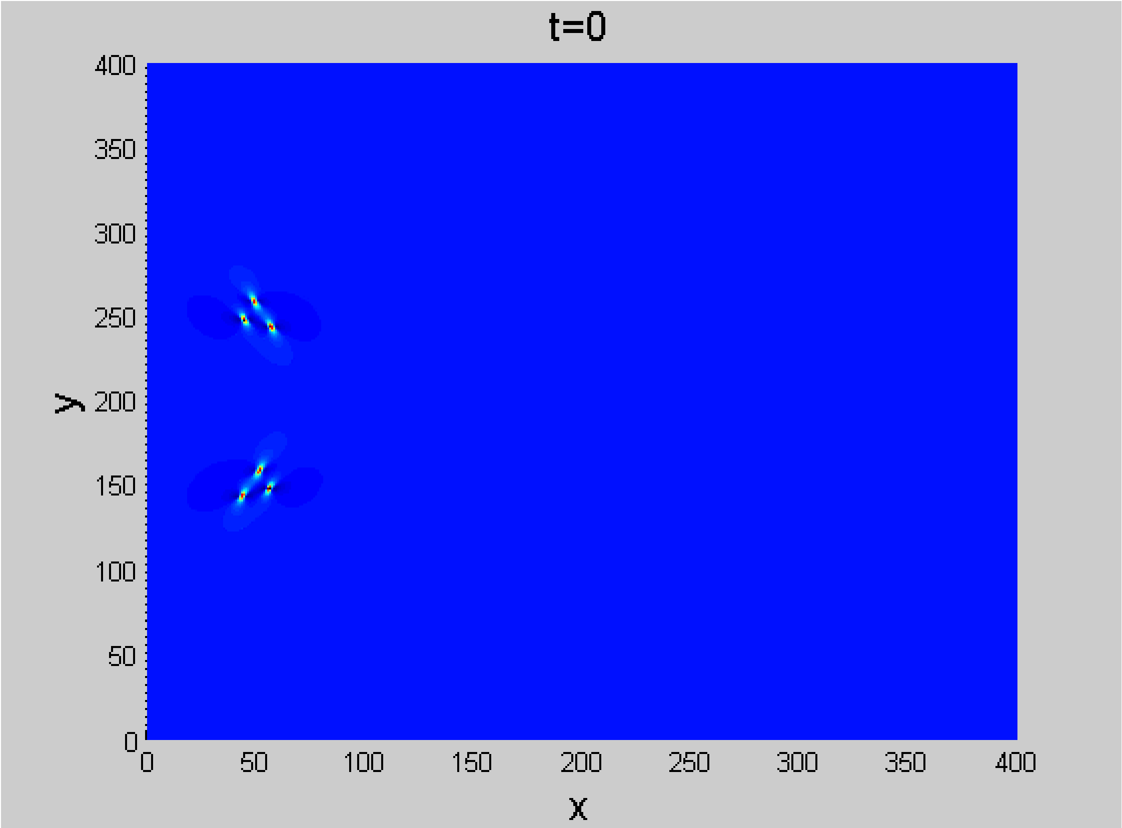


Line interaction of two triple lumps



Line interaction of two triple lumps

Oblique interaction of two triple-lumps



Oblique interaction of two triple-lumps

summary

1. Interaction of multi-lumps is different from the interaction of the single-lump case, in the case direct interaction it's much easier becoming to inelastic.

2. The component lumps of the multi-lump also can interact with themselves when interact with other multi-lump, which cause the interaction more complicated.

Conclusions

Conclusions

1. Multi-lump solutions with free parameters are obtained by the Hirota bilinear method. The dependence of stationary multi-lump structures on free parameters is discussed.

2. The multi-lump interactions resemble the processes in the interaction of elementary particles. Both elastic and inelastic collision may occur.

Thank you for your attention

Six-lump solution

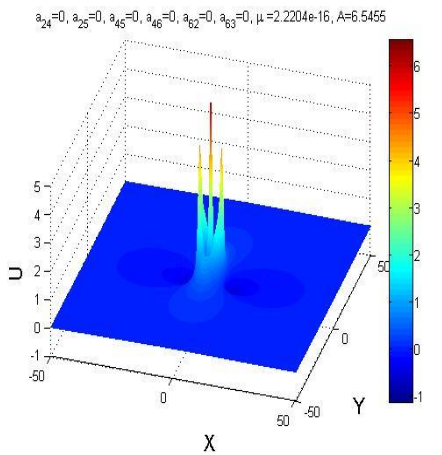
$$f = X^{12} + a_2 X^{11} Y + \cdots + a_{91}$$

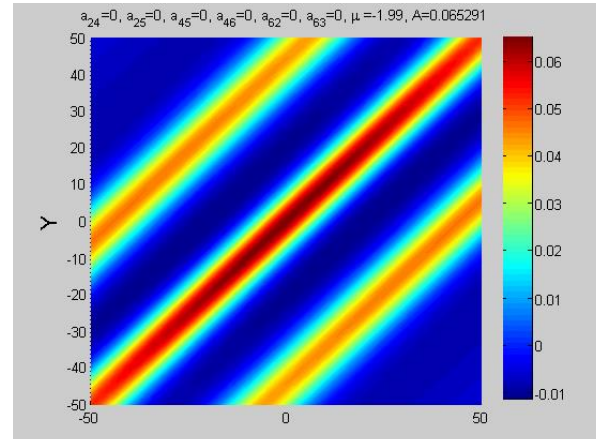
276 equations for 90 parameters, get six free parameters

$$a_{24}, a_{25}, a_{45}, a_{46}, a_{62}, a_{63}$$

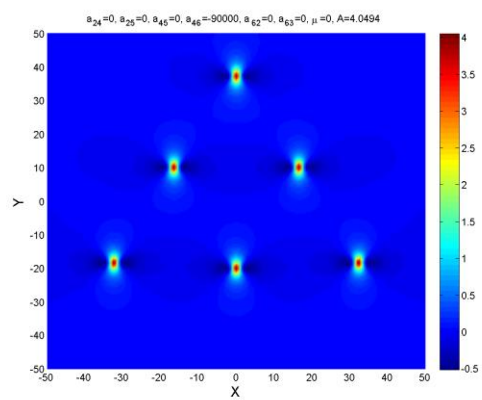
Lump solution to KPI

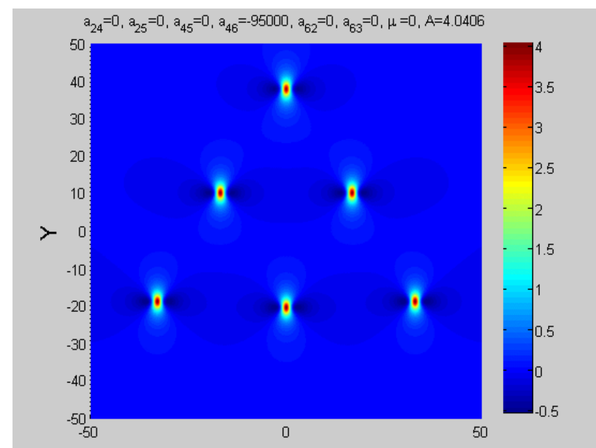
Triple-lump





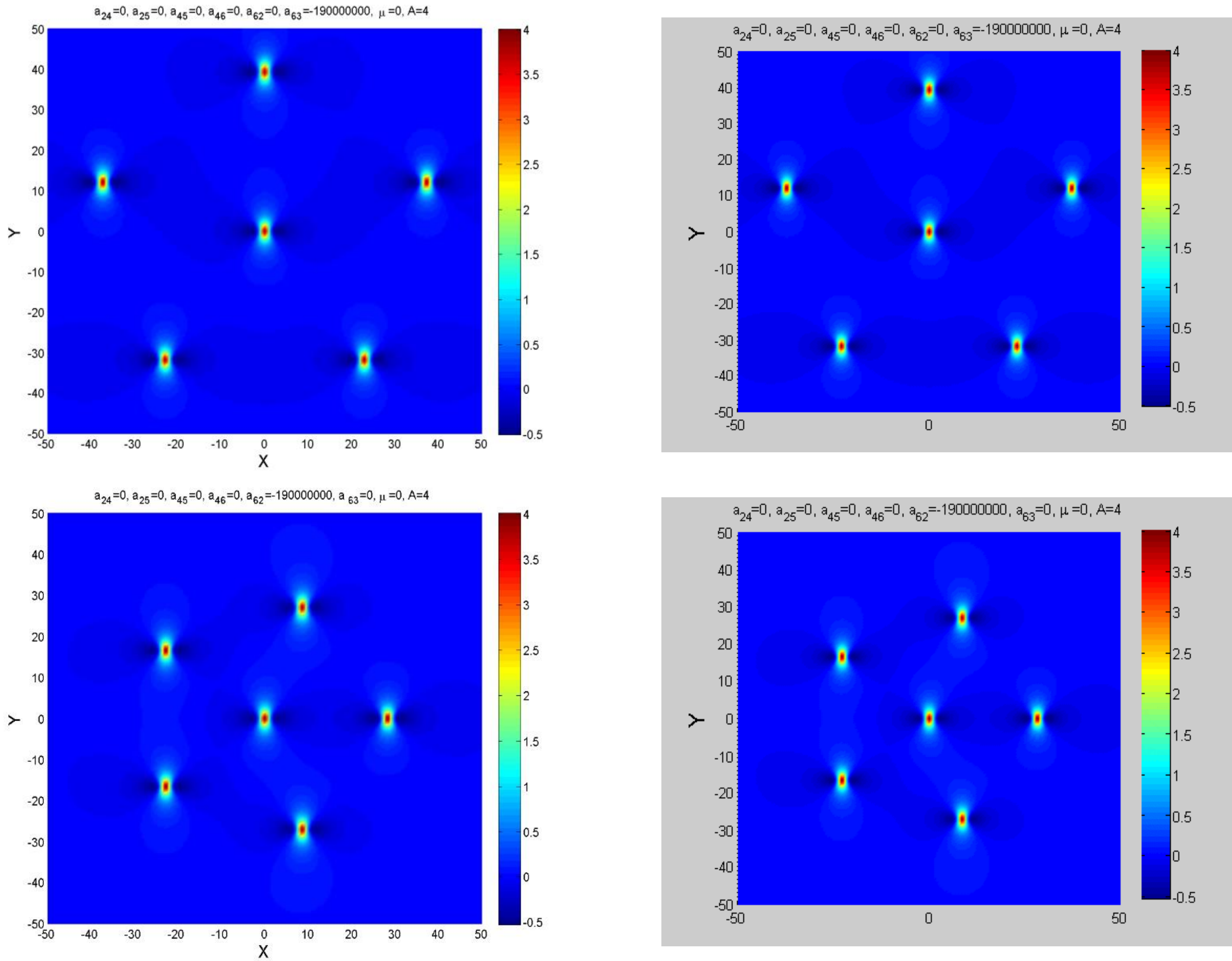
Triangular





Lump solution to KPI

Pentagrams



Lump solution to KPI

Skew case